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TYPES OF VARIETIES OF RECOGNIZABLE ω-LANGUAGES AND EILENBERG CORRESPONDENCES

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ABSTRACT. In this paper, we prove that the correspondences $\underline{V} \Rightarrow V^\omega$, $\underline{V} \Rightarrow \vec{V}$, proposed by D. Perrin (1982) between M-varieties $\underline{V}$'s of finite monoids and varieties of recognizable *omega*-languages V^ω 's and $\vec{V}$'s are one-to-one. New definitions of saturation and syntactic monoid of adherences of ω-languages basing on the limit operation are introduced. As consequence, a new type of varieties generated by adherences of ω-languages is defined and studied.

1. Introduction

S. Eilenberg (1976) established the two famous one-to-one correspondences between the M-varieties of finite monoids (S-varieties of finite semigroups) and the $*$-varieties ($+$-varieties) of regular languages. To obtain these correspondences, the important notions of syntactic monoids (syntactic semigroup) of languages are defined. These one-to-one correspondences immediately play a very fundamental role in the theory of varieties of regular languages and varieties of finite monoids (semigroups) (see [Ei]). Many deep works have been developed from Eilenberg's results in period 80's-90's. In the case of infinite words, recognizable ω-languages has been considered by J. R. Büchi 1962 [Bu], D. Müller 1963 [Mu]. D. Perrin 1982-84 [Pe82, Pe84] considered some types of classes of recognizable ω-languages basing on the ω-operation, limit operation, and the notion of saturation of recognizable ω-languages. These classes are defined by M-varieties and S-varieties and provided some kinds of Eilenberg correspondences which are subjects of this paper.

To achieve deeper researchs on this subject, one has to face with the following natural questions:

• Does a one-to-one correspondence between M-varieties (also, S-variesties) and some type of varieties of recognizable ω-languages exist ?

• Do the Eilenberg Correspondences introduced by D. Perrin are one-to-one ?

However, the situation seems to be more complicated than that of regular languages (see [Pec, HV93]). Considering the correspondences $\underline{V} \Rightarrow V^N$, V^ω, $\vec{V}$ between varieties of finite monoids (or finite semigroups) $\underline{V}$'s and three types of varieties of recognizable ω-languages V^N's, V^ω's, $\vec{V}$'s established by D. Perrin

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[Pe82]. J. Pecuchet 1986 [Pec] showed that in the case of S-varieties of finite semigroups, these correspondences are not one-to-one.

To overcome these situations, T. Wilke 1991 [Wi] proposed another approach based on a new kind of algebraic structure called *binoids* instead of traditional semigroups. This allows Wilke to define a new kind of variety of finite binoids and variety of infinitary languages (i.e. the languages consist of finite words and infinite words) and to establish a one-to-one Eilenberg correspondence between those varieties. Essentially, the one-to-one property in Wilke's result is induced from the one-to-one property in Eilenberg's result, because in each variety of infinitary languages considered by T. Wilke, the component consists of languages of finite words is nothing but a $+$-variety of languages in Eilenberg's work. It may be that some varieties of infinitary languages possess the same component of ω-languages of infinite words (see [Wi,Pec]). Hence, one can see that Wilke's result can not provide a direct proof of the one-to-one correspondence for the case of pure ω-languages considered by Perrin if we take monoid structure instead of binoid structure.

Dealing with the case of monoids, in [HV93] we showed that the Perrin correspondence $\underline{V} \Rightarrow V^N$ for the case of finite monoids is one-to-one. This result is obtained by studying thoroughly (see [HLV]) the notion of syntactic monoids of recognizable ω-languages which has been proposed by A. Arnold 1985 [Ar]. The correspondence $\underline{V} \Rightarrow V^N$ between the M-varieties, which is not a G-variety, and the N-varieties of recognizable ω-languages is one-to-one (for the case of all G-varieties, as shown by Perrin [Pe84], all corresponding N-varieties identify with the same trivial N-variety).

For the case of traditional monoids, this paper gives a positive answer to all cases mentioned above by showing that the correspondences $\underline{V} \Rightarrow V^\omega$, $\vec{V}$ and $\underline{V} \Rightarrow \overline{V^\omega}$ (the boolean closure of V^ω) are one-to-one without any restriction on the M-varieties (Theorems 2.7, 2.9 and Corollary 2.8). These results make use of the notion of *the trace of ω-languages* which has been considered in [HV93] and some properties of traces (Lemma 2.5, 2.6). To study adherences of languages (which are considered by L. Boasson and M. Nivat [BN]), a type of varieties of recognizable ω-languages generated by adherences of languages is proposed and one can see (Theorem 2.12) that these varieties are closely related to N-varieties. For this, a new notion of saturation of ω-languages based on the limit operation $^{\rightarrow}$ is introduced and studied.

In this paper, we consider only finite or free objects: finite alphabets, finite (or free) monoids, taking finite number of operations etc. For the basis notions and definitions we refer to [Ei,La,Pe82,Pe84].

For each alphabet A, denote by A^* the free monoid on A with the unit ε, $RecA^*$ ($RecA^\omega$) the set of regular (recognizable ω-) languages on A. Let $X \subseteq A^*$. We define the sets:

$X^\omega = \{x_1x_2 \cdots \in A^\omega | x_1, x_2, \cdots \in X - \{\varepsilon\}\}$;

$\vec{X} = \{x_1x_2 \cdots \in A^\omega | x_1x_2 \cdots x_k \in X - \{\varepsilon\}, k \geq 1\}$.

For monoids M, N, we denote $M \prec N$ whenever M is a homomorphic image of a submonoid of N. A class $\underline{V}$ of finite monoids is called an *M-variety* if for any M, $N \in \underline{V}$, for every monoid P, $P \prec M \times N$ implies $P \in \underline{V}$. Given a morphism $h : A^* \to M$, for the sake of brevity, for any $B \subseteq M$ we denote by h_B the sets $h^{-1}(B)$, and for $e, f \in M$, and by $h_{e,f}$ the set $h^{-1}(e)[h^{-1}(f)]^\omega$. The morphism h is said to *saturate* a language $L \subseteq A^*$ (resp. $W \subseteq A^\omega$) if for every $e, f \in M$, $h_e \cap L \neq \emptyset$ implies $h_e \subseteq L$ (resp. $h_{e,f} \cap W \neq \emptyset$ implies $h_{e,f} \subseteq W$). We then also say that the kernel congruence $\overset{h}{\sim}$ and the monoid M *saturate* L (resp. *saturate* W). The largest congruence which saturates L (resp. W), denoted by $\sim_L$ (resp. by $\sim_W$), is called the *syntactic congruence* of L (resp. of W). It is well-known (see [Ei]) that $\sim_L$ is defined by

$$(1.1) \qquad \forall\, u, v \in A^* : \ u \sim_L v \text{ iff } \text{“}\forall\, x, y \in A^* : \ xuy \in L \Leftrightarrow xvy \in L\text{”},$$

and due to [Ar,HLV], $\sim_W$ is defined by the following congruences on A^*:

(1.2)

$$\begin{cases} R_W = \{(u, v) \in A^* \times A^* |\ \forall x, y, z \in A^* : xuyz^\omega \in W \Leftrightarrow xvyz^\omega \in W\} \\ T_W = \{(u, v) \in A^* \times A^* |\ \forall x \in A^*, y, z \in A^+ : x(yuz)^\omega \in W \Leftrightarrow x(yvz)^\omega \in W\} \\ \sim_W = R_W \cap T_W. \end{cases}$$

Denote $A^*/\sim_L$ by M_L and $A^*/\sim_W$ by I_W. We call them the *syntactic monoid* of L and of W respectively. Given $W \in RecA^\omega$, for each $v \in A^*$ we set

$$(1.3) \quad W(v, -) = \{u \in A^+ |\ vu^\omega \in W\ \} \cup \{\varepsilon\} : \ W(-, v) = \{u \in A^* |\ uv^\omega \in W\ \}$$

with a *convention* that $W(-, \varepsilon) = \emptyset$ and $x\varepsilon^\omega \notin A^\omega$ for any $x \in A^*$. These sets are nothing but languages in the *trace of ω-language* W which is considered in [HV93]. Let M be a monoid. Define

$$E(M) = \{e \in M | e^2 = e\},$$
$$P(M) = \{(e, f) \in M \times M |\ ef = e, f^2 = f\}.$$

We say that a morphism $h : A^* \to M$ *recognizes* an ω-language $W \in A^\omega$ if $W = \bigcup_{(e,f) \in I} h_{e,f}$ for some $I \subseteq P(M)$ (or that W *is recognized* by M). Given an M-variety $\underline{V}$, for each alphabet A, due to Eilenberg [Ei] and D. Perrin [Pe82,84] we define

$$(1.4) \qquad AV^* \ = \ \{\ X \subseteq A^* |\ M_X \in \underline{V}\ \},$$

$$(1.5) \qquad \begin{cases} AV^\omega = \{W \in RecA^\omega | W \text{ is recognized by some } M \in \underline{V}\} \\ A\vec{V} = \{\vec{X} |\ X \in AV^*\}^{\mathcal{B}} \text{ the boolean closure of } \vec{X}'s, \ X \in AV^*, \\ A\overline{V^\omega} = (AV^\omega)^{\mathcal{B}} \text{ the boolean closure of the set } AV^\omega. \end{cases}$$

Then we call the family $V^\omega = \{AV^\omega | \forall\, A\}$ an *ω-variety*, the family $\vec{V} = \{A\vec{V} |\ \forall\, A\}$ an *L-variety* and $\overline{V^\omega} = \{A\overline{V^\omega} | \forall\, A\}$ an *$\overline{\omega}$-variety.* For each $X, Y \subseteq$

$A^{\infty} = A^* \cup A^{\omega}$ we define the *shuffle product* of X and Y by

$$X \text{ Ш } Y = \{ \ x_1y_1x_2y_2 \cdots \in A^{\infty} | \ x_1x_2 \cdots \in X, \ y_1y_2 \cdots \in Y \ \}.$$

Similar to (1.1), we associate to each subset A of a monoid M a *congruence* $\sim_A$ on M defined by:

$$\forall \, a, b \in M : a \sim_A b \text{ iff } \text{“}\forall \, p, q \in M : paq \in A \Leftrightarrow pbq \in A\text{”}$$

and denote $M/\sim_A$ by $M//A$ (and by $M//a$ if $A = \{a\}$).

2. Main results

We first need some lemmas.

Lemma 2.1. [Ei] *Let M be a monoid and $X \subseteq A^*$. The following conditions are equivalent:*

(1) $M_A \prec M$;

(2) *There exist a morphism $h : A^* \to M$ and $B \subseteq M$ such that $h_B = X$.*

Lemma 2.2. [Ei] *Let $h : S \to T$ be a surjective morphism and M a monoid. Then*

(1) $\forall B \subseteq T : \ S//h_B \cong T//B$.

(2) $\forall a \in M : \ M//a \prec M \prec \prod_{e \in M} M//e$.

Given $X \subseteq A^*$. We define the left and right quotients of X by an element v in A^*, the sets

$$v^{-1}X = \{u \in A^* | \ vu \in X\},$$
$$Xv^{-1} = \{u \in A^* | \ uv \in X\}.$$

Lemma 2.3. [Ei] *Let $X, Y \in RecA^*$. Then*

(1) *X, Y are saturated by $\sim_X \cap \sim_Y$;*

(2) *The family of regular languages saturated by h is closed under the Boolean operations and the formation of “left, right quotients” by elements of A^*.*

Due to Ramsey-Büchi we can deduce

Lemma 2.4. *Let $h : A^* \to M$ be a surjective morphism and n the index of M (i.e. $e^n \in E(M)$ for every $e \in M$). Then*

(1) $A^{\omega} = \bigcup_{(e,f) \in P(M)} h_{e,f}$;

(2) *For any $x, v \in A^*, v \neq \varepsilon$, $(e, f) \in P(M)$, then $xv^{\omega} \in h_{e,f}$ iff v admits a factorization $v^n = ab$ such that $xv^n a \in h_e$, $ba \in h_f$.*

Proof. (1) This is a well-known result.

(2) Since the “if” part is clear, it suffices to check the “only if” part. Suppose that $xv^{\omega} \in h_{e,f}$ for some $(e, f) \in P(M)$. It implies that there exist $m, k \in \mathbb{N}$, $1 \leq m, k; z, t \in A^*$ such that $v = zt$ and $xv^m z$, $xv^{m+k} z \in h_e$, $(tz)^k \in h_f$. Then

$(tz)^{kn} = ((tz)^n)^k$ implies $(tz)^n \in h_f$. The proof then is completed by taking $a = z$, $b = (tz)^{n-1}t$. $\square$

By the definition (1.3), one immediately has the following result:

Lemma 2.5. *For any $X, Y \in RecA^\omega$ and $v \in A^+$, we have*

(1) $(X \cup Y)(-, v) = X(-, v) \cup Y(-, v)$;

(2) $(X \cap Y)(-, v) = X(-, v) \cap Y(-, v)$;

(3) $(X - Y)(-, v) = X(-, v) - Y(-, v)$.

Lemma 2.6. *Let $h : A^* \to M$ be a surjective morphism, $X \subseteq A^*$ a language saturated by h and n the index of M. Then*

(1) *For any $(e, f) \in P(M)$*

$$h_{e,f}(-, v) = \bigcup_{v^n = ab, ba \in h_f} h_e(v^n a)^{-1}; \tag{2.1}$$

(2) *For any $v \in A^+$ and $(e, f) \in P(M)$, $h_{e,f}(-, v)$ is saturated by h;*

(3) *$\vec{X}(-, v)$ is saturated by h.*

Proof. (1) Denote by S the right-hand side of (2.1). If $x \in S$, then v admits some factorization $v^n = ab$ so that $ba \in h_f$ and $x \in h_e(v^n a)^{-1}$, i.e. $x(v^n a) \in h_e$. Consequently, $xv^\omega \in h_{e,f}$. Thus $x \in h_{e,f}(-, v)$. Conversely, if $x \in h_{e,f}(-, v)$, then $xv^\omega \in h_{e,f}$. Using Lemma 2.4 one deduces $xv^n a \in h_e$, $ba \in h_f$ for some $a, b \in A^*$ with $v^n = ab$. Hence $x \in h_e(v^n a)^{-1}$ and x belongs to S.

(2) This is a direct consequence of the equality (2.1) and Lemma 2.3.

(3) By lemma 2.4, we obtain

$$\vec{X} = \bigcup_{h_e \subseteq X} \overset{longrightarrow}{h_e} = \bigcup_{h_e \subseteq X, (e,f) \in P(M)} h_{e,f}.$$

From Lemma 2.5 we have

$$\vec{X}(-, v) = \bigcup_{h_e \subseteq X, (e,f) \in P(M)} h_{e,f}(-, v). \tag{2.2}$$

By Lemma 2.3 and the part (2) above, we deduce that $\vec{X}(-, v)$ is saturated by h. $\square$

For each M-variety $\underline{V}$, by (1.5) we define the varieties V^ω, $\vec{V}$, $\overline{V^\omega}$ and then the *Eilenberg Correspondences* $\underline{V} \Rightarrow V^\omega$, $\underline{V} \Rightarrow \vec{V}$, $\underline{V} \Rightarrow \overline{V^\omega}$. We now state the main results for Eilenberg correspondences.

Theorem 2.1. *The correspondence $\underline{V} \Rightarrow V^\omega$ from the class of all M-varieties to the class of all ω-varieties is one-to-one.*

Proof. By definition, it suffices to shows that if $\underline{V}_1 \Rightarrow V_1^\omega$, $\underline{V}_2 \Rightarrow V_2^\omega$ and $V_1^\omega = V_2^\omega$, then $\underline{V}_1 = \underline{V}_2$. Letting $M \in \underline{V}_1$, we prove that $M \in \underline{V}_2$. Choose an alphabet A_0 with a surjective morphism $h' : A_0^* \to M$ (for example $A_0 = \{a_m | m \in M\}$

with $h'(a_m) = m$). Take $A = A_0 \cup \{a\}$ for some $a \notin A_0$ and $h : A^* \to M$ is an extension of h' defined by $h|A_0 = h'$, $h(a) = 1$. Then $(e, 1) \in P(M)$ for each $e \in M$. It follows from (2.1) that

$$h_{e,1}(-, a) = \bigcup_{u_i v_i = a, v_i u_i \in h_1} h_e(au_i)^{-1}.$$

Since $h(a) = 1$ and either $u_i = \varepsilon$ or $u_i = a$, it implies that $h_e a^{-1} = h_e$ and then $h_e(au_i)^{-1} = h_e$. This shows that $h_{e,1}(-, a) = h_e$. Because $h_{e,1} \in AV_1^{\omega}$, by assumption $V_1^{\omega} = V_2^{\omega}$ one has $h_{e,1} \in AV_2^{\omega}$. Hence, there exists $N \in \underline{V}_2$ with a surjective morphism $g : A^* \to N$ such that

$$h_{e,1} = \bigcup_{(p,q) \in I} g_{p,q} \quad \text{for some } I \subseteq P(N). \tag{2.3}$$

By lemma 2.5 one gets $h_{e,1}(-, a) = \bigcup_{(p,q)\in I} g_{p,q}(-, a)$. It follows from Lemma 2.6 with the fact $h_e = h_{e,1}(-, a)$ that h_e is saturated by g. In turn, Lemmas 1, 2 yield $M//e = M_{h_e} \prec N$. Using Lemma 2.2 one obtains $M \prec \prod_{e\in M} M//e \prec N^{(m)}$ where $N^{(m)}$ is the m-fold Cartesian product of N and $m = cardM$. This shows that $M \in \underline{V}_2$, therefore $\underline{V}_1 \subseteq \underline{V}_2$. A similar verification gives $\underline{V}_2 \subseteq \underline{V}_1$. This completes the proof of $\underline{V}_1 = \underline{V}_2$. □

Corollary 2.1. *The correspondence $\underline{V} \Rightarrow \overline{V^{\omega}}$ from the class of all M-varieties of finite monoids to the class of all $\overline{\omega}$-varieties is one-to-one.*

Proof. It suffices to prove that $\overline{V^{\omega}}_1 = \overline{V^{\omega}}_2$ implies $\underline{V}_1 = \underline{V}_2$. Using the same method as the above proof, the only different thing one might meet is that instead of (2.3), in this proof $h_{e,1}$ is obtained from $g_{p,q}$, $(p, q) \in P(N)$ by taking a finite number of Boolean operations. But according to Lemma 2.5, the fact that g saturates $h_{e,1}(-, a)$ remains valid. Hence one also deduces that $\underline{V}_1 \subseteq \underline{V}_2$. A similar verification completes the proof of $\underline{V}_1 = \underline{V}_2$. □

Theorem 2.2. *The correspondence $\underline{V} \Rightarrow \vec{V}$ from the class of all M-varieties of finite monoids to the class of L-varieties of recognizable ω-languages is one-to-one.*

Proof. Supposing that $\underline{V}_1 \Rightarrow \vec{V}_1$, $\underline{V}_2 \Rightarrow \vec{V}_2$, $\vec{V}_1 = \vec{V}_2$, we have to prove that $\underline{V}_1 = \underline{V}_2$. First we proceed to check that $\underline{V}_1 \subseteq \underline{V}_2$. For this, considering an arbitrary $M \in \underline{V}_1$, we show that $M \in \underline{V}_2$. By a similar method of the proof of Theorem 2.1, we take an appropriate alphabet A_0 with a surjective morphism $h' : A_0^* \to M$ and the extension h' of $h : A^* \to M$ defined by $h|A_0 = h'$, $h(a) = 1$. For any $e \in M$, using the equality $h(a) = 1$ with some simple verifications we obtain $h_e = \vec{h}_e(-, a)$ and then

$$(h_e.a^{\omega})(-, a) = h_e = \vec{h}_e(-, a) \tag{2.4}$$

Since $\vec{h}_e \in A\vec{V}_1$, by assumption $\vec{V}_1 = \vec{V}_2$, one has $\vec{h}_e \in A\vec{V}_2$. Applying Lemma 2.3 one can verify that there exists a monoid $N \in \underline{V}_2$, a surjective morphism

$g : A^* \to N$ with some regular languages $L_1, L_2, \cdots, L_k$ saturated by g such that $\vec{h}_e$ is obtained from $\vec{L}_i$ by taking a finite number of Boolean operations. By Lemma 2.6, $\vec{L}_i(-,a)$ is saturated by g. It follows from Lemmas 2.3, 2.5 and (2.4) that h_e is saturated by g. This with Lemmas 2.1, 2.2 give $M//e = M_{h_e} \prec N$. Hence $M \prec \prod_{e\in M} M//e \prec N^{(m)}$ with $m = cardM$. Thus $M \in \underline{V}_2$, hence $\underline{V}_1 \subseteq \underline{V}_2$. In turn, a similar verification gives $\underline{V}_2 \subseteq \underline{V}_1$. The proof is completed. □

Next, we introduce a new type of varieties of recognizable ω-languages which is generated by adherences of languages. Given a morphism $h : A^* \to M$ and $W \in RecA^\omega$, we say that W *is l-saturated by* h if the following condition is satisfied

$$\vec{h_e} \cap W \neq \emptyset \Rightarrow \vec{h}_e \subseteq W \tag{2.5}$$

For any alphabet, we denote by $DRecA^\omega$ the subclass of $RecA^\omega$ containing all recognizable ω-languages of the form $\vec{X}$, $X \in RecA^*$. Let $\underline{V}$ be a U_1*-variety* (i.e. an M-variety containing the monoid $\{0,1\}$) (see [6]), for each alphabet A we define

$$\begin{cases} AV^N & = \{W \in RecA^\omega |\ I_W \in \underline{V}\}, \\ AV^L & = \{W \in RecA^\omega |\ W \text{ is l-saturated by some } M \in \underline{V}\}, \\ ADV^N & = AV^N \cap DRecA^\omega. \end{cases} \tag{2.6}$$

To get a relationship between AV^L, AV^N, $DRecA^\omega$ we first need some technical Lemmas. By definition with some simple verifications, one has

Lemma 2.7. *Let $h : A^* \to M$ be a surjective mmorphism. For any $e, f \in M$, $\vec{h}_e \neq \emptyset$, then $\vec{h}_e \cap \vec{h}_f \neq \emptyset$ iff $e\mathcal{R}f$, where $\mathcal{R}$ is the Green's relation on M.*

Using this lemma one gets

Lemma 2.8. *Let $W \in RecA^\omega$ and $h : A^* \to M$ be a surjective morphism saturating W. If W and $A^\omega - W$ in $DRecA^\omega$, then there exists $L \in RecA^*$ such that L is saturated by h and $\vec{L} = W$, $\overrightarrow{A^* - L} = A^\omega - W$.*

Proof. First, without loss of generality we may assume that W, $A^\omega - W \neq \emptyset$ and $W = \vec{X}$, $A^\omega - W = \vec{Y}$ for some $X, Y \in RecA^*$. Put

$$W' = A^\omega - W, \quad \sim = \sim_W \cap \sim_X \cap \sim_Y$$

where $\sim_W$, $\sim_X$, $\sim_Y$ are defined by (1.1), (1.2). Considering the quotient monoid $A^*/\sim$ we define the following subsets of A^*:

$$(X \sim \mathcal{R}) = \bigcup_{[y]_\sim \mathcal{R}[x]_\sim, x\in X} [y]_\sim; \quad U = \bigcup_{u\in A^*, \overrightarrow{[u]_\sim} = \emptyset} [u]_\sim, \tag{2.7}$$

where $[y]_\sim$ is the class of y modulo $\sim$ and $\mathcal{R}$ is the Green's relation on $A^*/\sim$. From Lemma 2.7 one can see that $U = (U \sim \mathcal{R})$. By definition, using again

Lemma 2.7 one deduces $(Y \sim \mathcal{R}) \cap (X \sim \mathcal{R}) - U = \emptyset$. Taking $Y_0 = (Y \sim \mathcal{R}) - U$, $X_0 = (X \sim \mathcal{R}) - U$ we obtain

$$X_0 = (X_0 \sim \mathcal{R}); \quad Y_0 = (Y_0 \sim \mathcal{R}); \quad X_0 \cap Y_0 = \emptyset. \tag{2.8}$$

Hence, $W' \subseteq \vec{Y_0}$, $W \subseteq \vec{X_0}$, this implies $\vec{Y_0} = W'$, $\vec{X_0} = W$. Furthermore, we have the following facts as obvious consequences of Lemma 2.7

(i) $\forall u \in X_0 \cup Y_0 :\ \overrightarrow{[u]_\sim} \neq \emptyset$.

(ii) If $u, v \in A^*,\ uv^\omega \in W$ (resp. $uv^\omega \in W'$), then $uv \sim u$ and $v^2 \sim v$ imply $u \in X_0$ (resp. $u \in Y_0$) (applying (2.8), $\vec{X_0} = W$, $\vec{Y_0} = W'$).

Now, we prove that $(X_0 \sim_W \mathcal{R}) \cap (Y_0 \sim_W \mathcal{R}) = \emptyset$. Indeed, assuming the contrary, there exists $u \in X_0$, $v \in Y_0$, $u', v' \in A^*$ such that $u \sim_W u'$, $v \sim_W v'$, $\exists\, \lambda, \gamma \in A^*$: $u'\lambda \sim_W v'$, $v'\gamma \sim_W u'$. Hence

$$u\lambda \sim_W v; \quad v\gamma \sim_W u, \quad \lambda, \gamma \neq \varepsilon. \tag{2.9}$$

Since $u \in X_0$, $v \in Y_0$, by (i) there exist $\alpha, \beta \in A^*$ such that

$$u\alpha^\omega \in W,\ u\alpha \sim u,\ \alpha^2 \sim \alpha;\ v\beta^\omega \in W',\ v\beta \sim v,\ \beta^2 \sim \beta. \tag{2.10}$$

Considering at the same time the two infinite words $w = u(\lambda\beta\gamma\alpha)^\omega$, $w_1 = v(\gamma\alpha\lambda\beta)^\omega$ we can assert that $w \in \vec{Y_0} \cap \vec{X_0}$. Indeed:

(iii) By (2.9), (2.10) we have $u\lambda\beta \sim_W v\beta \sim v$. Since $\sim \subseteq \sim_W$, then $u\lambda\beta \sim_W v$. Similarly, $v\gamma\alpha \sim_W u$.

(iv) Since $u\lambda \sim_W v,\ \ v\beta^\omega \in W',\ \sim_W \equiv \sim_{W'}$, it follows that $(u\lambda\beta)\beta^\omega \in W'$. By the fact (ii) one has $u_1 = u\lambda\beta \in Y_0$ and by analog fact, $v_1 = v\gamma\alpha \in X_0$.

(v) Using $(v\gamma\alpha)\lambda\beta \sim_W u\lambda\beta \sim_W v$, $v\beta^\omega \in W'$ one obtains $(v\gamma\alpha\lambda\beta)\beta^\omega \in W'$. Then the fact (ii) yields $v_2 = v\gamma\alpha\lambda\beta \in Y_0$. Analogously, $u_2 = u\lambda\beta\gamma\alpha \in X_0$.

By (iii), $v_2 \sim_W v$, $u_2 \sim_W u$. Applying again the same arguments as (iv), (v) for u_2, v_2 one gets

(vi) $u_3 = u_2\lambda\beta \in Y_0$, $v_3 = v_2\gamma\alpha \in X_0$,

(vii) $u_4 = u_2\lambda\beta\gamma\alpha \in X_0$, $v_4 = v_2\gamma\alpha\lambda\beta \in Y_0$,

and so on, using again and again the same arguments as (iv), (v) one obtains the infinite chains of left factors of w and w_1: $u, u_2, u_4 \cdots \in X_0$; $u_1, u_3, \cdots \in Y_0$; $v_1, v_3, \cdots \in X_0$; $v_2, v_4, \cdots \in Y_0$. This shows that $\vec{X_0} \cap \vec{Y_0} \neq \emptyset$, a contradiction. Thus

$$(X_0 \sim_W \mathcal{R}) \cap (Y_0 \sim_W \mathcal{R}) = \emptyset.$$

Put $L = (X_0 \sim_W \mathcal{R})$, $L' = (Y_0 \sim_W \mathcal{R})$. Then L, L' are saturated by h. By Lemma 2.7 it implies that

$$\vec{L} \cap \vec{L'} = \emptyset, \quad W = \vec{X_0} \subseteq \vec{L}, \quad W' = \vec{Y_0} \subseteq \vec{L'} \subseteq \overrightarrow{A^* - L}.$$

Consequently, $W = \vec{L}$ and $W' = \overrightarrow{A^* - L}$. □

We call a recognizable ω-language $W \in RecA^\omega$ an *adherence* if $W = \overrightarrow{LF(X)}$ for some $X \in RecA^*$, where $LF(X)$ is the set of all left factors of X. We have the following relationship between L-varieties, N-varieties and adherences.

Theorem 2.3. *Let $\underline{V}$ be a U_1-variety. For every alphabet A, the following conditions are equivalent:*

(i) $W \in AV^L$;

(ii) *W and $A^\omega - W$ belong to ADV^N;*

(iii) *W belongs to the boolean closure of all adherences in AV^N.*

Proof. (i) $\Rightarrow$ (ii) Let $W \in AV^L$. There exist a surjective morphism $h : A^* \to M$ with $M \in \underline{V}$ such that W is l-saturated by h. Taking

$$\begin{aligned}&I = \{e \in M |\ \vec{h}_e \cap W \neq \emptyset\},\\ &P = I\mathcal{R} = \{e \in M |\ \exists\, f \in P : e\mathcal{R}f\},\\ &Q = M - P,\ X = h_P,\ Y = h_Q\end{aligned}$$

by Lemma 2.7 we deduce that $\vec{X} = W$, $\vec{Y} = A^\omega - W$. Thus W, $A^\omega - W \in ADV^N$.

(ii) $\Rightarrow$ (iii) Suppose that W, $A^\omega - W \in ADV^N$. By virtue of Lemma 2.8, there exist $M \in \underline{V}$, $P \subseteq M$, $Q = M - P$ such that $P = P\mathcal{R}$, $W = \vec{h}_P$, $A^\omega - W = \vec{h}_Q$. For each $m \in M$, we write $m \leq_{\mathcal{R}} e$ whenerver $mM \subseteq eM$, $m <_{\mathcal{R}} e$ if $mM \subseteq eM$ and denote by $\langle m\rangle$ the set $\{e \in M |\ m \leq_{\mathcal{R}} e\}$. Then by a simple verification we get the equality

$$R_m = \langle m\rangle - \bigcup_{m <_{\mathcal{R}} e} \langle e\rangle. \tag{2.11}$$

where R_m is the *R-class of m modulo $\mathcal{R}$.* By Lemma 2.7 we obtain

$$\overrightarrow{h_{R_m}} = \overrightarrow{h_{\langle m\rangle}} - \bigcup_{m <_{\mathcal{R}} e} \overrightarrow{h_{\langle e\rangle}}. \tag{2.12}$$

Besides, one can verify that for each $e \in M$, $h_{\langle e\rangle} = LF(h_e)$ is the set of all left factors of h_e, i.e. $\overrightarrow{h_{\langle e\rangle}}$ is an adherence. This together with (2.12) implies (iii).

(iii) $\Rightarrow$ (i) First, let W be an adherence in AV^N and $X \in RecA^*$ such that $W = \overrightarrow{LF(X)}$. Consider the suntactic morphism $h : A^* \to M_X$ of X. By definition, $X = h_P$ for some $P \subseteq M_X$. By a direct verification we obtain

$$LF(X) = \bigcup_{e \in P} h_{<e>}, \quad W = \bigcup_{e \in P} \overrightarrow{h_{\langle e\rangle}} = \bigcup_{e \leq_{\mathcal{R}} m, e \in P} \overrightarrow{h_{R_m}},$$

hence from Lemma 2.7 it follows that W is l-saturated by h. Thus $W \in AV^L$. Second, using the fact that AV^L is closed under Boolean operations, we deduce that if W is in the boolean closure of adherences in AV^N, then $W \in AV^L$. $\square$

Remark. From Lemma 2.8 one can see that if W, $A^\omega - W \in DRecA^\omega$, then for any morphism $h : A^* \to M$, h saturates W iff h l-saturates W. Hence, we can obtain similar results as A. Arnold's results by using the same method in [Ar]. The main one is that the largest congruence, for which W is l-saturated, is notthing but the syntactic congruence of W. Moreover, one can verify that each A-variety is closed under boolean operations, inverse images of morphisms, the formation of "left quotient" by finite words.

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