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LOGARITHMIC INTEGRALS, SOBOLEV SPACES AND RADON TRANSFORM IN THE PLANE

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Abstract. We prove that the set $\{\varphi_0, \varphi_1, \varphi_4, \ldots, \varphi_{3k+1}, \ldots\}$ of Hermite functions is an orthogonal system in the Sobolev space $H^1(\mathbf{R}) = H_{(1)}(\mathbf{R})$. Furthermore, the logarithmic integral of a function f from the real Hardy space $\mathcal{H}^1(\mathbf{R})$ is exactly the primitive function of $-\tilde{f}$ (the Hilbert transform of f). And more interesting formulas are found for Radon transform of Hermite-like functions.

1. Hermite functions and logarithmic integrals

Consider the following power series

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \exp(2tx - t^2), \tag{1}$$

where H_n are the Hermite polynomials. Replace x by $-x$ we have

$$H_n(-x) = (-1)^n H_n(x).$$

The Hermite function φ_n is defined by setting

$$\varphi_n(x) = H_n(x)\exp(-x^2/2).$$

We have

$$\varphi_n(-x) = (-1)^n \varphi_n(x)$$

and

$$\sum_{n=0}^{\infty} \varphi_n(x)\frac{t^n}{n!} = \exp\Big(2tx - t^2 - \frac{x^2}{2}\Big). \tag{2}$$

It is very well-known that the system $\{\varphi_n\}_{n=0}^{\infty}$ is orthogonal in $L^2(\mathbf{R})$. Next we give a new method to prove this and get more results for Hermite functions. To

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this end we use the following formula from [4]:

(3)
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\bigl(-(Ax^2+Bx+C)\bigr)dx=\frac{1}{\sqrt{2A}}\exp\Bigl(\frac{B^2-4AC}{4A}\Bigr)\quad(\Re(A)>0).$$

Therefore, taking the Fourier transforms of both sides of (2) we have

$$\sum_{n=0}^{\infty}\hat{\varphi}_n(\xi)\frac{t^n}{n!}=\exp\Bigl(-2it\xi+t^2-\frac{\xi^2}{2}\Bigr). \tag{4}$$

Here the Fourier transform $\hat{\phi}$ of a function ϕ is defined by

$$\hat{\phi}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)e^{-ix\xi}dx.$$

On the other hand, note that if in (2) we replace x by ξ and t by $-it$ then

$$\sum_{n=0}^{\infty}(-i)^n\varphi_n(\xi)\frac{t^n}{n!}=\exp\Bigl(-2it\xi+t^2-\frac{\xi^2}{2}\Bigr).$$

Compare this series with (4) we get

$$\hat{\varphi}_n=(-i)^n\varphi_n.$$

If in (2) we replace t by s and take the product of these power series, then we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi_n(x)\varphi_m(x)\frac{t^ns^m}{n!m!}=\exp\bigl(2tx+2sx-t^2-s^2-x^2\bigr).$$

Integrate term by term according to x and apply (3) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Bigl(\int_{-\infty}^{\infty}\varphi_n(x)\varphi_m(x)dx\Bigr)\frac{t^ns^m}{n!m!}&=\int_{-\infty}^{\infty}\exp\bigl(2tx+2sx-t^2-s^2-x^2\bigr)dx\\&=\sqrt{\pi}\exp(2st)\\&=\sqrt{\pi}\sum_{n=0}^{\infty}\frac{2^ns^nt^n}{n!}\cdot\end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty}\varphi_n(x)\varphi_m(x)dx=\sqrt{\pi}2^nn!\delta(n-m), \tag{5}$$

here δ denotes the Kronecker-delta. This proves that the Hermite functions are orthogonal in $L^2(\mathbf{R})$. Next, we prove that the system of Hermite functions can be separated into 3 parts which are orthogonal in the Sobolev space $H_{(1)}(\mathbf{R})$. To this end we define the Sobolev norm $||\cdot||_{(1)}$ by letting

$$||u||_{(1)}^2 = \int\limits_{-\infty}^{\infty} |\hat{u}(x)|^2(1+x^2)dx = ||u||^2 + ||u'||^2.$$

Here, u' denotes the distributional derivative of u. The Sobolev space $H^1(\mathbf{R}) = H_{(1)}(\mathbf{R})$ is the set of all $u \in L^2(\mathbf{R})$ such that $||u||_{(1)} < \infty$. This is a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_{(1)}$ defined by setting

$$\langle u, v\rangle_{(1)} = \langle u, v\rangle + \langle u', v'\rangle.$$

Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbf{R})$. We deduce from (5) that $\langle\varphi_n, \varphi_m\rangle = \sqrt{\pi}2^n n!\delta(n-m)$. Now derivate (2) according to x we have

$$\sum_{n=0}^{\infty} \varphi_n'(x)\frac{t^n}{n!} = (2t-x)\exp\Big(2tx - t^2 - \frac{x^2}{2}\Big).$$

Multiple this equation with itself after replacing t by s we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varphi_n'(x)\varphi_m'(x)\frac{t^n s^m}{n!m!} = (2t-x)(2s-x)\exp\big(2tx + 2sx - t^2 - s^2 - x^2\big).$$

Integrating term by term according to x we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \langle\varphi_n', \varphi_m'\rangle\frac{t^n s^m}{n!m!} &= \int\limits_{-\infty}^{\infty} (2t-x)(2s-x)\exp\big(2tx + 2sx - t^2 - s^2 - x^2\big)dx \\ &= e^{2ts}[\Gamma(3/2) - (t-s)^2\Gamma(1/2)] \\ &= \sqrt{\pi}\sum_{n=0}^{\infty}\frac{2^n t^n s^n}{n!}\Big(\frac{1}{2} - t^2 - s^2 + 2ts\Big).\end{aligned}$$

This implies that

$$\begin{aligned}||\varphi_0'||^2 &= \frac{\sqrt{\pi}}{2} \\ ||\varphi_n'||^2 &= 2^{n-1}(n+2)n!\sqrt{\pi} \quad \text{for } n > 0, \\ \langle\varphi_n', \varphi_{n+2}'\rangle &= \langle\varphi_{n+2}', \varphi_n'\rangle = -2^n(n+2)!\sqrt{\pi} \\ \langle\varphi_n', \varphi_m'\rangle &= 0 \quad \text{if} \quad |n-m| = 1 \quad \text{or} \quad |n-m| > 2,\end{aligned}$$

and we obtain the following theorem.

Theorem 1. *The following systems of Hermite functions are orthogonal in the Sobolev space* $H_{(1)}(\mathbf{R})$*:*

(i) $\{\varphi_0, \varphi_1, \varphi_4, \ldots, \varphi_{3k+1}, \ldots\}$;
(ii) $\{\varphi_1, \varphi_2, \varphi_5, \ldots, \varphi_{3k+2}, \ldots\}$;
(iii) $\{\varphi_0, \varphi_3, \varphi_6, \ldots, \varphi_{3k}, \ldots\}$.

Next we define the Hilbert transform and the real Hardy space $\mathcal{H}^1(\mathbf{R})$. The Hilbert transform $Hf := \tilde{f}$ of a function $f \in L^p(\mathbf{R})$ ($p \in [1,\infty)$) is defined by the formula

$$Hf(x) = \tilde{f}(x) = \frac{1}{\pi}(\text{p.v.}) \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt.$$

It is well-known that

$$\widehat{Hf}(\xi) = -i \,\text{sign}(\xi) \hat{f}(\xi), \qquad H(Hf) = -f,$$

and

$$\langle f, \tilde{g} \rangle = -\langle \tilde{f}, g \rangle \quad \text{for } f \in L^p(\mathbf{R}) \quad \text{and } g \in L^q(\mathbf{R}), \quad \frac{1}{p} + \frac{1}{q} = 1,\ 1 < p < \infty.$$

Therefore, the Hilbert transform is a *unitary operator* in both Hilbert spaces $L^2(\mathbf{R})$ and $H_{(1)}(\mathbf{R})$. The Hilbert transform of the characteristic function $\chi_{(a,b)}$ of the interval (a,b) is

$$\tilde{\chi}_{(a,b)}(x) = \frac{1}{\pi} \ln \left| \frac{x-a}{x-b} \right|,$$

so we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \left| \frac{x-a}{x-b} \right| dx = - \int_a^b \tilde{f}(x) dx. \tag{6}$$

The *real Hardy space* $\mathcal{H}^1(\mathbf{R})$ is the set of all functions $f \in L^1(\mathbf{R})$ such that $\tilde{f} \in L^1(\mathbf{R})$. Functions in the real Hardy space are called *Hardy functions*. From [1] we have $\varphi_n \in \mathcal{H}^1(\mathbf{R})$ for every odd n. It is a well-known fact that the dual space of $\mathcal{H}^1(\mathbf{R})$ is $BMO(\mathbf{R})$ and the logarithmic function $\ln x$ is in BMO. So we can define the *logarithmic integral*

$$F(b) := \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \frac{1}{|x-b|} dx$$

for all function $f \in \mathcal{H}^1(\mathbf{R})$. But by (6) we have

$$F(b) - F(a) = - \int_a^b \tilde{f}(x) dx;$$

so the function F is absolutely continuous on the real line and $F'(b) = -\tilde{f}(b)$ for almost all $b \in \mathbf{R}$. Take the Fourier transform of F' in the distributional sense we have $it\hat{F}(t) = i \,\text{sign}(t)\hat{f}(t)$. This implies

$$\hat{F}(t) = \frac{\hat{f}(t)}{|t|} \in L^1(\mathbf{R}) \qquad \text{(by Hardy inequality)}.$$

Hence

$$F(b) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \hat{f}(t) \frac{e^{ibt}}{|t|} dt$$

and

$$\tilde{F}(b) = -\frac{i}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \hat{f}(t) \frac{e^{ibt}}{t} dt.$$

Consequently,

$$\lim_{|a|\to\infty} F(a) = \lim_{|a|\to\infty} \tilde{F}(a) = 0.$$

Thus we have the following result.

Theorem 2. *The logarithmic integral F of a Hardy function f is absolutely continuous and it can be rewritten in the form*

$$F(b) = -\int\limits_{-\infty}^{b} \tilde{f}(x) dx.$$

In [3] it is proved (in a complicated manner) that the logarithmic integral F is of bounded variation if the Hardy function f is of compact support. Our result is much stronger. Note that if φ is a function in $L^2(\mathbf{R})$ then

$$H(\varphi^2 - \tilde{\varphi}^2) = 2\varphi\tilde{\varphi};$$

so the function $f := \varphi^2 - \tilde{\varphi}^2$ is a Hardy function. This is the most important example for Hardy functions. For example, if

$$\varphi(x) = \frac{1}{x^2+1}$$

then, according to [1],

$$\tilde{\varphi}(x) = \frac{x}{x^2+1} \cdot$$

Thus

$$f(x) = \frac{1-x^2}{(x^2+1)^2}$$

is a Hardy function and we have

$$\tilde{f}(x) = \frac{2x}{(x^2+1)^2} = -\varphi'(x).$$

Apply Theorem 2 we obtain the following interesting formula

$$\frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{1-x^2}{(x^2+1)^2} \ln \frac{1}{|x-b|} dx = \varphi(b) = \frac{1}{b^2+1} \cdot$$

Next we compute the Hilbert transforms of the Hermite functions φ_n. To this end, note that (2) implies

$$\sum_{n=0}^{\infty} \varphi_n(x)\frac{t^n}{n!} = \exp\left(2tx - t^2 - \frac{x^2}{2}\right) = e^{t^2}\varphi_0(x-2t).$$

Take the Hilbert transform of both sides according to the variable x we have

$$\sum_{n=0}^{\infty} \tilde{\varphi}_n(x)\frac{t^n}{n!} = e^{t^2}\tilde{\varphi}_0(x-2t).$$

Therefore

$$\tilde{\varphi}_n(x) = \frac{d^n}{dt^n}\left\{e^{t^2}\tilde{\varphi}_0(x-2t)\right\}_{t=0}.$$

So we should compute the Hilbert transform of φ_0 first. Using the Inversion Theorem we have

$$\begin{aligned}
\tilde{\varphi}_0(x) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \widehat{H\varphi_0}(t)e^{ixt}dt \\
&= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} (-i)\ \mathrm{sign}(t)e^{-t^2/2}(\cos xt + i\sin xt)dt \\
&= \frac{2}{\sqrt{2\pi}}\int\limits_{0}^{\infty} e^{-t^2/2}\sin xt dt \\
&= \frac{2}{\sqrt{2\pi}}\int\limits_{0}^{\infty} e^{-t^2/2}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}t^{2k+1}dt \\
&= \frac{2}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\int\limits_{0}^{\infty} e^{-t^2/2}t^{2k+1}dt \\
&= \frac{2}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\int l;imits_0^{\infty} e^{-\tau}(2\tau)^k d\tau \\
&= \frac{2}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}2^k\Gamma(k+1) \\
&= \frac{2}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!!}\,.
\end{aligned}$$

Next we construct the recurrence relationship for Hermite functions and their Hilbert transforms. To this end, we derivate (2) according to t to obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi_{n+1}(x)\frac{t^n}{n!} &= (2x-2t)\exp\Big(2tx-t^2-\frac{x^2}{2}\Big)\\ &= (2x-2t)\sum_{n=0}^{\infty}\varphi_n(x)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\big(2x\varphi_n(x)-2n\varphi_{n-1}(x)\big)\frac{t^n}{n!}\end{aligned}$$

or, equivalently,

$$\varphi_{n+1}(x) = 2x\varphi_n(x) - 2n\varphi_{n-1}(x).$$

Let Φ_n denote the logarithmic integral of the Hermite function φ_n. For even n we have

$$\hat{\Phi}_{n+1}(t) = \frac{\hat{\varphi}_{n+1}(t)}{|t|} = \frac{(-i)^{n+1}\varphi_{n+1}(t)}{|t|}\,.$$

This implies that

$$\hat{\Phi}_{n+1}(t) - 2n\hat{\Phi}_{n-1}(t) = 2(-i)^{n+1}\varphi_n(t)\ \mathrm{sign}(t) = 2\widehat{H\varphi_n}(t),$$

and consequently,

$$\Phi_{n+1} - 2n\Phi_{n-1} = 2\tilde{\varphi}_n.$$

Derivate both sides term by term we have

$$\tilde{\varphi}_{n+1} - 2n\tilde{\varphi}_{n-1} = -2\tilde{\varphi}_n' \qquad \text{for even } n.$$

On the other hand, for odd n,

$$||\Phi_n||^2 = \int_{-\infty}^{\infty}\Big|\frac{\varphi_n(x)}{x}\Big|^2 dx < \infty,$$

so we have

Theorem 3. *For odd n the logarithmic integral Φ_n of the Hermite function φ_n belongs to the Sobolev space $H_{(1)}(\mathbf{R})$ and*

$$||\Phi_n||_{(1)}^2 = \int_{-\infty}^{\infty}\Big|\frac{\varphi_n(x)}{x}\Big|^2 dx + ||\varphi_n||^2.$$

2. Radon transform in the plane

For a Schwart function $f(x,y)$ we define the Radon transform $\mathcal{R}f(r,\theta) = \mathcal{R}(f,r,\theta)$ of f as follows

$$\mathcal{R}f(r,\theta) = \int\limits_{x\cos\theta + y\sin\theta = r} f(x,y)d\ell,$$

here $d\ell$ is the Lebesgue measure in the line $x\cos\theta + y\sin\theta = r$. If $u(x,y) = \phi\big(\sqrt{x^2+y^2}\big)$ is a radial function, the Radon transform of f is the same, i.e.,

$$\mathcal{R}u(r,\theta) = \int\limits_{-\infty}^{\infty} u(r,y)dy = 2\int\limits_{0}^{\infty} u(r,y)dy = 2\int\limits_{|r|}^{\infty} \phi(t)\frac{tdt}{\sqrt{t^2-r^2}}\,\cdot$$

For example, if $u(x,y) = \exp\big(-(x^2+y^2)\big)$ then we have

$$\mathcal{R}u(r,\theta) = \int\limits_{-\infty}^{\infty} e^{-(r^2+y^2)}dy = \sqrt{\pi}e^{-r^2}.$$

Now let $f(x,y) = u(x-t, y-s) = \exp\big\{-[(x-t)^2 + (y-s)^2]\big\}$, where s and t are fixed. Then

$$\begin{aligned}\mathcal{R}f(r,\theta) &= \int\limits_{x\cos\theta + y\sin\theta = r} u(x-t, y-s)d\ell \\ &= \int\limits_{x\cos\theta + y\sin\theta = r - t\cos\theta - s\sin\theta} u(x,y)d\ell \\ &= \sqrt{\pi}\exp\Big(-(r - t\cos\theta - s\sin\theta)^2\Big).\end{aligned}$$

Put

$$\psi_n(x) = H_n(x)e^{-x^2}.$$

Then from (1) we have

$$\sum_{n=0}^{\infty}\psi_n(x)\frac{t^n}{n!} = \exp\big[-(x-t)^2\big]. \tag{7}$$

Replace t by $t+s$ we have

$$\begin{aligned}\exp\big[-(x-t-s)^2\big] &= \sum_{n=0}^{\infty}\psi_n(x)\frac{(t+s)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\psi_{n+m}(x)\frac{t^n s^m}{n!m!}\,\cdot \qquad (8)\end{aligned}$$

Replace x by y and t by s in (7) then multiple it with itself we obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\psi_n(x)\psi_m(y)\frac{t^n s^m}{n!m!} = \exp\big\{-[(x-t)^2+(y-s)^2]\big\} =: f(x,y).$$

Now take the Radon transform in variables x and y in both sides term by term, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{R}(\psi_n\otimes\psi_m,r,\theta)\frac{t^n s^m}{n!m!} &= \sqrt{\pi}\exp\big\{-(r-t\cos\theta-s\sin\theta)^2\big\}\\ &= \sqrt{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\psi_{n+m}(r)\cos^n\theta\sin^m\theta\frac{t^n s^m}{n!m!} \quad \text{(by (8))}.\end{aligned}$$

Therefore

$$\mathcal{R}(\psi_n\otimes\psi_m,r,\theta) = \sqrt{\pi}\psi_{n+m}(r)\cos^n\theta\sin^m\theta.$$

The inversion formula (for a Schwart function f) from [2] reads as follows

$$f(x,y) = \frac{1}{4\pi^2}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}\frac{\partial}{\partial r}\mathcal{R}f(r,\theta)\frac{dr}{x\cos\theta+y\sin\theta-r}\,.$$

Apply this formula for functions $\psi_n\otimes\psi_m$ we have

$$\psi_n(x)\psi_m(y) = \frac{\sqrt{\pi}}{4\pi^2}\int_0^{2\pi}\tilde{\psi}'_{n+m}(x\cos\theta+y\sin\theta)\cos^n\theta\sin^m\theta d\theta.$$

To compute the norm of ψ_n note that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\psi_n(x)\psi_m(x)\frac{t^n s^m}{n!m!} = \exp\big[-(x-t)^2-(x-s)^2\big].$$

Integrating side by side we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{-\infty}^{\infty}\psi_n(x)\psi_m(x)dx\frac{t^n s^m}{n!m!} &= \int_{-\infty}^{\infty}\exp\big[-(x-t)^2-(x-s)^2\big]dx\\ &= \sqrt{\frac{\pi}{2}}\varphi_0(t-s).\end{aligned}$$

Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_n(x)\psi_m(x)dx &= \sqrt{\frac{\pi}{2}}\frac{\partial^{n+m}}{\partial t^n \partial s^m}\varphi_0(t-s)\big|_{t=s=0}\\ &= \sqrt{\frac{\pi}{2}}(-1)^n \varphi_0^{(n+m)}(0)\\ &= 0 \qquad \text{if } n+m \text{ is odd},\\ &= \sqrt{\frac{\pi}{2}}(-1)^{|n-m|/2}(n+m-1)!! \quad \text{if } n+m \text{ is even}.\end{aligned}$$

Thus

$$||\psi_n||^2 = \sqrt{\frac{\pi}{2}}(2n-1)!!.$$

To compute the Hilbert transform of ψ_n we observe that (7) implies

$$\sum_{n=0}^{\infty} \psi_n(x)\frac{t^n}{n!} = \exp\big[-(x-t)^2\big] = \psi_0(x-t).$$

Therefore

$$\psi_n(x) = (-1)^n \frac{d^n}{dx^n}\psi_0(x)$$

and

$$\tilde{\psi}_n(x) = (-1)^n \frac{d^n}{dx^n}\tilde{\psi}_0(x).$$

Consequently,

$$\psi_{n+1}(x) = -\psi_n'(x), \quad \tilde{\psi}_{n+1}(x) = -\tilde{\psi}_n'(x),$$

and

$$||\psi_n||^2_{(1)} = ||\psi_n||^2 + ||\psi_{n+1}||^2 = \sqrt{2\pi}(2n-1)!!(n+1).$$

On the other hand, we have $\psi_0(x) = \varphi_0(\sqrt{2}\,x)$, so

$$\tilde{\psi}_0(x) = \tilde{\varphi}_0(x\sqrt{2}) = \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{k+1}x^{2k+1}}{(2k+1)!!}\,.$$

Let Ψ_n be the logarithmic integral of ψ_n (this function is a Hardy function for odd n). Then

$$\begin{aligned}\Psi_n(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \psi_n(t) \ln \frac{1}{|x-t|} dt \\ &= -\int_0^x \tilde{\psi}_n(t)dt = \int_0^x \tilde{\psi}'_{n-1}(t)dt \\ &= \tilde{\psi}_{n-1}(x) \\ &= \frac{1}{\pi}(\text{p.v.})\int_{-\infty}^{\infty} \psi_{n-1}(t)\frac{dt}{x-t}.\end{aligned}$$

Therefore we have

Theorem 4. *For odd n the logarithmic integral Ψ_n of the function*

$$\psi_n(x) = H_n(x)\exp(-x^2)$$

belongs to the Sobolev space $H_{(1)}(\mathbf{R})$ with the norm

$$||\Psi_n||^2_{(1)} = ||\psi_{n-1}||^2_{(1)} = ||\psi_{n-1}||^2 + ||\psi_n||^2 = \sqrt{2\pi}n(2n-3)!!.$$

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