LOGARITHMIC INTEGRALS, SOBOLEV SPACES
AND RADON TRANSFORM IN THE PLANE

DANG VU GIANG

Abstract. We prove that the set \{\varphi_0, \varphi_1, \varphi_4, \ldots, \varphi_{3k+1}, \ldots\} of Hermite functions is an orthogonal system in the Sobolev space \(H^1(\mathbb{R}) = H_{13}(\mathbb{R})\). Furthermore, the logarithmic integral of a function \(f\) from the real Hardy space \(H^1(\mathbb{R})\) is exactly the primitive function of \(-\tilde{f}\) (the Hilbert transform of \(f\)). And more interesting formulas are found for Radon transform of Hermite-like functions.

1. Hermite functions and logarithmic integrals

Consider the following power series

\[
\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp(2tx - t^2),
\]

where \(H_n\) are the Hermite polynomials. Replace \(x\) by \(-x\) we have

\[H_n(-x) = (-1)^n H_n(x).\]

The Hermite function \(\varphi_n\) is defined by setting

\[\varphi_n(x) = H_n(x) \exp(-x^2/2).\]

We have

\[\varphi_n(-x) = (-1)^n \varphi_n(x)\]

and

\[
\sum_{n=0}^{\infty} \varphi_n(x) \frac{t^n}{n!} = \exp \left(2tx - t^2 - \frac{x^2}{2} \right).
\]

It is very well-known that the system \(\{\varphi_n\}_{n=0}^{\infty}\) is orthogonal in \(L^2(\mathbb{R})\). Next we give a new method to prove this and get more results for Hermite functions. To
this end we use the following formula from [4]:

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(Ax^2 + Bx + C)}{2}\right) dx = \frac{1}{\sqrt{2A}} \exp\left(\frac{B^2 - 4AC}{4A}\right) \quad (\Re(A) > 0).
\]

Therefore, taking the Fourier transforms of both sides of (2) we have

\[
\sum_{n=0}^{\infty} \hat{\varphi}_n(\xi) \frac{t^n}{n!} = \exp\left(-2it\xi + t^2 - \frac{\xi^2}{2}\right).
\]

Here the Fourier transform \( \hat{\varphi} \) of a function \( \varphi \) is defined by

\[
\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-ix\xi} dx.
\]

On the other hand, note that if in (2) we replace \( x \) by \( \xi \) and \( t \) by \( -it \) then

\[
\sum_{n=0}^{\infty} (-i)^n \varphi_n(\xi) \frac{t^n}{n!} = \exp\left(-2it\xi + t^2 - \frac{\xi^2}{2}\right).
\]

Compare this series with (4) we get

\[ \hat{\varphi}_n = (-i)^n \varphi_n. \]

If in (2) we replace \( t \) by \( s \) and take the product of these power series, then we get

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi_n(x) \varphi_m(x) \frac{t^n s^m}{n!m!} = \exp(2tx + 2sx - t^2 - s^2 - x^2).
\]

Integrate term by term according to \( x \) and apply (3) we obtain

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \int_{-\infty}^{\infty} \varphi_n(x) \varphi_m(x) dx \right) \frac{t^n s^m}{n!m!} = \int_{-\infty}^{\infty} \exp(2tx + 2sx - t^2 - s^2 - x^2) dx
\]

\[ = \sqrt{\pi} \exp(2st) \]

\[ = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n s^n t^n}{n!} .
\]

Therefore

\[
\int_{-\infty}^{\infty} \varphi_n(x) \varphi_m(x) dx = \sqrt{\pi} 2^n n! \delta(n - m),
\]

here \( \delta \) denotes the Kronecker-delta. This proves that the Hermite functions are orthogonal in \( L^2(\mathbb{R}) \). Next, we prove that the system of Hermite functions can
be separated into 3 parts which are orthogonal in the Sobolev space $H^1(R)$. To this end we define the Sobolev norm $|| \cdot ||_{(1)}$ by letting

$$||u||^2_{(1)} = \int_{-\infty}^{\infty} |\hat{u}(x)|^2(1 + x^2)dx = ||u||^2 + ||u'||^2.$$  

Here, $u'$ denotes the distributional derivative of $u$. The Sobolev space $H^1(R) = H_{(1)}(R)$ is the set of all $u \in L^2(R)$ such that $||u||_{(1)} < \infty$. This is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_{(1)}$ defined by setting

$$\langle u, v \rangle_{(1)} = \langle u, v \rangle_{L^2} + \langle u', v' \rangle.$$  

Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the scalar product in $L^2(R)$. We deduce from (5) that $\langle \varphi_n, \varphi_m \rangle = \sqrt{\pi} 2^n n! \delta(n - m)$. Now derive (2) according to $x$ we have

$$\sum_{n=0}^{\infty} \varphi'_n(x) t^n n! = (2t - x) \exp \left( 2tx - t^2 - \frac{x^2}{2} \right).$$  

Multiple this equation with itself after replacing $t$ by $s$ we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi'_n(x) \varphi'_m(x) \frac{t^n s^m}{n! m!} = (2t - x)(2s - x) \exp(2tx + 2sx - t^2 - s^2 - x^2).$$  

Integrating term by term according to $x$ we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle \varphi'_n, \varphi'_m \rangle \frac{t^n s^m}{n! m!} = \int_{-\infty}^{\infty} (2t - x)(2s - x) \exp(2tx + 2sx - t^2 - s^2 - x^2) dx$$

$$= e^{2ts} \Gamma(3/2) - (t - s)^2 \Gamma(1/2)$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n t^n s^n}{n!} \left( \frac{1}{2} - t^2 - s^2 + 2ts \right).$$  

This implies that

$$||\varphi'_0||^2 = \frac{\sqrt{\pi}}{2}$$

$$||\varphi'_n||^2 = 2^{n-1} (n+2)! \sqrt{\pi} \quad \text{for } n > 0,$$

$$\langle \varphi'_n, \varphi'_{n+2} \rangle = \langle \varphi'_{n+2}, \varphi'_n \rangle = -2^n (n+2)! \sqrt{\pi}$$

$$\langle \varphi'_n, \varphi'_m \rangle = 0 \quad \text{if } |n - m| = 1 \quad \text{or} \quad |n - m| > 2,$$

and we obtain the following theorem.

**Theorem 1.** The following systems of Hermite functions are orthogonal in the Sobolev space $H^1(R)$:

(i) $\{ \varphi_0, \varphi_1, \varphi_4, \ldots, \varphi_{3k+1}, \ldots \}$;

(ii) $\{ \varphi_1, \varphi_2, \varphi_5, \ldots, \varphi_{3k+2}, \ldots \}$;

(iii) $\{ \varphi_0, \varphi_3, \varphi_6, \ldots, \varphi_{3k}, \ldots \}$. 
Next we define the Hilbert transform and the real Hardy space $\mathcal{H}^1(\mathbb{R})$. The Hilbert transform $Hf := \tilde{f}$ of a function $f \in L^p(\mathbb{R})$ $(p \in [1, \infty))$ is defined by the formula

$$Hf(x) = \tilde{f}(x) = \frac{1}{\pi} (\text{p.v.}) \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt.$$ 

It is well-known that

$$\widehat{Hf}(\xi) = -i \text{sign}(\xi) \hat{f}(\xi), \quad H(Hf) = -f,$$

and

$$\langle f, g \rangle = -\langle \tilde{f}, g \rangle \quad \text{for } f \in L^p(\mathbb{R}) \text{ and } g \in L^q(\mathbb{R}), \quad \frac{1}{p} + \frac{1}{q} = 1, \ 1 < p < \infty.$$

Therefore, the Hilbert transform is a unitary operator in both Hilbert spaces $L^2(\mathbb{R})$ and $\mathcal{H}^1(\mathbb{R})$. The Hilbert transform of the characteristic function $\chi_{(a,b)}$ of the interval $(a, b)$ is

$$\tilde{\chi}_{(a,b)}(x) = \frac{1}{\pi} \ln \left| \frac{x-a}{x-b} \right|,$$

so we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \left| \frac{x-a}{x-b} \right| dx = -\int_{a}^{b} \tilde{f}(x)dx.$$ \hspace{1cm} (6)

The real Hardy space $\mathcal{H}^1(\mathbb{R})$ is the set of all functions $f \in L^1(\mathbb{R})$ such that $\tilde{f} \in L^1(\mathbb{R})$. Functions in the real Hardy space are called Hardy functions. From [1] we have $\varphi_n \in \mathcal{H}^1(\mathbb{R})$ for every odd $n$. It is a well-known fact that the dual space of $\mathcal{H}^1(\mathbb{R})$ is $BMO(\mathbb{R})$ and the logarithmic function $\ln x$ is in $BMO$. So we can define the logarithmic integral

$$F(b) := \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \left| \frac{1}{x-b} \right| dx$$

for all function $f \in \mathcal{H}^1(\mathbb{R})$. But by (6) we have

$$F(b) - F(a) = -\int_{a}^{b} \tilde{f}(x)dx;$$

so the function $F$ is absolutely continuous on the real line and $F'(b) = -\tilde{f}(b)$ for almost all $b \in \mathbb{R}$. Take the Fourier transform of $F'$ in the distributional sense we have $it\hat{F}(t) = i \text{sign}(t)\hat{f}(t)$. This implies

$$\hat{\tilde{f}}(t) = \frac{\hat{f}(t)}{|t|} \in L^1(\mathbb{R}) \quad \text{(by Hardy inequality)}.$$
Hence

$$F(b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(t) \frac{e^{ibt}}{|t|} dt$$

and

$$\tilde{F}(b) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(t) \frac{e^{ibt}}{t} dt.$$ 

Consequently,

$$\lim_{|a| \to \infty} F(a) = \lim_{|a| \to \infty} \tilde{F}(a) = 0.$$ 

Thus we have the following result.

**Theorem 2.** The logarithmic integral $F$ of a Hardy function $f$ is absolutely continuous and it can be rewritten in the form

$$F(b) = -\int_{-\infty}^{b} \hat{f}(x) dx.$$ 

In [3] it is proved (in a complicated manner) that the logarithmic integral $F$ is of bounded variation if the Hardy function $f$ is of compact support. Our result is much stronger. Note that if $\varphi$ is a function in $L^2(\mathbb{R})$ then

$$H(\varphi^2 - \tilde{\varphi}^2) = 2\varphi \tilde{\varphi};$$

so the function $f := \varphi^2 - \tilde{\varphi}^2$ is a Hardy function. This is the most important example for Hardy functions. For example, if

$$\varphi(x) = \frac{1}{x^2 + 1}$$

then, according to [1],

$$\tilde{\varphi}(x) = \frac{x}{x^2 + 1}.$$ 

Thus

$$f(x) = \frac{1 - x^2}{(x^2 + 1)^2}$$

is a Hardy function and we have

$$\hat{f}(x) = \frac{2x}{(x^2 + 1)^2} = -\varphi'(x).$$

Apply Theorem 2 we obtain the following interesting formula

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - x^2}{(x^2 + 1)^2} \ln \frac{1}{|x-b|} dx = \varphi(b) = \frac{1}{b^2 + 1}.$$
Next we compute the Hilbert transforms of the Hermite functions $\varphi_n$. To this end, note that (2) implies
\[
\sum_{n=0}^{\infty} \varphi_n(x) \frac{t^n}{n!} = \exp \left( 2tx - t^2 - \frac{x^2}{2} \right) = e^t \varphi_0(x - 2t).
\]

Take the Hilbert transform of both sides according to the variable $x$ we have
\[
\sum_{n=0}^{\infty} \tilde{\varphi}_n(x) \frac{t^n}{n!} = e^t \tilde{\varphi}_0(x - 2t).
\]

Therefore
\[
\tilde{\varphi}_n(x) = \frac{d^n}{dt^n} \left\{ e^t \tilde{\varphi}_0(x - 2t) \right\}_{t=0}.
\]

So we should compute the Hilbert transform of $\varphi_0$ first. Using the Inversion Theorem we have
\[
\tilde{\varphi}_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_{\tilde{\varphi}_0}(t) e^{i xt} dt
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-i \text{ sign}(t)) e^{-t^2/2} (\cos xt + i \sin xt) dt
\]
\[
= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t^2/2} \sin xt dt
\]
\[
= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t^2/2} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} t^{2k+1} dt
\]
\[
= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \int_{0}^{\infty} e^{-t^2/2} t^{2k+1} dt
\]
\[
= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \int_{0}^{\infty} \exp(-\tau^2) (2\tau)^k d\tau
\]
\[
= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} 2^k \Gamma(k + 1)
\]
\[
= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} 2^k.
\]
Next we construct the recurrence relationship for Hermite functions and their Hilbert transforms. To this end, we derivate (2) according to $t$ to obtain

$$\sum_{n=0}^{\infty} \varphi_{n+1}(x) \frac{t^n}{n!} = (2x - 2t) \exp \left( 2tx - t^2 - \frac{x^2}{2} \right)$$

or, equivalently,

$$\varphi_{n+1}(x) = 2x \varphi_n(x) - 2n \varphi_{n-1}(x).$$

Let $\Phi_n$ denote the logarithmic integral of the Hermite function $\varphi_n$. For even $n$ we have

$$\hat{\Phi}_{n+1}(t) = \frac{\hat{\varphi}_{n+1}(t)}{|t|} = \frac{(-i)^{n+1} \varphi_{n+1}(t)}{|t|}.$$

This implies that

$$\hat{\Phi}_{n+1}(t) - 2n \hat{\Phi}_{n-1}(t) = 2(-i)^{n+1} \varphi_n(t) \text{ sign}(t) = 2\hat{H}\varphi_n(t),$$

and consequently,

$$\Phi_{n+1} - 2n \Phi_{n-1} = 2\hat{\varphi}_n.$$

Derivate both sides term by term we have

$$\hat{\varphi}_{n+1} - 2n \hat{\varphi}_{n-1} = -2\hat{\varphi}_n' \quad \text{for even } n.$$

On the other hand, for odd $n,$

$$||\Phi_n||^2 = \int_{-\infty}^{\infty} \left| \frac{\varphi_n(x)}{x} \right|^2 dx < \infty,$$

so we have

**Theorem 3.** For odd $n$ the logarithmic integral $\Phi_n$ of the Hermite function $\varphi_n$ belongs to the Sobolev space $H_{(1)}(\mathbb{R})$ and

$$||\Phi_n||^2_{(1)} = \int_{-\infty}^{\infty} \left| \frac{\varphi_n(x)}{x} \right|^2 dx + ||\varphi_n||^2.$$
2. Radon transform in the plane

For a Schwartz function \( f(x, y) \) we define the Radon transform \( \mathcal{R}f(r, \theta) = \mathcal{R}(f, r, \theta) \) of \( f \) as follows

\[
\mathcal{R}f(r, \theta) = \int_{x \cos \theta + y \sin \theta = r} f(x, y) d\ell,
\]

here \( d\ell \) is the Lebesgue measure in the line \( x \cos \theta + y \sin \theta = r \). If \( u(x, y) = \phi(\sqrt{x^2 + y^2}) \) is a radial function, the Radon transform of \( f \) is the same, i.e.,

\[
\mathcal{R}u(r, \theta) = \int_{-\infty}^{\infty} u(r, y) dy = 2 \int_{0}^{\infty} u(r, y) dy = 2 \int_{|r|}^{\infty} \phi(t) \frac{tdt}{\sqrt{t^2 - r^2}}.
\]

For example, if \( u(x, y) = \exp(-x^2 - y^2) \) then we have

\[
\mathcal{R}u(r, \theta) = \int_{-\infty}^{\infty} e^{-(r^2 + y^2)} dy = \sqrt{\pi} e^{-r^2}.
\]

Now let \( f(x, y) = u(x - t, y - s) = \exp\{-(x - t)^2 + (y - s)^2\} \), where \( s \) and \( t \) are fixed. Then

\[
\mathcal{R}f(r, \theta) = \int_{x \cos \theta + y \sin \theta = r} u(x - t, y - s) d\ell
\]

\[
= \int_{x \cos \theta + y \sin \theta = r - t \cos \theta - s \sin \theta} u(x, y) d\ell
\]

\[
= \sqrt{\pi} \exp\left(-(r - t \cos \theta - s \sin \theta)^2\right).
\]

Put

\[
\psi_n(x) = H_n(x) e^{-x^2}.
\]

Then from (1) we have

\[
\sum_{n=0}^{\infty} \psi_n(x) \frac{t^n}{n!} = \exp\left[-(x - t)^2\right].
\]

Replace \( t \) by \( t + s \) we have

\[
\exp\left[-(x - t - s)^2\right] = \sum_{n=0}^{\infty} \psi_n(x) \frac{(t + s)^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \psi_{n+m}(x) \frac{t^n s^m}{n!m!}.
\]
Replace $x$ by $y$ and $t$ by $s$ in (7) then multiple it with itself we obtain

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \psi_n(x) \psi_m(y) \frac{t^n s^m}{n!m!} = \exp\{-[(x-t)^2 + (y-s)^2]\} =: f(x, y).$$

Now take the Radon transform in variables $x$ and $y$ in both sides term by term, we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{R}(\psi_n \otimes \psi_m, r, \theta) \frac{t^n s^m}{n!m!} = \sqrt{\pi} \exp\{-r^2 - s^2\} = \sqrt{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \psi_{n+m}(r) \cos^n \theta \sin^m \theta \frac{t^n s^m}{n!m!} \quad \text{(by (8))}.$$ 

Therefore

$$\mathcal{R}(\psi_n \otimes \psi_m, r, \theta) = \sqrt{\pi} \psi_{n+m}(r) \cos^n \theta \sin^m \theta.$$ 

The inversion formula (for a Schwartz function $f$) from [2] reads as follows

$$f(x, y) = \frac{1}{4\pi^2} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} \frac{\partial}{\partial r} \mathcal{R}f(r, \theta) \frac{dr}{x \cos \theta + y \sin \theta - r}. $$

Apply this formula for functions $\psi_n \otimes \psi_m$ we have

$$\psi_n(x) \psi_m(y) = \frac{\sqrt{\pi}}{4\pi^2} \int_0^{2\pi} \psi'_{n+m}(x \cos \theta + y \sin \theta) \cos^n \theta \sin^m \theta d\theta.$$ 

To compute the norm of $\psi_n$ note that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \psi_n(x) \psi_m(x) \frac{t^n s^m}{n!m!} = \exp\{-(x-t)^2 - (x-s)^2\}. $$

Integrating side by side we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \psi_n(x) \psi_m(x) dx \frac{t^n s^m}{n!m!} = \int_{-\infty}^{\infty} \exp\{-(x-t)^2 - (x-s)^2\} dx = \sqrt{\frac{\pi}{2}} \varphi_0(t - s).$$
Therefore
\[
\int_{-\infty}^{\infty} \psi_n(x)\psi_m(x)dx = \sqrt{\frac{\pi}{2}} \frac{\partial^{n+m}}{\partial t^n \partial s^m} \varphi_0(t-s) \bigg|_{t=s=0}
\]
\[
= \sqrt{\frac{\pi}{2}} (-1)^n \varphi_0^{(n+m)}(0)
\]
\[
= 0 \quad \text{if } n + m \text{ is odd},
\]
\[
= \sqrt{\frac{\pi}{2}} (-1)^{|n-m|/2} (n + m)!! \quad \text{if } n + m \text{ is even}.
\]
Thus
\[
||\psi_n||^2 = \sqrt{\frac{\pi}{2}} (2n - 1)!!.
\]
To compute the Hilbert transform of \(\psi_n\) we observe that (7) implies
\[
\sum_{n=0}^{\infty} \psi_n(x) \frac{I_n}{n!} = \exp[-(x-t)^2] = \psi_0(x-t).
\]
Therefore
\[
\psi_n(x) = (-1)^n \frac{d^n}{dx^n} \psi_0(x)
\]
and
\[
\tilde{\psi}_n(x) = (-1)^n \frac{d^n}{dx^n} \tilde{\psi}_0(x).
\]
Consequently,
\[
\psi_{n+1}(x) = -\psi_n'(x), \quad \tilde{\psi}_{n+1}(x) = -\tilde{\psi}_n'(x),
\]
and
\[
||\psi_n||^2_{(1)} = ||\psi_n||^2 + ||\psi_{n+1}||^2 = \sqrt{2\pi} (2n - 1)!! (n + 1).
\]
On the other hand, we have \(\psi_0(x) = \varphi_0(\sqrt{2}x)\), so
\[
\tilde{\psi}_0(x) = \tilde{\varphi}_0(x\sqrt{2}) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{k+1} x^{2k+1}}{(2k + 1)!!}.
\]
Let $\Psi_n$ be the logarithmic integral of $\psi_n$ (this function is a Hardy function for odd $n$). Then

$$
\Psi_n(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_n(t) \ln \frac{1}{|x-t|} dt
$$

$$
= -\int_{0}^{x} \tilde{\psi}_n(t) dt = \int_{0}^{x} \tilde{\psi}'_{n-1}(t) dt
$$

$$
= \tilde{\psi}_{n-1}(x)
$$

$$
= \frac{1}{\pi} \left( \text{p.v.} \right) \int_{-\infty}^{\infty} \psi_{n-1}(t) \frac{dt}{x-t}.
$$

Therefore we have

**Theorem 4.** For odd $n$ the logarithmic integral $\Psi_n$ of the function

$$
\psi_n(x) = H_n(x) \exp(-x^2)
$$

belongs to the Sobolev space $H_{1/2}(\mathbb{R})$ with the norm

$$
||\Psi_n||^2_{1/2} = ||\psi_{n-1}||^2_{1/2} = ||\psi_{n-1}||^2 + ||\psi_n||^2 = \sqrt{2\pi n(2n-3)!}.
$$

**References**


**Institute of Mathematics**
18 Hoang Quoc Viet Road
10307 Hanoi, Vietnam

**E-mail address:** dvgiang@math.ac.vn