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ON HAMILTON CYCLES IN CONNECTED TETRAVALENT METACIRCULANT GRAPHS WITH NON-EMPTY FIRST SYMBOL

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ABSTRACT. In this paper, we show that every connected tetravalent metacirculant graph $MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ with $S_0 \neq \emptyset$ possesses a Hamilton cycle if $m = 1$ or 2 or $m > 2$ and both m and n are odd.

1. INTRODUCTION

Thomassen (and others) conjectured that there are only finitely many connected vertex-transitive nonhamiltonian graphs (see [8]). At present, only four such graphs are known to exist: the Petersen graph, the Coxeter graph and the two graphs obtained from them by replacing each vertex by a triangle. The readers can see [7] for more information about the Petersen and Coxeter graphs.

Metacirculant graphs were introduced by Alspach and Parsons in [3] as an interesting class of vertex-transitive graphs, in which there might be some new connected nonhamiltonian graphs. A natural question raised here is to find hamiltonian metacirculant graphs.

Connectedness of cubic metacirculant graphs has been considered in [10]. The obtained results there were used successfully to prove the existence of a Hamilton cycle in many connected cubic metacirculant graphs [9, 11]. Motivated by this, we apply here the results obtained in [13] to prove the existence of a Hamilton cycle in some connected tetravalent metacirculant graphs. Namely, we will prove that every connected tetravalent metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ with $S_0 \neq \emptyset$ are hamiltonian whenever $m = 1$ or $m = 2$ (Theorem 3.1) or $m > 2$ and both m and n are odd (Theorem 3.2).

2. PRELIMINARIES

All graphs considered in this paper are finite undirected graphs without loops and multiple edges. Unless otherwise indicated, our graph-theoretic terminology will follow [6], and our group-theoretic terminology will follow [14]. For a graph G we denote by $V(G)$, $E(G)$ and $\mathrm{Aut}(G)$ the vertex-set, the edge-set and the automorphism group of G, respectively. For a positive integer n, we will denote

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the ring of integers modulo n by $\mathbf{Z}_n$ and the multiplicative group of units in $\mathbf{Z}_n$ by $\mathbf{Z}_n^*$.

A graph G is called *vertex-transitive* if for any two vertices $u, v \in V(G)$ there exists an automorphism $\varphi \in \text{Aut}(G)$ such that $\varphi(u) = v$. If a graph G is both vertex-transitive and connected, then it is called *connected vertex-transitive*. Vertex-transitive graphs possess a high symmetry. So it is probable that they have many pleasant properties.

Let S be a subset of a group Γ such that $1 \notin S = S^{-1}$, where $S^{-1} = \{s^{-1} | s \in S\}$. Then the *Cayley graph on* Γ *respect to* S, denoted by $Cay(\Gamma, S)$, is defined to be the graph with vertex-set $V(Cay(\Gamma, S)) = \Gamma$ and two elements $x, y \in \Gamma$ are adjacent in $Cay(\Gamma, S)$ if and only if $x^{-1}y \in S$.

Circulant graphs are Cayley graphs on cyclic groups. But for abelian groups one usually use additive notation. So we must reformulate the definition for circulant graphs as follows. Let n be a positive integer and S be a subset of $\boldsymbol{Z}_n$ such that $0 \notin S = -S$. Then we define the *circulant graph* $G = C(n, S)$ to be the graph with vertex-set $V(G) = \{v_y \mid y \in \boldsymbol{Z}_n\}$ and edge-set $E(G) = \{v_y v_h \mid y, h \in \boldsymbol{Z}_n; (h - y) \in S\}$, where subscripts are always reduced modulo n. The subset S is called the *symbol* of $C(n, S)$.

The following class of graphs called metacirculants was introduced by Alspach and Parsons in [3]. This class of graphs is of interest because it properly contains the class of circulant graphs. Therefore, many problems for vertex-transitive graphs can be verified nontrivially first in this class.

Let m and n be two positive integers, $\alpha \in \boldsymbol{Z}_n^*$, $\mu = \lfloor n/2 \rfloor$ and $S_0, S_1, \ldots, S_\mu$ be subsets of $\boldsymbol{Z}_n$, satisfying the following conditions:

1) $0 \notin S_0 = -S_0$;

2) $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$;

3) If m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Then we define the *metacirculant graph* $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ to be the graph with vertex-set

$$V(G) = \{v_j^i \mid i \in \boldsymbol{Z}_m;\ j \in \boldsymbol{Z}_n\}$$

and edge-set

$$E(G) = \{v_j^i v_h^{i+r} \mid 0 \leq r \leq \mu;\ i \in \boldsymbol{Z}_m;\ j, h \in \boldsymbol{Z}_n \ \&\ (h - j) \in \alpha^i S_r\},$$

where superscripts and subscripts are always reduced modulo m and modulo n, respectively. The subset S_i is called $(i+1)$*-th symbol* of G.

Let ρ and τ be two permutations on $V(G)$ defined by $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$. Then ρ and τ are automorphisms of G and the subgroup $\langle \rho, \tau \rangle$ of $\text{Aut}(G)$ generated by ρ and τ is a transitive subgroup of $\text{Aut}(G)$. Thus, metacirculant graphs are vertex-transitive.

Denote the *degree* of a vertex v of a graph G by $\deg(v)$. It is easy to see that for any vertex $v \in V(G)$ of a metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$

$$\deg(v) = \begin{cases} |S_0| + 2|S_1| + \cdots + 2|S_\mu| & \text{if } m \text{ is odd,} \\ |S_0| + 2|S_1| + \cdots + 2|S_{\mu-1}| + |S_\mu| & \text{if } m \text{ is even.} \end{cases}$$

A graph G is called *cubic* (resp. *tetravalent*) if for any vertex $v \in V(G)$, $\deg(v) = 3$ (resp. $\deg(v) = 4$).

The following results have been proved in [12] and [13], respectively.

Lemma 2.1. [12] *A metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ with $S_0 \neq \emptyset$ is tetravalent if and only if one of the following cases holds:*

(1) $|S_0| = 4$ *and* $S_1 = \ldots = S_\mu = \emptyset$;
(2) m *and* n *are even,* $|S_0| = 3$, $S_j = \emptyset$ *for any* $j \in \{1, 2, \ldots, \mu - 1\}$ *and* $|S_\mu| = 1$;
(3) m *is even,* $|S_0| = 2$, $S_i = \emptyset$ *for any* $i \in \{1, 2, \ldots, \mu - 1\}$ *and* $|S_\mu| = 2$;
(4) $m > 2$ *is odd,* $|S_0| = 2$, $|S_i| = 1$ *for some* $i \in \{1, 2, \ldots, \mu\}$ *and* $S_j = \emptyset$ *for any* $i \neq j \in \{1, 2, \ldots, \mu\}$;
(5) $m > 2$ *is even,* $|S_0| = 2$, $|S_i| = 1$ *for some* $i \in \{1, 2, \ldots, \mu - 1\}$ *and* $S_j = \emptyset$ *for any* $i \neq j \in \{1, 2, \ldots, \mu\}$;
(6) m *and* n *are even,* $|S_0| = 1$, $S_i = \emptyset$ *for any* $i \in \{1, 2, \ldots, \mu - 1\}$ *and* $|S_\mu| = 3$;
(7) $m > 2$, m *and* n *are even,* $|S_0| = 1$, $|S_i| = 1$ *for some* $i \in \{1, 2, \ldots, \mu - 1\}$, $S_j = \emptyset$ *for any* $i \neq j \in \{1, 2, \ldots, \mu - 1\}$ *and* $|S_\mu| = 1$.

Theorem 2.1. [13] *Let $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ be a tetravalent metacirculant graph with $S_0 \neq \emptyset$. Then G is connected if and only if one of the following conditions holds:*

(1) $m = 1$, $S_0 = \{\pm s, \pm r\}$ *and* $gcd(s, r, n) = 1$;
(2) $m = 2$, n *is even,* $S_0 = \left\{\pm s, \frac{n}{2}\right\}$, $S_1 = \{k\}$ *and* $gcd\left(s, \frac{n}{2}\right) = 1$;
(3) $m = 2$, $S_0 = \{\pm s\}$, $S_1 = \{k, l\}$ *and* $gcd(s, k - l, n) = 1$;
(4) $m > 2$ *is odd,* $S_0 = \{\pm s\}$, $S_i = \{k\}$ *for some* $i \in \{1, 2, \ldots, \mu\}$ *such that* $gcd(i, m) = 1$, $S_j = \emptyset$ *for any* $i \neq j \in \{1, 2, \ldots, \mu\}$ *and* $gcd(s, r, n) = 1$ *where* $r = k(1 + \alpha^i + \cdots + \alpha^{(m-1)i})$;
(5) $m > 2$ *is even,* $S_0 = \{\pm s\}$, $S_i = \{k\}$ *for some* $i \in \{1, 2, \ldots, \mu - 1\}$ *such that* $gcd(i, m) = 1$, $S_j = \emptyset$ *for any* $i \neq j \in \{1, 2, \ldots, \mu\}$ *and* $gcd(s, r, n) = 1$ *where* $r = k(1 + \alpha^i + \cdots + \alpha^{(m-1)i})$;
(6) $m = 2$, n *is even,* $S_0 = \left\{\frac{n}{2}\right\}$, $S_1 = \{h, k, l\}$ *and* $gcd\left(h - k, k - l, \frac{n}{2}\right) = 1$;
(7) $m > 2$ *is even,* n *is even,* $S_0 = \left\{\frac{n}{2}\right\}$, $S_i = \{s\}$ *where* i *is odd and* $gcd(i, m) = 1$, $S_j = \emptyset$ *for any* $i \neq j \in \{1, 2, \ldots, \mu - 1\}$, $S_\mu = \{r\}$ *and* $gcd\left(p, \frac{n}{2}\right) = 1$, *where* p *is* $\left[r - s(1 + \alpha^i + \alpha^{2i} + \cdots + \alpha^{(\mu-1)i})\right]$ *reduced modulo* n;
(8) $m > 2$ *is even but* $\mu = \frac{m}{2}$ *is odd,* n *is even,* $S_0 = \left\{\frac{n}{2}\right\}$, $S_i = \{s\}$ *where* i *is even and* $gcd(i, m) = 2$, $S_j = \emptyset$ *for any* $i \neq j \in \{1, 2, \ldots, \mu - 1\}$, $S_\mu = \{r\}$ *and* $gcd\left(q, \frac{n}{2}\right) = 1$, *where* $i = 2^t i'$ *with* $t \geq 1$ *and* i' *odd and* q *is*

$\big[r(1+\alpha^{i'}+\alpha^{2i'}+\cdots+\alpha^{(2^t-1)i'})-s(1+\alpha^{i'}+\alpha^{2i'}+\cdots+\alpha^{(\mu-1)i'})\big]$ *reduced modulo* n.

Let $n>1$ be an integer. The *dihedral group* D_n is the group generated by two elements α and β satisfying the relations $\alpha^n=\beta^2=1$ and $\beta\alpha\beta=\alpha^{-1}$.

The following theorem has been proved in [5].

Theorem 2.2. [5] *Every connected cubic Cayley graph on a dihedral group has a Hamilton cycle.*

Let $n>1$ be an integer. Then the *generalized Petersen graph* $GP(n,k)$, $1\le k\le n-1$, is defined to be the graph with vertex-set

$$V(GP(n,k))=\{u_0,u_1,\dots,u_{n-1},v_0,v_1,\dots,v_{n-1}\}$$

and edge-set

$$E(GP(n,k))=\{u_iu_{i+1},\ u_iv_i,\ v_iv_{i+k}\mid 0\le i\le n-1\},$$

where subscripts are always reduced modulo n.

The following result was proved by Alspach [1] for generalized Petersen graphs.

Theorem 2.3. [1] *The generalized Petersen graph* $GP(n,k)$ *is hamiltonian if and only if it is not one of the following:*

(1) $GP(n,2)\cong GP(n,n-2)\cong GP\Big(n,\dfrac{(n-1)}{2}\Big)\cong GP\Big(n,\dfrac{(n+1)}{2}\Big)$, *where* $n\equiv 5$ *(mod 6),*
(2) $GP(4m,2m)$, $m\ge 2$.

A permutation α is said to be *semiregular* if all cycles in the disjoint cycle decomposition of α have the same length. If $\mathrm{Aut}(G)$ of a graph G contains a semiregular element α, then the *quotient graph* G/α can be defined as follows: the vertices of G/α are orbits of $\langle\alpha\rangle$ and two such vertices are adjacent in G/α if and only if there is an edge in G joining a vertex in one corresponding orbit to a vertex in the other orbit.

The following result will be useful for this work.

Theorem 2.4. [2] *Let* G *be a graph that admits a semiregular automorphism* α *of order* $t\ge 3$ *and let* $G_1,G_2,\dots,G_k$ *be the subgraphs induced by* G *on the orbits of* $\langle\alpha\rangle$. *Let each* G_i *be connected and have degree* 2. *Then the graph* G *has a Hamilton cycle if either of the following statements is true:*

(1) G_r *and* G_s *have the same symbol and there is a Hamilton path of* G/α *joining them;*
(2) *There is a Hamilton cycle in* G/α *and* k *is odd.*

3. Results

First we prove the following lemmas.

Lemma 3.1. *Let* $G=MC(m,n,\alpha,S_0,S_1)$ *be a metacirculant graph with* $m=1$, $S_0=\{\pm s,\pm r\}$ *and* $gcd(s,r,n)=1$. *Then* G *possesses a Hamilton cycle.*

Proof. It is clear that G is a connected circulant graph. So G has a Hamilton cycle [4]. □

Lemma 3.2. *Let $G = GP(n,k)$ be the generalized Petersen graph with $gcd(n,k) = 1$. Then there is a Hamilton path in G joining v_i to v_{i+1}.*

Proof. Let $V = \{v_i \mid i \in \boldsymbol{Z}_n\}$ and $U = \{u_i \mid i \in \boldsymbol{Z}_n\}$. Then $G[V]$ and $G[U]$ are isomorphic to the circulant graphs $C(n, \{\pm k\})$ and $C(n, \{\pm 1\})$, respectively. So $G[V]$ is the cycle $C_0 = v_i v_{i+k} v_{i+2k} \ldots v_{i+(n-1)k} v_i$ and $G[U]$ is the cycle $C_1 = u_i u_{i+1} u_{i+2} \ldots u_{i+(n-1)} u_i$. Denote

$P_{10} = v_i v_{i+k} v_{i+2k} \ldots v_{(i+1)-k}$,

$P_{11} = v_{i+1} v_{(i+1)+k} v_{(i+1)+2k} \ldots v_{i-k}$,

$P_0 \ = v_{(i+1)-k} u_{(i+1)-k} u_{(i+2)-k} u_{(i+3)-k} \ldots u_{i-k} v_{i-k}$.

If $k \neq 1$, then $v_{i+k} \neq v_{i+1}$, $v_{(i+1)-k} \neq v_i$ and $P = P_{10} \cup P_0 \cup P_{11}$ is a Hamilton path in G with the endvertices v_i and v_{i+1}. If $k = 1$, then $P = v_i v_{i-1} v_{i-2} \ldots v_{i+2} u_{i+2} u_{i+3} \ldots u_i \ u_{i+1} v_{i+1}$ is a Hamilton path in G with the endvertices v_i and v_{i+1}. □

Lemma 3.3. *Let $G = MC(2, n, \alpha, S_0, S_1)$ be a metacirculant graph with n even, $S_0 = \Big\{ \pm s, \dfrac{n}{2} \Big\}$, $S_1 = \{k\}$ and $gcd\Big(s, \dfrac{n}{2}\Big) = 1$. Then G possesses a Hamilton cycle.*

Proof. By Lemma 2.1 and Theorem 2.1, it is clear that G is a connected tetravalent metacirculant graph.

Let $V(G) = \{v_j^i \mid i \in \boldsymbol{Z}_2,\ j \in \boldsymbol{Z}_n\}$ and $G' = MC(2, n, \alpha, S'_0, S'_1)$ be a metacirculant graph with vertex-set

$$V(G') = \{w_j^i \mid i \in \boldsymbol{Z}_2,\ j \in \boldsymbol{Z}_n\}$$

and $S'_0 = S_0$, $S'_1 = \{0\}$. It is easy to see that the mapping $\varphi : V(G) \to V(G')$, defined by $\varphi(v_j^0) = w_j^0$; $\varphi(v_j^1) = w_{j-k}^1$, is an isomorphism between the graphs G and G'. Therefore, without loss of generality we may assume that the graph G has the second symbol $S_1 = \{0\}$. Let H be a spanning subgraph of G with the edge-set

$$E(H) = E(G) \setminus \{v_j^0 v_{j+\frac{n}{2}}^0,\ v_j^1 v_{j+\frac{n}{2}}^1 \mid j \in \boldsymbol{Z}_n\}.$$

We consider separately two cases.

Case 1: $\gcd\Big(s, \dfrac{n}{2}\Big) = 1$ and $\gcd(s,n) = 1$.

Since $\gcd(s,n) = 1$ and $\alpha \in \boldsymbol{Z}_n^*$, we can see that

$$\begin{aligned} \{0, s, 2s, \ldots, (n-1)s\} &= \{0, \alpha s, 2\alpha s, \ldots, (n-1)\alpha s\} \\ &= \{0, 1, 2, \ldots, n-1\} = \boldsymbol{Z}_n. \end{aligned}$$

Therefore,

$$E(H) = \{v_j^0 v_{j+s}^0, v_j^1 v_{j+\alpha s}^1,\ v_j^0 v_j^1 \mid j \in \boldsymbol{Z}_n\}.$$

Now we define the map $\psi : V(H) \to V(GP(n,\alpha))$ by $\psi(v^0_{is}) = u_i$; $\psi(v^1_{\alpha is}) = v_{\alpha i}$, $i \in \{0,1,2,\dots,n-1\}$. Then ψ is a bijection from $V(H)$ onto $V(GP(n,\alpha))$. Furthermore, it is not difficult to see that ψ is an isomorphism between H and $GP(n,\alpha)$.

Since n is even and $\alpha \in \boldsymbol{Z}^*_n$, the generalized Petersen graph $GP(n,\alpha)$ is neither exclusion *(1)* nor exclusion *(2)* in Theorem 2.3. Therefore $GP(n,\alpha)$ has a Hamilton cycle. But H is isomorphic to $GP(n,\alpha)$ and is a spanning subgraph of G. So G also has a Hamilton cycle.

Case 2: $\gcd\Big(s,\dfrac{n}{2}\Big) = 1$ but $\gcd(s,n) = 2$.

It is clear that n and s are even, α and $\dfrac{n}{2}$ are odd. Then

$$\begin{aligned}\Big\{0,s,2s,\dots,(\frac{n}{2}-1)s\Big\} &= \{0,\alpha s,2\alpha s,\dots,(\frac{n}{2}-1)\alpha s\Big\}\\ &= \{0,2,\dots,n-2\},\end{aligned}$$

$$\begin{aligned}&\Big\{\frac{n}{2},\frac{n}{2}+s,\frac{n}{2}+2s,\dots,\frac{n}{2}+\big(\frac{n}{2}-1\big)s\Big\}\\ &=\Big\{\frac{n}{2},\frac{n}{2}+\alpha s,\frac{n}{2}+2\alpha s,\dots,\frac{n}{2}+\big(\frac{n}{2}-1\big)\alpha s\Big\}\\ &=\{1,3,\dots,n-1\}.\end{aligned}$$

Let

$$\begin{aligned}V_{even} &= \Big\{v^0_0, v^0_s,\dots,v^0_{(\frac{n}{2}-1)s}, v^1_0, v^1_{\alpha s},\dots,v^1_{(\frac{n}{2}-1)\alpha s}\Big\},\\ H_e &= H\big[V_{even}\big]\\ V_{odd} &= \Big\{v^0_{\frac{n}{2}}, v^0_{\frac{n}{2}+s},\dots,v^0_{\frac{n}{2}+(\frac{n}{2}-1)s}, v^1_{\frac{n}{2}}, v^1_{\frac{n}{2}+\alpha s},\dots,v^1_{\frac{n}{2}+(\frac{n}{2}-1)\alpha s}\Big\},\\ H_0 &= H\big[V_{odd}\big].\end{aligned}$$

Then both H_e and H_0 are isomorphic to the generalized Petersen graph $GP\Big(\dfrac{n}{2},\alpha'\Big)$, where α' is the integer satisfying $1 \le \alpha' \le \dfrac{n}{2}$ and $\alpha' \equiv \alpha \pmod{\dfrac{n}{2}}$. So we may identify them with the graph $GP\Big(\dfrac{n}{2},\alpha'\Big)$.

Since $\gcd(n,\alpha) = 1$, we have $\gcd\Big(\dfrac{n}{2},\alpha'\Big) = 1$. By Lemma 3.2, there exist a Hamilton path P_e in H_e joining v^1_0 to v^1_s and a Hamilton path P_0 in H_0 joining $v^1_{\frac{n}{2}}$ to $v^1_{\frac{n}{2}+s}$. Since α is odd and $\alpha\dfrac{n}{2} \equiv \dfrac{n}{2} \pmod{n}$, the vertex v^1_0 is adjacent to $v^1_{\frac{n}{2}}$ and v^1_s is adjacent to $v^1_{\frac{n}{2}+s}$. Therefore, we can construct a Hamilton cycle C in G as follows: Starting C at v^1_0, we go along the Hamilton path P_e in H_e to the vertex v^1_s. Further, by $v^1_s v^1_{s+\frac{n}{2}}$ we go to the vertex $v^1_{s+\frac{n}{2}}$. Then from $v^1_{s+\frac{n}{2}}$ we go along the Hamilton path P_0 in H_0 to the vertex $v^1_{\frac{n}{2}}$. Finally, we return to v^1_0 from $v^1_{\frac{n}{2}}$ by the edge $v^1_0 v^1_{\frac{n}{2}}$. Lemma 3.3 is proved. □

Lemma 3.4. *Let $G = MC(2, n, \alpha, S_0, S_1)$ be a metacirculant graph with $S_0 = \{\pm s\}$, $S_1 = \{h, k\}$ and $gcd(s, h-k, n) = 1$. Then G possesses a Hamilton cycle.*

Proof. By Lemma 2.1 and Theorem 2.1, it is clear that G is a connected tetravalent metacirculant graph. Consider the automorphism ρ of G defined by $\rho(v^i_j) = v^i_{j+1}$. We can see that ρ is semiregular. Let $\gcd(s, n) = d$. Then the automorphism $\beta = \rho^d$ of G is also semiregular. The orbit of $\langle\beta\rangle$ containing the vertex v^i_j is $V^i_j = \{v^i_j, v^i_{j+d}, \dots, v^i_{j+\left(\frac{n}{d}-1\right)d}\}$ for $i = 0, 1$ and $j = 0, 1, \dots, (d-1)$.

On the other hand, the subsets $\{0, d, \dots, (\frac{n}{d}-1)d\}$, $\{0, s, \dots, (\frac{n}{d}-1)s\}$ and $\{0, \alpha s, \dots, (\frac{n}{d}-1)\alpha s\}$ of $\boldsymbol{Z}_n$ coincide with each other. So $G[V^i_j]$ is the cycle

$$v^i_j v^i_{j+\alpha^i s} v^i_{j+2\alpha^i s} \dots v^i_{j+(\frac{n}{d}-1)\alpha^i s} v^i_j$$

for any $i \in \boldsymbol{Z}_2$ and $j \in \boldsymbol{Z}_d$.

Consider the quotient graph G/β. It has the vertex-set

$$V(G/\beta) = \{V^i_j \mid i \in \boldsymbol{Z}_2;\ j \in \boldsymbol{Z}_d\}$$

and two vertices of G/β are adjacent in G/β if and only if there is an edge in G joining a vertex in one corresponding orbit of $< \beta >$ to a vertex in the other orbit. Since G is a connected tetravalent graph $MC(2, n, \alpha, S_0, S_1)$ with $S_0 = \{\pm s\}$, it is not difficult to see that G/β is the cycle

$$V^0_0 V^1_h V^0_{h-k} V^1_{(h-k)+h} V^0_{2(h-k)} \dots V^0_{(d-1)(h-k)} V^1_{(d-1)(h-k)+h} V^0_0.$$

In G, each vertex $v^0_{x_i} \in V^0_x$ is adjacent to $v^1_{x_i+h} \in V^1_{x+h}$ and each vertex $v^1_{y_i} \in V^1_y$ is adjacent to $v^0_{y_i-k} \in V^0_{y-k}$.

Let $H_j = G[V^0_{j(h-k)} \cup V^1_{j(h-k)+h}]$, $j \in \boldsymbol{Z}_d$. Then

$$V(H_j) = \{v^0_{j+ts}, v^1_{j+h+t\alpha s} \mid t = 0, 1, \dots \frac{n}{d} - 1\},$$

$$E(H_j) = \{v^0_{j+ts} v^0_{j+(t+1)s}, v^1_{j+h+t\alpha s} v^1_{j+h+(t+1)\alpha s}, v^0_{j+ts} v^1_{j+h+ts} \mid t = 0, 1, \dots \frac{n}{d} - 1\}.$$

Let α' be the integer satisfying $1 \le \alpha' \le \frac{n}{d}$ and $\alpha' \equiv \alpha \pmod{\frac{n}{d}}$. Then the bijection

$$\varphi\ : V(H_j) \to V\big(GP(\frac{n}{d}, \alpha')\big) :$$
$$v^0_{j+ts} \mapsto u_t,\ v^1_{j+h+t\alpha s} \mapsto v_{t\alpha'},\ t \in \{0, 1, \dots, (\frac{n}{d} - 1)\}$$

is an isomorphism between H_j and $GP\Big(\frac{n}{d}, \alpha'\Big)$.

We rename the vertices of H_j, $j = 0, 1, \dots, d-1$, as follows: v^0_{j+ts} is renamed with $u_{j,t}$; $v^1_{j+h+t\alpha s}$ is renamed with $v_{j,t\alpha'}$ for $t = 0, 1, \dots, \frac{n}{d} - 1$. We can see that $GP\Big(\frac{n}{d}, \alpha'\Big)$ is neither exclusion (1) nor exclusion (2) in Theorem 2.3. Therefore

H_{d-1} has a Hamilton cycle C_1, containing the edge $u_{d-1,0}u_{d-1,1}$. Let $v_{d-2,i}$ be adjacent in G to $u_{d-1,0}$. Then it is not difficult to see that $v_{d-2,i+1}$ is adjacent to $u_{d-1,1}$. On the other hand, since $\gcd\big(\frac{n}{d}, \alpha'\big) = 1$, by Lemma 3.2 there exists a Hamilton path P_{d-2} in H_{d-2} joining $v_{d-2,i}$ and $v_{d-2,i+1}$. Now replacing the edge $u_{d-1,0}u_{d-1,1}$ in C_1 by the path

$$\{u_{d-1,0}v_{d-2,i}\} \cup P_{d-2} \cup \{v_{d-2,i+1}u_{d-1,1}\}$$

we can obtain a Hamilton cycle in $G\big[V(H_{d-2}) \cup V(H_{d-1})\big]$. This procedure can be continued to obtain a Hamilton cycle in

$$G = G\big[V(H_0) \cup V(H_1) \cup \ldots \cup V(H_{d-1})\big].$$

□

Lemma 3.5. *Let $G = MC(2, n, \alpha, S_0, S_1)$ be a metacirculant graph with n even, $S_0 = \{\frac{n}{2}\}, S_1 = \{h, k, l\}$ and gcd$\big(h-k, k-l, \frac{n}{2}\big) = 1$. Then G possesses a Hamilton cycle.*

Proof. Let $G' = MC(2, n, -1, S_0, S_1)$ where $S_0 = \{\frac{n}{2}\}$, $S_1 = \{h, k, l\}$ and the vertex-set $V(G') = \{u^i_j \mid i \in \boldsymbol{Z}_2, j \in \boldsymbol{Z}_n\}$. Let φ be a bijection from $V(G)$ onto $V(G')$, defined by $\varphi(v^i_j) = u^i_j$. Then it is not difficult to verify that φ is an isomorphism between G and G'. Therefore, without loss of generality , we may assume that the graph G is $MC(2, n, -1, S_0, S_1)$ where n is even, $S_0 = \{\frac{n}{2}\}$, $S_1 = \{h, k, l\}$ and $\gcd\big(h-k, k-l, \frac{n}{2}\big) = 1$. There are two cases to consider.

Case 1: $\gcd(h-k, k-l, n) = 1$.

Let G' be a spanning subgraph of G isomorphic to $H = MC(2, n, -1, S'_0, S'_1)$ with $S'_0 = \emptyset$ and $S'_1 = S_1 = \{h, k, l\}$. It is clear that H is a cubic metacirculant graph. Since $\gcd(h-k, k-l, n) = 1$, by [10, Theorem 2], the graph H is connected. By [3, Theorem 9], H is a Cayley graph on the group $\langle\rho, \tau\rangle$, where ρ and τ are the automorphisms of H with $\rho(v^i_j) = v^i_{j+1}$ and $\tau(v^i_j) = v^{i+1}_{\alpha j}$. It is not difficult to see that ρ and τ satisfy the relations $\tau\rho\tau^{-1} = \rho^{-1}$ and $\rho^n = \tau^2 = 1$. Therefore $\langle\rho, \tau\rangle$ is a dihedral group. Thus H is a connected cubic Cayley graph on the dihedral group $\langle\rho, \tau\rangle$. By Theorem 2.2, we conclude H has a Hamilton cycle. Since H is isomorphic to the spanning subgraph G' of G, G possesses a Hamilton cycle.

Case 2: $\gcd(h-k, k-l, n) = 2$.

Let G be a tetravalent metacirculant graph $MC(2, n, -1, S_0, S_1)$ with n even, $S_0 = \{\frac{n}{2}\}$, $S_1 = \{h, k, l\}$ and $\gcd\big(h-k, k-l, \frac{n}{2}\big) = 1$ but $\gcd(h-k, k-l, n) = 2$. It is clear that $\gcd(h-k, k-l) = d$ is even. It follows that $h-k, k-l$ are even. So either all h, k, l are even or all of them are odd. Since $\gcd\big(h-k, k-l, \frac{n}{2}\big) = 1$, the number $\frac{n}{2}$ must be odd.

Consider two subsets $A_1 = \{0, 2, \dots, n-2\}$ and $A_2 = \{1, 3, \dots, n-1\}$ of $\boldsymbol{Z}_n$. Since $\frac{n}{2}$ is odd, $A_2 = A_1 + \frac{n}{2}$. Let

$$V_{11} = \{v_i^0, v_i^1 \mid i \in A_1\};$$
$$V_{22} = \{v_j^0, v_j^1 \mid j \in A_2\};$$
$$V_{12} = \{v_i^0, v_j^1 \mid i \in A_1, j \in A_2\};$$
$$V_{21} = \{v_j^0, v_i^1 \mid i \in A_1, j \in A_2\}.$$

It is clear that $V_{11} \cap V_{22} = \emptyset$ and $V_{11} \cup V_{22} = V(G)$; $V_{12} \cap V_{21} = \emptyset$ and $V(G) = V_{12} \cup V_{21}$.

First assume that all h, k, l are even. Let $G_{11} = G[V_{11}]$ and $G_{22} = G[V_{22}]$. Then it is not difficult to verify that $\psi : V_{11} \to V_{22}$, $v_j^i \mapsto v_{j+\frac{n}{2}}^i$ is an isomorphism between G_{11} and G_{22}. Furthermore, G_{11} and G_{22} are isomorphic to the cubic metacirculant graph $H = MC\big(2, \frac{n}{2}, -1, S_0', S_1'\big)$ with $S_0' = \emptyset$, $S_1' = \{h', k', l'\}$, where $h' = \frac{h}{2}$, $k' = \frac{k}{2}$, $l' = \frac{l}{2}$. Since $\gcd\big(h-k, k-l, \frac{n}{2}\big) = 1$, we have $\gcd\big(h'-k', k'-l', \frac{n}{2}\big) = 1$. Therefore the graph H is connected. As in Case 1, we can show that H is a Cayley graph on a dihedral group of order $\frac{n}{2}$. By Theorem 2.2, H has a Hamilton cycle. This implies that G_{11} has a Hamilton path P with the endvertices v_i^0 and v_j^1, where $j - i \in S_1$. Then $\psi(P)$ is a Hamilton path of G_{22} with the endvertices $\psi(v_i^0) = v_{i+\frac{n}{2}}^0$ and $\psi(v_j^1) = v_{j+\frac{n}{2}}^1$. Since in G v_i^0 is adjacent to $v_{i+\frac{n}{2}}^0$ and v_j^1 is adjacent to $v_{j+\frac{n}{2}}^1$, it is not difficult to construct a Hamilton cycle of G from P, $\psi(P)$ and the edges $v_i^0 v_{i+\frac{n}{2}}^0$, $v_j^1 v_{j+\frac{n}{2}}^1$.

Now assume that all h, k, l are odd. Let $G_{12} = G[V_{12}]$ and $G_{21} = G[V_{21}]$. By considering G_{12} and G_{21} with arguments similar to those above, we can show that the graph G has a Hamilton cycle. Lemma 3.5 has been proved completely. □

Next we consider which connected tetravalent metacirculant graph $G = MC(m, n, \alpha, S_0, \dots, S_\mu)$ with $S_0 \neq \emptyset$ and $m = 1$ or 2 has a Hamilton cycle.

Theorem 3.1. *Let $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ be a connected tetravalent metacirculant graph with $S_0 \neq \emptyset$ and $m = 1$ or 2. Then G possesses a Hamilton cycle.*

Proof. Let $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ be a connected tetravalent metacirculant graph with $S_0 \neq \emptyset$ and $m = 1$ or 2. By Theorem 2.1, only one of the following cases may happen:

(1) $m = 1$, $S_0 = \{\pm s, \pm r\}$ and $\gcd(s, r, n) = 1$;

(2) $m = 2$, n is even, $S_0 = \big\{\pm s, \frac{n}{2}\big\}$, $S_1 = \{k\}$ and $\gcd\big(s, \frac{n}{2}\big) = 1$;

(3) $m = 2$, $S_0 = \{\pm s\}$, $S_1 = \{k, l\}$ and $\gcd(s, k-l, n) = 1$;

(4) $m = 2$, n is even, $S_0 = \{\frac{n}{2}\}$, $S_1 = \{h, k, l\}$ and $\gcd\big(h-k, k-l, \frac{n}{2}\big) = 1$.

Now Theorem 3.1 is implied from Lemmas 3.1, 3.3, 3.4, 3.5. □

Finally we consider which connected tetravalent metacirculant graph $G = MC(m,\ n, \alpha, S_0, S_1, \dots, S_\mu)$ with $S_0 \neq \emptyset$ and $m > 2$ has a Hamilton cycle. For this case, we obtain the following result.

Theorem 3.2. *Let $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ be a connected tetravalent metacirculant graph with $S_0 \neq \emptyset$, $m > 2$ and both m and n are odd. Then G possesses a Hamilton cycle.*

Proof. Let $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ be a graph satisfying the hypothesis. Then $m \geq 3$. By Theorem 2.1, we must have $S_0 = \{\pm s\}$, $S_i = \{k\}$ for some $i \in \{1, 2, \dots, \mu\}$ such that $\gcd(i, m) = 1$, $S_j = \emptyset$ for any $i \neq j \in \{1, 2, \dots, \mu\}$ and $\gcd(s, r, n) = 1$ where

$$r = k(1 + \alpha^i + \alpha^{2i} + \cdots + \alpha^{(m-1)i}).$$

Let $G' = MC(m, n, \alpha', S'_0, S'_1, \dots, S'_\mu)$ be a metacirculant graph with

$$V(G') = \{u^x_y \mid x \in \boldsymbol{Z}_m,\ y \in \boldsymbol{Z}_n\}$$

and $\alpha' = \alpha^i$, $S'_0 = S_0$, $S'_1 = S_i$, $S'_2 = S'_3 = \cdots = S'_\mu = \emptyset$. We will prove that the graph G is isomorphic to the graph G'.

Consider the mapping $\varphi\ :\ V(G) \to V(G')$, $v^{xi}_y \mapsto u^x_y$. Since $\gcd(i, m) = 1$, we can see that φ is a bijection. Further, let $v^{xi}_y v^{xi+r}_h \in E(G)$. Then we must have either $r = i$ and $(h - y) \in \alpha^{xi} S_i$ or $r = 0$ and $(h - y) \in \alpha^{xi} S_0$.

If $r = i$ and $(h-y) \in \alpha^{xi} S_i$, then $\varphi(v^{xi}_y)\varphi(v^{xi+i}_h) = u^x_y u^{x+1}_h$ with $(h-y) \in \alpha^{xi} S_i$. This means $(h - y) \in (\alpha^i)^x S_i$. So $(h - y) \in (\alpha')^x S'_1$. Thus $u^x_y u^{x+1}_h$ is an edge of G'. If $r = 0$ and $(h - y) \in \alpha^{xi} S_0$, then we have

$$\varphi(v^{xi}_y)\varphi(v^{xi+0}_h) = u^x_y u^x_h$$

with $(h-y) \in \alpha^{xi} S_0 = (\alpha')^x S'_0$. Thus $u^x_y u^x_h$ is also an edge of G'. Similarly, we can verify that if $u^x_y u^{x+r}_h$ is an edge of G' then $\varphi^{-1}(u^x_y)\varphi^{-1}(u^{x+r}_y)$ is also an edge of G. Thus, φ is an isomorphism from G onto G'. So, without loss of generality, we may assume that the graph G is the graph $MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ with $m > 2$ odd, n odd, $S_0 = \{\pm s\}$, $S_1 = \{k\}$, $S_2 = S_3 = \ldots = S_\mu = \emptyset$ and $\gcd(s, r, n) = 1$, where r is $k(1 + \alpha + \alpha^2 + \ldots + \alpha^{(m-1)})$.

Let ρ be the automorphism of G defined by $\rho(v^i_j) = v^i_{j+1}$. Then ρ is semiregular. If $\gcd(s, n) = d$ then the automorphism $\beta = \rho^d$ is also semiregular. The orbit of $\langle\beta\rangle$ containing the vertex v^i_j is

$$V^i_j = \{v^i_j, v^i_{j+d}, v^i_{j+2d}, \dots, v^i_{j+(\frac{n}{d}-1)d}\}.$$

On the other hand, the subsets

$\{0, d, 2d, \dots, (\frac{n}{d} - 1)d\}$ and $\{0, \alpha^i s, 2\alpha^i s, \dots, (\frac{n}{d}\ -\ 1)\ \alpha^i s\}$

of $\boldsymbol{Z}_n$ coincide with each other. So $G[V_j^i]$ is the cycle

$$v_j^i v_{j+\alpha^i s}^i v_{j+2\alpha^i s}^i \cdots v_{j+(\frac{n}{d}-1)\alpha^i s}^i v_j^i$$

for any $i = 0, 1, \dots, (m-1)$ and $j = 0, 1, \dots, (d-1)$.

If the automorphism β has order 2, then $\rho^{2d}(v_j^i) = v_j^i$. This means $v_{j+2d}^i = v_j^i \Leftrightarrow 2d \equiv 0 \pmod{n}$. This is impossible because n is odd and d is a proper divisor of n.

Consider the quotient graph G/β. We have $V(G/\beta) = \{V_j^i \mid i \in \boldsymbol{Z}_m,\ j \in \boldsymbol{Z}_d\}$ and two vertices of G/β are adjacent in G/β if and only if there is an edge in G joining a vertex of one corresponding orbit to a vertex of the other orbit of $\langle\beta\rangle$. Since G is connected, the graph G/β is also connected. Moreover, since $G[V_j^i]$ is a cycle and G is tetravalent, G/β is a regular graph of degree 2. It follows that G/β is a cycle. We have $|V(G/\beta)| = md$ with m odd and d a divisor of n. So $|V(G/\beta)|$ is odd. By Theorem 2.4 we conclude that G has a Hamilton cycle. The proof of Theorem 3.2 is complete. □

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