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GENERAL FORM OF LINEAR FUNCTIONALS ON HARDY SPACES

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Dedicated to Prof. Nguyen Duy Tien on the oscasion of his 60th birthday

Abstract. In this note, we give a general form of linear functionals on martingale Hardy spaces $\mathcal{H}_\Phi$, where the Young function Φ has finite power and its conjugate function Ψ is of the form $\Psi(x) = \Psi_1(x^2)$ with Ψ_1 being also a Young function. Moreover, we discuss the relationship between the martingale Hardy space $\mathcal{H}_\Phi$ and the spaces $\mathcal{K}_\Psi$.

1. Introduction

Fefferman [4] has proved that the dual space of the Hardy space $\mathcal{H}_1$ is the BMO-space (the space of functions of bounded mean oscillation, which in turn, has been treated by John and Nirenberg [7]). The generalizations of the above result were obtained by Garsia [5], who investigated the Hardy spaces $\mathcal{H}_p$ with $1 \leq p < +\infty$. He has shown that the dual of $\mathcal{H}_p$ is the so called $\mathcal{K}_q$ space, where $1 \leq p \leq 2$ and $p^{-1} + q^{-1} = 1$. Later on, Mogyorodi [9] and Bui K. D. [1], [2] have generalized these results to the martingale Hardy space $\mathcal{H}_\Phi$. In [1], it was showed that the dual space of the martingale Hardy space $\mathcal{H}_\Phi$ is the space $\mathcal{H}_\Psi$, where Φ and Ψ are conjugate Young functions and both have finite power. However, the condition saying that the function Ψ has finite power can be omitted if we suppose that $\Phi(x) = \Phi_1(x^2)$ with Φ_1 is also Young function having finite power (see [2]). For a martingale Hardy space with the continuous time, it was treated by Dellacherie and Meyer [3], and Weise [12].

In this paper, we give a general form of a bounded, linear functional on the Hardy space $\mathcal{H}_\Phi$ under the condition that Φ has finite power and the conjugate function Ψ has the form $\Psi(x) = \Psi_1(x^2)$ with Ψ_1 being a Young function (need not to have finite power). Moreover, we investigate the relationship between $\mathcal{H}_\Phi$ and $\mathcal{K}_\Psi$ in some special cases.

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2. Preliminaries and notations

Let $\phi(t)$ be a nondecreasing and left-continuous function defined on the interval $[0,\infty)$ such that $\phi(0)=0$ and $\lim\limits_{t\to+\infty}\phi(t)=+\infty$. For $x\geq 0$ define

$$\Phi(x)=\int_0^x \phi(t)dt.$$

Then Φ is a continuous convex increasing function on R_+. It is called a Young function. The power p of Φ will be defined as follows:

$$p=\sup_{x>0}\frac{x\phi(x)}{\Phi(x)}\cdot$$

We say that Φ has finite power if $p<\infty$.

Now we define the generalized inverse function ψ of the function ϕ as follows:

$$\psi(u)=\sup(s>0:\phi(s)<u) \qquad \text{if } \ u>0$$

and

$$\psi(0)=0.$$

It is easy to see that ψ is also nondecreasing, left-continuous function and $\lim\limits_{t\to+\infty}\psi(t)=+\infty$. The indefinite integral of this function

$$\Psi(x)=\int_0^x \psi(u)du,$$

is also a continuous convex increasing function on R_+ (it is called the conjugate Young function of the function Φ). A pair of such functions (Φ,Ψ) is called a pair of Young's functions and the function Ψ (resp. Φ) is said to be conjugate to Φ (resp. Ψ). The following lemma shows that the function Φ has finite power if and only if it satisfies the condition Δ_2 (see [11]).

Lemma 2.1. *For every Young function Φ, the following conditions are equivalent:*

(a) $\sup\limits_{t>0}(\Phi(a.t)/\Phi(t))<\infty$ *for some* $a>1$,

(b) $\sup\limits_{t>0}(\phi(a.t)/\phi(t))<\infty$ *for some* $a>1$,

(c) $\sup\limits_{t>0}(t.\phi(t)/\Phi(t))<\infty$.

We give two examples concerning Young's functions.

Example 2.1. a) If we put $\Phi_\alpha(t)=(1+\alpha)^{-1}.t^{1+\alpha}$ for $t\in R_+$ and for every $\alpha\in[0,\infty)$, then the functions Φ_α and $\Phi_{1/\alpha}$ form a pair of Young functions (note that $\phi_\alpha(t)=t^\alpha$).

b) The pair

$$\Phi_1(t)=e^t-1-t, \quad \Psi_1(v)=(1+v)\ln(1+v)-v;\ t,v\in R_+$$

are conjugate Young functions.

Note that Φ has finite power, but the exponential function Φ_1 does not.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$ be a sequence of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_\infty = \sigma\big(\bigcup_{n=0}^{\infty} \mathcal{F}_n\big) = \mathcal{F}$. Consider a random variable $X \in L_1 = L_1(\Omega, \mathcal{F}, P)$ and the martingale $X_n = E(X|\mathcal{F}_n)$, $n = 0, 1, \dots$. For the sake of commodity we suppose that $X_0 = 0$ a.s.. Denote by $d_0 = 0$, $d_i = X_i - X_{i-1}$, $i = 1, 2, \dots$, the corresponding martingale differences.

Definition 2.1. The random variable X belongs to the family $L_\Phi(\Omega, \mathcal{F}, P)$ if there exists a positive constant a such that

$$E[\Phi(a^{-1}|X|)] \leq 1.$$

In this case we define

$$\|X\|_\Phi = \inf\{a > 0 : E[\Phi(a^{-1}|X|)] \leq 1\}.$$

Note that $\|.\|_\Phi$ is a norm on $L_\Phi(\Omega, \mathcal{F}, P)$ and the normed vector space $L_\Phi(\Omega, \mathcal{F}, P)$ is called an Orlicz space.

Definition 2.2. We say that the random variable $X \in L_1$ belongs to the Hardy space $\mathcal{H}_\Phi$ generated by the Young function Φ if the random variable

$$S = S(X) = \left(\sum_{i=1}^{\infty} d_i^2\right)^{1/2}$$

belongs to L_Φ, where L_Φ is the Orlicz space generated by the Young function Φ.

It is easy to show that $\mathcal{H}_\Phi$ with the norm

$$\|X\|_{\mathcal{H}_\Phi} = \|S(X)\|_\Phi$$

is a Banach space.

Definition 2.3. For $X \in L_1$ define the families Γ_X^Φ and $\widetilde{\Gamma}_X^\Phi$ of the random variables by setting

$$\Gamma_X^\Phi = \{\gamma : \gamma \in L_\Phi, E(|X - X_{n-1}||\mathcal{F}_n) \leq E(\gamma|\mathcal{F}_n) \quad \text{a.s.} \quad \forall n \geq 1\},$$
$$\widetilde{\Gamma}_X^\Phi = \{\gamma : \gamma \in L_\Phi, E(|X - X_{n-1}|^2|\mathcal{F}_n) \leq E(\gamma^2|\mathcal{F}_n) \quad \text{a.s.} \quad \forall n \geq 1\}.$$

Let

$$\|X\|_{\mathcal{K}_\Phi} = \inf_{\gamma \in \Gamma_X^\Phi} \|\gamma\|_\Phi$$

if Γ_X^Φ is not empty and $\|X\|_{\mathcal{K}_\Phi} = +\infty$ if $\Gamma_X^\Phi = \emptyset$.

Similarly, let

$$\|X\|_{\tilde{\mathcal{K}}_\Phi} = \inf_{\gamma \in \tilde{\Gamma}_X^\Phi} \|\gamma\|_\Phi$$

if $\widetilde{\Gamma}_X^\Phi$ is not empty and $\|X\|_{\tilde{\mathcal{K}}_\Phi} = +\infty$ if $\widetilde{\Gamma}_X^\Phi = \emptyset$.

We say that $X \in \mathcal{K}_{\Phi}$ (resp. $X \in \widetilde{\mathcal{K}}_{\Phi}$) with the norm $\|X\|_{\mathcal{K}_{\Phi}}$ (resp. $\|X\|_{\widetilde{\mathcal{K}}_{\Phi}}$) if Γ_X^{Φ} (resp. $\widetilde{\Gamma}_X^{\Phi}$) is nonempty.

Remark 2.1. In the case $\Phi(x) = x^p$ with $2 \le p \le +\infty$ the definition of the spaces $\mathcal{K}_{\Phi}$ is equivalent to the definition of $\widetilde{\mathcal{K}}_{\Phi}$ space.

The problem of finding a general form of lineal functional on Hardy space was solved for such a cases, as both functions Φ and Ψ have finite power or Φ has finite power and $\Phi(x) = \Phi_1(x^2)$ with $\Phi_1(x)$ being also Young function but Ψ does not. These results can be found in [1], [2], [9].

Theorem 2.1. (See [1, p. 291]) *Let Φ, Ψ be a pair of conjugate Young functions and suppose that both of them have finite power. Then the dual space of the Hardy space $\mathcal{H}_{\Phi}$ is the space $\mathcal{K}_{\Psi}$.*

Theorem 2.2. (See [2, Theorem 3.2., p. 59]) *Let Φ, Ψ be a pair of conjugate Young functions.*

i) *For every $X \in \mathcal{H}_{\Phi}$ and $Y \in \mathcal{H}_{\Psi}$ we have*

$$|E(X_nY_n)| \le 2\|X_n\|_{\mathcal{H}_{\Phi}}\|Y_n\|_{\mathcal{H}_{\Psi}} \qquad \forall n \ge 1.$$

Further, $\lim_{n\to+\infty} E(X_nY_n)$ exists, it is finite and

$$|\lim_{n\to+\infty} E(X_nY_n)| \le 2\|X\|_{\mathcal{H}_{\Phi}}\|Y\|_{\mathcal{H}_{\Psi}}.$$

ii) *Suppose that Φ is of the form $\Phi(x) = \Phi_1(x^2)$, where $\Phi_1(x)$ itself is a Young function having finite power p. If F is a bounded, linear functional on $\mathcal{H}_{\Phi}$, i.e.,*

$$|F(X)| \le B\|X\|_{\mathcal{H}_{\Phi}}$$

for some constant $B > 0$. Then there existx $Y \in \mathcal{H}_{\Psi}$ such that

$$\|Y\|_{\mathcal{H}_{\Psi}} \le \sqrt{2}(2\sqrt{p} + 1)B$$

and

$$\lim_{n\to+\infty} E(X_nY_n) = F(X), \quad \forall X \in \mathcal{H}_{\Phi}.$$

Theorem 2.3. (See [9]) *Suppose that Φ, Ψ is a pair of conjugate Young functions such that Φ has finite power and Ψ is of the form $\Psi(x) = \Psi_2(x^2)$ where $\Psi_2(x)$ itself is also a Young function. Then the dual space of the Hardy space $\mathcal{H}_{\Phi}$ is the space $\widetilde{\mathcal{K}}_{\Psi}$.*

3. Results

Now we present the main results.

Theorem 3.1. *Let Φ, Ψ be a pair of conjugate Young functions such that*

(i) *Φ has finite power,*

(ii) *Ψ is of the form $\Psi(x) = \Psi_2(x^2)$ where Ψ_2 itself is also Young function and Ψ has a continuous derivative.*

Then the dual space of the Hardy space $\mathcal{H}_{\Phi}$ is the space $\mathcal{K}_{\Psi}$.

Proof. First of all, we show that the function Φ_2, which is conjugate function of the function Ψ_2, has finite power. Note that Ψ_2 has also continuous derivative since Ψ has. Furthermore,

$$\psi_2(t) = \frac{\psi(\sqrt{t})}{2\sqrt{t}}, \quad t > 0.$$

By Theorem 4.3 of [7, p. 27] this implies

$$\frac{t\psi_2(t)}{\Psi_2(t)} = \frac{1}{2} \cdot \frac{\sqrt{t}\psi(\sqrt{t})}{\Psi(\sqrt{t})} > a, \quad a > 0,$$

since Φ has finite power. Again, by the above equality and Theorem 4.3 of [7] we deduce that Φ_2 has finite power.

Let us denote the power of Φ (resp. of Φ_2) by p (resp. p_2). Suppose $X \in \mathcal{H}_\Phi$, $Y \in \mathcal{K}_\Psi$. Denote $X_n = E(X|\mathcal{F}_n)$ and $Y_n = E(Y|\mathcal{F}_n)$. Then there exists a positive constant C_Φ depending only on Φ such that the following Feffermann-type inequality is true (see [8]):

$$|E(X_nY_n)| \leq C_\Phi.\|X_n\|_{\mathcal{H}_\Phi}\|Y_n\|_{\mathcal{K}_\Psi}.$$

Moreover, the limit $\lim\limits_{n\to+\infty} E(X_nY_n)$ exists and satisfies the condition

$$|\lim_{n\to+\infty} E(X_nY_n)\,| \leq C_\Phi.\|X\|_{\mathcal{H}_\Phi}\|Y\|_{\mathcal{K}_\Psi}.$$

So if $Y \in \mathcal{K}_\Psi$ is fixed and X runs on $\mathcal{H}_\Phi$, then $F(X) = \lim\limits_{n\to+\infty} E(X_nY_n)$ is a continuous linear functional on $\mathcal{H}_\Phi$ with a norm not exceeding $\|Y\|_{\mathcal{K}_\Psi}$.

Conversely, let F be a bounded linear functional on $\mathcal{H}_\Phi$, i.e.,

$$|F(X)| \leq B.\|X\|_{\mathcal{H}_\Phi}, \quad X \in \mathcal{H}_\Phi, \quad 0 < B < +\infty.$$

By Theorem 2 of [9], there exists $Y \in \widetilde{\mathcal{K}}_\Psi$ with $\|Y\|_{\tilde{\mathcal{K}}_\Psi} \leq 2.p.B$ such that

$$F(X) = \lim_{n\to+\infty} E(X_nY_n)$$

for all $X \in \mathcal{H}_\Phi$.

Now we show that Y also belongs to the $\mathcal{K}_\Psi$-space. Let $\gamma \in \widetilde{\Gamma}_Y^\Psi$. Since $\gamma \in L_\Psi(R)$, there exists a real number $a > 0$ such that $E\Psi(a^{-1}|\gamma|) \leq 1$. But $\Psi(x) = \Psi_2(x^2)$, thus $E\Psi_2(a^{-2}\gamma^2) \leq 1$. So $\gamma^2 \in L_{\Psi_2}(R)$ and $\|\gamma^2\|_{\Psi_2} \leq \|\gamma\|_\Psi^2$. Consider the martingale $\{E(\gamma^2|\mathcal{F}_n)\}_{n\geq 0}$ and denote

$$\gamma^\star = \sup_{n\geq 0} E(\gamma^2|\mathcal{F}_n).$$

Using the maximal inequality for martingales on the Orlicz space L_{Ψ_2} with the note that the function Φ_2, the conjugate function of the function Ψ_2, has the power p_2 (see [10]), we have

$$\|\gamma^\star\|_{\Psi_2} \leq p_2.\|\gamma^2\|_{\Psi_2} \leq p_2.\|\gamma\|_\Psi^2.$$

It is easy to see that

$$E(|Y - Y_{n-1}||\mathcal{F}_n) \leq (E(|Y - Y_{n-1}|^2|\mathcal{F}_n))^{1/2} \leq$$
$$\leq (E(\gamma^2|\mathcal{F}_n))^{1/2} \leq (\gamma^\star)^{1/2} \quad \text{a.s.,} \quad n \geq 1.$$

Taking conditional expectation of both sides of the above inequality, we get

$$E(|Y - Y_{n-1}||\mathcal{F}_n) \leq E(\sqrt{\gamma^\star}|\mathcal{F}_n) \quad \text{a.s.} \quad n \geq 1,$$

On the other hand, $\gamma^\star \in L_{\Psi_2}$; thus $\sqrt{\gamma^\star} \in L_\Psi(R)$ and

$$\|\sqrt{\gamma^\star}\|_\Psi \leq \sqrt{\|\gamma^\star\|_{\Psi_2}} \leq \sqrt{p_2}\|\gamma\|_\Psi.$$

Consequently, $Y \in \mathcal{K}_\Psi$ and

$$\|Y\|_{\mathcal{K}_\Psi} \leq \sqrt{\|\gamma^\star\|_\Psi} \leq \sqrt{p_2}\|\gamma\|_\Psi.$$

Since this inequality holds for all $\gamma \in \widetilde{\Gamma}_Y^\Psi$, we have

$$\|Y\|_{\mathcal{K}_\Psi} \leq \sqrt{p_2}.\|Y\|_{\tilde{\mathcal{K}}_\Psi} \leq 2p\sqrt{p_2}B.$$

The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Let Φ, Ψ be a pair of conjugate Young functions such that Φ has finite power.*

(i) *If Φ is of the form $\Phi(x) = \Phi_1(x^2)$, where Φ_1 is also Young function, then $\mathcal{K}_\Psi$ is a subspace of the space $\mathcal{H}_\Psi$ and*

$$\|Y\|_{\mathcal{H}_\Psi} \leq A_\Psi.\|Y\|_{\mathcal{K}_\Psi} \quad \textit{for all} \quad Y \in \mathcal{K}_\Psi$$

where A_Ψ is a constant depending only on Ψ

(ii) *If Ψ is of the form $\Psi(x) = \Psi_2(x^2)$, where Ψ_2 is also Young function and Ψ has continuous derivative, then $\mathcal{H}_\Psi$ is a subspace of the space $\mathcal{K}_\Psi$ and*

$$\|Y\|_{\mathcal{K}_\Psi} \leq B_\Psi.\|Y\|_{\mathcal{H}_\Psi} \quad \textit{for all} \quad Y \in \mathcal{H}_\Psi,$$

with B_Ψ being a constant depending only on Ψ.

Proof. (i) Let $Y \in \mathcal{K}_\Psi$. The formula

$$F_Y(X) = \lim_{n\to+\infty} E(X_n Y_n), \quad (X \in \mathcal{H}_\Phi)$$

defines a bounded linear functional on $\mathcal{H}_\Phi$ and $|F_Y(X)| \leq C_\Phi.\|Y\|_{\mathcal{K}_\Psi}\|X\|_{\mathcal{H}_\Phi}$ for all $X \in \mathcal{H}_\Phi$, where C_Φ is a constant depending only on Φ. By Theorem 2.1, there exists $Z \in \mathcal{H}_\Psi$ such that

$$F_Y(X) = \lim_{n\to+\infty} E(X_n Z_n)$$

for all $X \in \mathcal{H}_\Phi$ with $Z_n = E(Z|\mathcal{F}_n)$ $\forall n \geq 1$ and $\|Z\|_{\mathcal{H}_\Psi} \leq \sqrt{2}(2\sqrt{2p}+1)\|F_Y\|$.

The above inequalities imply that $Y = Z$ a.s.. So $\mathcal{K}_\Psi \subset \mathcal{H}_\Psi$ and

$$\|Y\|_{\mathcal{H}_\Psi} \leq \sqrt{2}C_\Phi(2\sqrt{2p}+1)\|Y\|_{\mathcal{K}_\Psi}.$$

(ii) Let $Y \in \mathcal{H}_\Psi$ be fixed and let $X \in \mathcal{H}_\Phi$. Denote

$$\Delta X_i = X_i - X_{i-1}, \quad i = 1, 2, ... : \Delta Y_j = X_j - X_{j-1}, \quad j = 1, 2, ...$$

Using the Cauchy-Schwartz inequality, we have

$$\Big(\sum_{i=1}^{n}\Delta X_i\Delta Y_i\Big)^2 \le \Big[\sum_{i=1}^{n}(\Delta X_i)^2\Big]\Big[\sum_{i=1}^{n}(\Delta Y_i)^2\Big].$$

Combining the above inequality with Hölder's inequality for conjugate Young functions, we get

$$\begin{aligned}|E(X_nY_n)| &= \Big|\sum_{i=1}^{n}E(\Delta X_i\Delta Y_i)\Big| = \Big|E\Big(\sum_{i=1}^{n}\Delta X_i\Delta Y_i\Big)\Big| \\ &\le E\Big(\Big[\sum_{i=1}^{n}(\Delta X_i)^2\Big]^{1/2}\Big[\sum_{j=1}^{n}(\Delta Y_j)^2\Big]^{1/2}\Big) \\ &\le 2\Big\|\Big[\sum_{i=1}^{n}(\Delta X_i)^2\Big]^{1/2}\Big\|_{\Phi}\Big\|\Big[\sum_{j=1}^{n}(\Delta Y_j)^2\Big]^{1/2}\Big\|_{\Psi}.\end{aligned}$$

Note that $\{E(X_nY_n)\}_{n\ge 1}$ is a Cauchy sequence since for $m \ge n$ we have

$$|E(X_mY_m) - E(X_nY_n)| = \Big|E\Big(\sum_{i=n+1}^{m}\Delta X_i\Delta Y_i\Big)\Big| \le$$

$$\le \Big\|\Big[\sum_{i=n+1}^{m}(\Delta X_i)^2\Big]^{1/2}\Big\|_{\Phi}\Big\|\Big[\sum_{j=n+1}^{m}(\Delta Y_j)^2\Big]^{1/2}\Big\|_{\Psi} \to 0$$

as $n \to +\infty$. Therefore $\lim\limits_{n\to+\infty} E(X_nY_n)$ exists and

$$\Big|\lim_{n\to+\infty}E(X_nY_n)\Big| \le 2\|X\|_{\mathcal{H}_\Phi}\|Y\|_{\mathcal{K}_\Psi}.$$

Define the functional

$$F_Y(X) = \lim_{n\to+\infty}E(X_nY_n), \quad X \in \mathcal{H}_\Phi.$$

It is easy to see that $F_Y(.)$ is a linear and bounded functional on $\mathcal{H}_\Phi$ and $\|F_Y\| \le 2\|Y\|_{\mathcal{H}_\Psi}$.

Using Theorem 3.1 we deduce that there exists $Z \in \mathcal{K}_\Psi$ such that

$$F_Y(X) = \lim_{n\to+\infty}E(X_nZ_n), \quad X \in \mathcal{H}_\Phi.$$

Combining the above equalities, we get

$$\lim_{n\to+\infty}E(X_nY_n) = \lim_{n\to+\infty}E(X_nZ_n) \quad \forall X \in \mathcal{H}_\Phi.$$

This implies $Y = Z$ a.s.. So $\mathcal{H}_\Psi \subset \mathcal{K}_\Psi$.

Our proof is completed. □

Remark 3.1. From the part (i) we have

$$\|Y\|_{\mathcal{H}_\Psi} = \|S(Y)\|_\Psi \le 2.A_\Psi\|Y^\star\|_\Psi$$

for all $Y \in \mathcal{K}_\Psi$.

References

[1] Bui Khoi Dam, *The dual of the martingale Hardy space $\mathcal{H}_\Phi$ with general Young function* Φ, Analysis Math. **14** (1988), 287-294.

[2] Bui Khoi Dam, *Connection between the BMO- and the $\mathcal{K}_\Phi$-spaces*, Annals Univ. Sci. Budapest, Sect. Comp. **9** (1988), 51-66.

[3] C. Dellacherie, P. A. Meyer, *Probabilities and potential* A., North-Holland Math. Studies **29**, North-Holland, 1978.

[4] C. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. **77** (1971), 587-588.

[5] A. M. Garsia, *Martingale Inequalities*, Benjamin, Reading, Massachusetts, 1973.

[6] S. Ishak and J. Mogyorodi, *On the generalization of the Fefferman-Garsia inequality*, Stochastic Differential Systems, Lecture Notes In Control and Information Sciences, Vol. **36**, Springer, Berlin, pp. 85-97, 1981.

[7] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **17** (1961), 415-426.

[8] M. A. Krasnoselskii and Ya. B. Rutickii, *Convex functions and Orlicz spaces*, Groningen, Noordhoff, 1961.

[9] J. Mogyorodi, *Linear functional on Hardy spaces*, Annals Univ. Sci. Budapest, Sect. Math. **26** (1983), 161-174.

[10] J. Mogyrodi and T. F. Mori, *Necessary and sufficient condition for the maximal inequality of convex Young functions*, Acta Sci. Math. **45** (1983), 325-332.

[11] J. Neveu, *Discrete parameter martingales*, North-Holland, Amsterdam 1973.

[12] F. Weisz, *Martingales Hardy spaces with continuous time*, In "Probability Theory and Applications, Essays to the Memory of Jozsef Mogyorodi", (J. Galambos and I. Katai, Eds.) Kluwer Academic Publishers, Dordrecht, pp. 47-75, 1992.

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