ACTA MATHEMATICA VIETNAMICA
Volume 28, Number 2, 2003, pp. 175-183

GROWTH OF A CLASS OF COMPOSITE ENTIRE FUNCTIONS

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ABSTRACT. In this paper, we obtain the following results:

Let f_1, f_2 and g_1, g_2 be four transcendental entire functions with $T(r, f_1) = O^*((\log r)^\nu e^{(\log r)^\alpha})$ and $T(r, g_1) = O^*((\log r)^\beta)$ (i.e., there exist four positive constants K_1, K_2, K_3 and K_4 such that $K_1 \le \frac{T(r,f_1)}{(\log r)^\nu e^{(\log r)^\alpha}} \le K_2$ and $K_3 \le \frac{T(r,g_1)}{(\log r)^\beta} \le K_4$).

If $T(r, f_1) \sim T(r, f_2)$, $T(r, g_1) \sim T(r, g_2)$ $(r \to \infty)$, then

$$T(r, f_1(g_1)) \sim T(r, f_2(g_2)) \quad (r \to \infty,\ r \notin E)$$

where $\nu > 0$, $0 < \alpha < 1$, $\beta > 1$ and $\alpha\beta < 1$ and E is a set of finite logarithmic measure.

We solved a problem due to C. C. Yang concerning the characteristic functions of the composite functions.

1. Introduction

Chitai Chuang and C. C. Yang [2] proposed the following problem: Let f_1, f_2 and g_1, g_2 be entire functions. If $T(r, f_1) \sim T(r, f_2)$, $T(r, g_1) \sim T(r, g_2)$ $(r \to \infty)$, whether or not the relation

$$T(r, f_1(g_1)) \sim T(r, f_2(g_2)) \quad (r \to \infty), \tag{1}$$

holds?

If (1) does not hold, what conditions can assure that (1) holds?

Obviously, if f_1 is a polynomial, then (1) holds. However, we point out that (1) does not hold the general case.

Example 1. Let $f_1(z) = e^z$, $f_2(z) = 2e^z$ and $g_1(z) = z^n$, $g_2(z) = 2z^n$. Then we have

Received March 25, 2002.
1991 *Mathematics Subject Classification.* 30D35.
Key words and phrases. Entire function, composition, growth.

$$f_1(g_1) = e^{z_n}, \quad f_2(g_2) = 2e^{2z^n}$$
$$m(r, f_1) = \frac{r}{\pi}, \quad m(r, f_2) = \frac{r}{\pi} + \log 2,$$
$$m(r, g_1) = n \log r, \quad m(r, g_2) = n \log r + \log 2.$$

Thus

$$T(r, g_1) \sim T(r, g_2), \quad T(r, f_1) \sim T(r, f_2) \quad (r \to \infty).$$

But

$$m(r, f_1(g_1)) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |e^{r^n \varepsilon^{in\theta}}| d\theta = \frac{r^n}{\pi},$$

and

$$m(r, f_2(g_2)) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |2e^{2r^n \varepsilon^{in\theta}}| d\theta = \frac{2r^n}{\pi} + \log 2.$$

Thus

$$\lim_{r\to\infty} \frac{T(r, f_1(g_1))}{T(r, f_2(g_2))} = \lim_{r\to\infty} \frac{m(r, f_1(g_1))}{m(r, f_2(g_2))} = 2.$$

This shows that $T(r, f_1(g_1))$ is not equivalent to $T(r, f_2(g_2))$ when $r \to \infty$.

We now give sufficient conditions for (1) to hold.

Theorem 1. *Let f_1, f_2 and g_1, g_2 be four transcendental entire functions with $T(r, f_1) = O^*((\log r)^\nu e^{(\log r)^\alpha})$ and $T(r, g_1) = O^*((\log r^\beta)$ (i.e., there exist four positive constants K_1, K_2, K_3 and K_4 such that $K_1 \le \frac{T(r, f_1)}{(\log r)^\nu \varepsilon^{(\log r)^\alpha}} \le K_2$ and $K_3 \le \frac{T(r, g_1)}{(\log r)^\beta} \le K_4$). If $T(r, f_1) \sim T(r, f_2)$ and $T(r, g_1) \sim T(r, g_2) \quad (r \to \infty)$, then*

$$T(r, f_1(g_1)) \sim T(r, f_2(g_2)) \quad (r \to \infty,\ r \notin E),$$

where $\nu > 0$, $0 < \alpha < 1$, $\alpha\beta < 1$, and E is a set of finite logarithmic measure.

2. Some lemmas

Lemma 1 ([4]). *Let $f(z)$ be an entire function. For $0 \le r < R < \infty$, we have*

$$T(r, f) \le \log^+ M(r, f) \le \frac{R + r}{R - r} T(R, f).$$

Lemma 2 ([5]). *Let $f(z)$ and $g(z)$ be two entire functions and $g(0) = 0$. Then for all $r > 0$ we have*

$$T(r, f(g)) \le T(M(r, g), f).$$

Lemma 3 ([3]). *Let f and g be two entire functions and $g(0)=0$. Then*

$$M(r,f(g)) \geq M((1-o(1)M(r,g),f) \quad (r\to\infty,\ r\notin E),$$

where E is a set of finite logarithmic measure of r.

Lemma 4 ([1]). *Let f be an entire function of order zero and $z=re^{i\theta}$. Then, for any $\zeta>0$ and $\eta>0$, there exist $R_0=R_0(\zeta,\eta)$ and $k=k(\zeta,\eta)$ such that for all $R>R_0$ it holds*

$$\log|f(re^{i\theta})| - N(2R) - \log|c| > -kQ(2R), \quad \zeta R\leq r\leq R,$$

except in a set of circles enclosing the zeros of f, the sum of whose radii is at most ηR. Here

$$Q(r) = r\int_r^\infty \frac{n(t,1/f)}{t^2}dt \quad \text{and} \quad N(r)=\int_0^r \frac{n(t,1/f)}{t}dt.$$

Lemma 5. *Let f be a transcendental entire function with $T(r,f)=O^*((\log r)^\beta e^{(\log r)^\alpha})$ ($0<\alpha<1$, $\beta>0$) (i.e., there exist two positive constants K_1 and K_2 such that $K_1 \leq \dfrac{T(r,f_1)}{(\log r)^\beta e^{(\log r)^\alpha}} \leq K_2$. Then*

1. $T(r,f) \sim \log M(r,f) \quad (r\to\infty,\ r\notin E)$,
2. $T(\sigma r,f)\sim T(r,f) \quad (r\to\infty,\ \sigma\geq 2,\ r\notin E)$,

where E is a set of finite logarithmic measure.

Proof. We may assume $f(0)=1$ (otherwise, we only need to make the transformation $F(z)=f(z)-f(0)+1$). By Jeesen's theorem,

$$N(r,1/f) = \int_0^r \frac{n(t,1/f)}{t}dt = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta \leq \log M(r,f) \tag{2}$$

for $r>1$ and $A>1$. By (2) we have

$$n(r,1/f)\log A \leq \int_r^{Ar}\frac{n(t,1/f)}{t}dt \leq N(Ar,1/f)\leq \log M(Ar,f).$$

So

$$n(r,1/f) \leq \frac{\log M(Ar,f)}{\log A}\,. \tag{3}$$

Since $T(r,f)=O^*((\log r)^\beta e^{(\log r)^\alpha})$ $(0<\alpha<1,\ \beta>1)$, by Lemma 1 we get

$$\log M(r,f) = O^*((\log r)^\beta e^{(\log r)^\alpha}). \tag{4}$$

Take $A = r^{\sigma(r)}$ and $\sigma(r) = \dfrac{1}{(\log r)^\alpha}$. By (3) we have

$$n(r, 1/f) \le \frac{\log M(r^{1+\sigma(r)}, f)}{\sigma(r)\log r}\,. \tag{5}$$

Therefore, putting $r = e^u$ we obtain

$$\begin{aligned}\frac{(\log r^{1+\sigma(r)})^\beta e^{(\log r^{1+\sigma(r)})^\alpha}}{r^{1/2}\sigma(r)\log r} &= \frac{\Big(1+\dfrac{1}{(\log r)^\alpha}\Big)^\beta (\log r)^\beta e^{(1+\frac{1}{(\log r)^\alpha})^\alpha(\log r)^\alpha}}{r^{1/2}(\log r)^{1-\alpha}}\\ &= \frac{(1+1/u^\alpha)^\beta u^\beta e^{(1+1/u^\alpha)^\alpha u^\alpha}}{(e^u)^{1/2}u^{1-\alpha}}\\ &= \frac{(1+1/u^\alpha)^\beta}{e^{u^\alpha(\frac{1}{2}u^{1-\alpha}-(1+1/u^\alpha)^\alpha-(\alpha+\beta-1)u^{-\alpha}\log u)}}\,. \end{aligned} \tag{6}$$

Since $0 < \alpha < 1$ and $\beta > 1$, for sufficiently large values of u we have

$$\frac{1}{2}u^{1-\alpha} - (1+1/u^\alpha)^\alpha - (\alpha+\beta-1)u^{-\alpha}\log u > 0$$

and $\frac{1}{2}u^{1-\alpha} - (1+1/u^\alpha)^\alpha - (\alpha+\beta-1)u^{-\alpha}\log u$ increases. By (6), for sufficiently large values of r, $\dfrac{(\log r^{1+\sigma(r)})^\beta e^{(\log r^{1+\sigma(r)})^\alpha}}{r^{1/2}\sigma(r)\log r}$ decreases.

By (1) and (5) we have

$$\begin{aligned} Q(r) &= r\int_r^{+\infty}\frac{n(t,1/f)}{t^2}dt \le r\int_r^{+\infty}\frac{\log M(t^{1+\sigma(t)},f)}{t^2\sigma(t)\log t}dt\\ &= r\int_r^{+\infty}\frac{O^*((\log t^{1+\sigma(t)})^\beta e^{(\log t^{1+\sigma(t)})^\alpha})}{t^2\sigma(t)\log t}dt\\ &= O^*\Big(r\int_r^{+\infty}\frac{(\log t^{1+\sigma(t)})^\beta e^{(\log t^{1+\sigma(t)})^\alpha}}{t^2\sigma(t)\log t}dt\Big)\\ &\le \frac{r^{1/2}O^*((\log r^{1+\sigma(r)})^\beta e^{(\log r^{1+\sigma(r)})^\alpha})}{\sigma(r)\log r}\int_r^{+\infty}t^{-3/2}dt\\ &= \frac{2\log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r}\,. \end{aligned} \tag{7}$$

Note that

$$
\begin{aligned}
\frac{(\log r^{1+\sigma(r)})^{\beta} e^{(\log r^{1+\sigma(r)})^{\alpha}}}{(\log r)^{\beta} e^{(\log r)^{\alpha}}} &= (1+\sigma(r))^{\beta} e^{(\log r)^{\alpha}[(1+\sigma(r))^{\alpha}-1]} \\
&= (1+(\sigma(r))^{\beta} e^{(\log r)^{\alpha}\alpha\sigma(r)(1+o(1))} \\
&= \Big(1+\frac{1}{(\log r)^{\alpha}}\Big)^{\beta} e^{(\log r)^{\alpha}\alpha\frac{1}{(\log r)^{\alpha}}(1+o(1))} \\
&\to e^{\alpha}(\geq 1) \quad (r\to\infty). \qquad (8)
\end{aligned}
$$

From (7) and (8) it follows that

$$
\begin{aligned}
\frac{Q(r)}{\log M(r,f)} &\leq \frac{2\log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r\log M(r,f)} \\
&\leq \frac{2K_2(\log r^{1+\sigma(r)})^{\beta} e^{(\log r^{1+\sigma(r)})^{\alpha}}}{K_1\sigma(r)\log r(\log r)^{\beta} e^{(\log r)^{\alpha}}} \\
&= \frac{2K_2}{K_1}\cdot\frac{1}{(\log r)^{1-\alpha}}\cdot\frac{(\log r^{1+\sigma(r)})^{\beta} e^{(\log r^{1+\sigma(r)})^{\alpha}}}{(\log r)^{\beta} e^{(\log r)^{\alpha}}} \\
&\to 0 \quad (r\to\infty).
\end{aligned}
$$

So

$$
Q(r) = o(\log M(r,f)). \qquad (9)
$$

Since $T(r,f) = O^{*}(\log r)^{\beta} e^{(\log r)^{\alpha}})$, the order ρ of f is equal to zero, $n(r,1/f) = o(r)$ and

$$
\begin{aligned}
\log M(r,f) &\leq \log\prod_{n=1}^{+\infty}(1+r/r_n) = \int_0^{+\infty}\log(1+r/\ell)dn(\ell,1/f) \\
&\leq \int_0^{+\infty}\frac{r}{t}dn(t,1/f) = r\int_0^{+\infty}\frac{n(t,1/f)}{t(t+r)}dt \\
&= r\Big(\int_0^{r}+\int_r^{+\infty}\Big)\frac{n(t,1/f)}{t(t+r)}dt \\
&\leq r\frac{1}{r}\int_0^{r}\frac{n(t,1/f)}{t}dt + r\int_r^{+\infty}\frac{n(t,1/f)}{t^2}dt \\
&= N(r)+Q(r). \qquad (10)
\end{aligned}
$$

So, from Lemma 4 and (9), (10) we obtain

$$\begin{aligned}\log|f(re^{i\theta})| &> N(2R) - kQ(2R) \quad (\zeta R \le r \le R,\ r \notin E)\\ &= N(2R) + Q(2R) - (k+1)Q(2R)\\ &\ge \log M(2R,f) - (k+1)\circ(\log M(2R,f))\\ &= \log M(2R,f)(1-o(1)) \qquad (11)\\ &\ge \log M(r,f)(1-o(1)), \qquad (12)\end{aligned}$$

where E is a set of finite logarithmic measure.

On the other hand,

$$\log|f(z)| \le \log M(r,f) \le \log M(\sigma r,f) \quad (|z| = r, \sigma \ge 2). \tag{13}$$

In (11), let $2R = \sigma r$, $\sigma \ge 2$. Then from (11), (12) and (13) we get

$$\log|f(z)| \sim \log M(\sigma r,f) \quad (r \to \infty, r \notin E), \tag{14}$$

$$\log|f(z)| \sim \log M(r,t) \quad (r \to \infty), r \notin E). \tag{15}$$

By (15), for sufficiently large values of r, we have

$$\begin{aligned}m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \log M(r,f)(1+o(1))d\theta\\ = \log M(r,f)(1+o(1)) &\quad (r \to \infty, r \notin E).\end{aligned}$$

So

$$\lim_{r\to\infty} \frac{T(r,f)}{\log M(r,f)} = 1 \quad (r \notin E). \tag{16}$$

By (14) and (15), we get

$$\log M(\sigma r,f) \sim \log M(r,f) \quad (r \to \infty, r \notin E, \sigma \ge 2). \tag{17}$$

Hence, from (16) and (17) we obtain

$$T(\sigma r,f) \sim T(r,f) \quad (r \to \infty, r \notin E, \sigma \ge 2). \tag{18}$$

From (16) and (18) we get the desired conclusion. □

3. Proof of Theorem 1

By Lemma 2 we have

$$T(r, f_1(g_1)) \le T(M(r,g_1), f_1) = O^*((\log M(r,g_1))^{\nu} e^{(\log M(r,g_1))^{\alpha}}). \tag{19}$$

Since $T(r,g_1) = O^*((\log r)^{\beta})$, by Lemma 1 we obtain

$$\log M(r,g_1) = O^*((\log r)^{\beta}). \tag{20}$$

So

$$\begin{aligned}T(r, f_1(g_1)) &\le O^*((\log M(r, g_1))^{\nu} e^{(\log M(r,g_1))^{\alpha}})\\ &= O^*(O^*((\log r)^{\beta\nu} e^{(O^*((\log r)^{\beta}))^{\alpha}}))\\ &= O^*(O^*((\log r)^{\beta\nu} e^{O^*((\log r)^{\alpha\beta})})).\end{aligned}$$

Since $O^*((\log r)^{\alpha\beta}) \le K(\log r)^{\alpha\beta}$ $(K > 0)$, there exist $r_0 > 1$ and $\mu > 0$ $(\alpha\beta < \mu < 1)$ such that for $r > r_0$ we have $K(\log r)^{\alpha\beta} < (\log r)^{\mu}$. So

$$O^*((\log r)^{\alpha\beta}) \le (\log r)^{\mu} \quad (\alpha\beta < \mu < 1).$$

Similarly, we have

$$O^*((\log r)^{\beta\nu}) \le (\log r)^{\sigma} \quad (\beta\nu < \sigma).$$

Thus

$$T(r, f_1(g_1)) < O^*((\log r)^{\sigma} e^{(\log r)^{\mu}}).$$

This implies that

$$T(r, f_1(g_1)) = O^{**}((\log r)^{\sigma} e^{(\log r)^{\mu}}) \quad (0 < \beta\nu < \sigma,\ 0 < \alpha\beta < \mu < 1).$$

(i.e., there exist two positive constants K', K'' such that $K' \le \dfrac{T(r, f_1)}{(\log r)^{\sigma} e^{(\log r)^{\mu}}} \le K''$).

Hence, by Lemma 5 we have

$$T(r, f_1(g_1)) \sim \log M(r, f_1(g_1)) \quad (r \to \infty, r \notin E), \tag{21}$$

where E is a set of finite logarithmic measure, and

$$\lim_{r\to\infty} T\Big(\frac{1}{8}M(r, g_1), f_1\Big)/T(M(r, g_1), f_1) = 1 \quad (r \notin E). \tag{22}$$

On the other hand, we may assume that $g_1(0) = b$, $G(z) = g_1(z) - b$ and $F(z) = f_1(z + b)$. Then

$$\begin{aligned}G(0) &= g_1(0) - b = 0,\\ F(G(z)) &= f_1(G(z) + b) = f_1(g_1(z)).\end{aligned}$$

By (21), (22), Lemma 3 and Lemma 5, for sufficiently large values of r, we have

$$\begin{aligned} T(r,f_1(g_1)) &= T(r,F(G)) = \log M(r,F(G))(1+o(1)) \\ &\geq \log M(1-o(1))M(r,G),F)(1+o(1)) \\ &\geq \log M\Big(\frac{1}{4}M(r,G),F\Big)(1+o(1)) \\ &= \log M\Big(\frac{1}{4}M(r,g_1-b),F\Big)(1+o(1)) \\ &\geq \log M\Big(\frac{1}{8}M(r,g_1),f_1\Big)(1+o(1)) \\ &= T\Big(\frac{1}{8}M(r,g_1),f_1\Big)(1+o(1)) \\ &= T(M(r,g_1),f_1)(1+o(1)) \quad (r \notin E). \end{aligned} \tag{23}$$

Thus, from (19) and (20) it follows that

$$T(r,f_1(g_1)) \sim T(M(r,g_1),f_1) \quad (r\to\infty, r\notin R). \tag{24}$$

Since $T(r,f_2)\sim T(r,f_1)$, $T(r,g_2)\sim T(r,g_1)$ $(r\to\infty)$, we have

$$T(r,f_2) = O^*((\log r)^{\nu} e^{(\log r)^{\alpha}})(1+o(1)),$$
$$T(r,g_2) = O^*((\log r)^{\beta})(1+o(1)).$$

Similarly,

$$T(r,f_2(g_2)) \sim T(M(r,g_2),f_2) \quad (r\to\infty, r\notin E). \tag{25}$$

Since $T(r,g_2) = O^*((\log r)^{\beta})$, by Lemma 5 we obtain

$$\log M(r,g_2) = O^*((\log r)^{\beta}). \tag{26}$$

Then there exist two constants K_5 and K_6 $(K_6 > K_5 > 0)$, $K_6 > 1$, such that

$$K_5 \leq \log M(r,g_2)/(\log r)^{\beta} \leq K_6.$$

Then

$$e^{K_5(\log r)^{\beta}} \leq M(r,g_2) \leq e^{K_6(\log r)^{\beta}}. \tag{27}$$

Since $T(r,g_2)\sim T(r,g_1)$ $(r\to\infty)$ and $T(r,g_1) = O^*((\log r)^{\beta})$, by Lemma 5 we have

$$\log M(r,g_1) \sin T(r,g_1) \sim T(r,g_2) \sim \log M(r,g_2) \quad (r\to\infty).$$

Therefore, for sufficiently small $\varepsilon > 0$, there exist $r_1 > r_0 > 0$ such that for $r > r_1$ it holds

$$1-\varepsilon < \frac{\log M(r,g_1)}{\log M(r,g_2)} < 1+\varepsilon.$$

Take $\varepsilon = 1/(\log r)^{\beta}$. By (27),

$$M(r,g_1) < (M(r,g_2))^{1+\varepsilon} \leq M(r,g_2)e^{K_6\varepsilon(\log r)^{\beta}} = e^{K_6}M(r,g_2),$$

and

$$M(r, g_1) > (M(r, g_2))^{1-\varepsilon} \geq e^{-K_5} M(r, g_2) > 1/2(e^{-K_5} M(r, g_2)).$$

Put $\delta = e^{K_6}$ $(\delta > 2)$ and $\delta' = 1/2(e^{-K_5})$ $(0 < \delta' < 1/2)$. We have

$$\delta' M(r, g_2) < M(r, g_1) < \delta M(r, g_2). \tag{28}$$

By (28) and Lamma 5, we get

$$T(M(r, g_1), f_1) \leq T(\delta M(r, g_2), f_1) = T(M(r, g_2), f_1)(1 + o(1)),$$

and

$$\begin{aligned} T(M(r, g_1), f_1) &\geq T(\delta' M(r, g_2), f_1) \\ &= T\Big(\frac{1}{\delta'}\delta' M(r, g_2), f_1\Big)(1 + o(1)) \\ &= T(M(r, g_2), f_1)(1 + o(1)). \end{aligned}$$

So

$$T(M(r, g_1), f_1) \sim T(M(r, g_2), f_1) \quad (r \to \infty). \tag{29}$$

Similarly we have

$$T(M(r, g_1), f_2) \sim T(M(r, g_2), f_2) \quad (r \to \infty). \tag{30}$$

Since $T(r, f_1) \sim T(r, f_2)$ $(r \to \infty)$, it holds

$$T(M(r, g_1), f_1) \sim T(M(r, g_1), f_2) \quad (r \to \infty).$$

By (29) and (30),

$$T(M(r, g_1), f_1) \sim T(M(r, g_2), f_2) \quad (r \to \infty). \tag{31}$$

Combining (24), (25) and (31) we get

$$T(r, f_1(g_1)) \sim T(r, f_2(g_2)) \quad (r \to \infty, r \notin E).$$

This completes the proof of Theorem 1. □

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