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ON THE GENERALIZED CONVOLUTION FOR I-TRANSFORM

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Abstract. An I-transform and a generalized convolution for this transform are introduced and their properties are considered

1. Introduction

Let us consider the integral transform $K : U(X) \to V(Y)$, where $U(X)$ is a linear space, $V(Y)$ is an algebraic one. The convolution of two functions f, g for transform K defined by the symbol $f * g$, is an operator such that the following factorization property is valid:

$$K(f * g)(y) = (Kf)(y) \cdot (Kg)(y), \quad y \in Y.$$

In 1942, Churchill initiated the convolution of functions f, g for the Fourier transform in the famous form as follows

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-t)g(t)dt$$

(see [18]). Analogously, the convolutions for the Mellin and Laplace transform have been investigated (see [18]):

$$(f * g)(x) = \int_{0}^{+\infty} f\Big(\frac{x}{t}\Big) g(t) \frac{dt}{t}\,,$$

$$(f * g)(x) = \int_{0}^{x} f(x-t)g(t)dt.$$

Further, the convolution with a weight-function γ of f, g for the transform K is an expression such that the following factorization property hold valid:

$$K(f \overset{\gamma}{*} g)(x) = \gamma(x) \cdot (Kf)(x) \cdot (Kg)(x), \quad x \in Y.$$

In 1958, for the first time Vilenkin [21] studied convolution of the above type for the generalized Mehler-Fox transform, where the weight-function is the following

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function:

$$\gamma(x) = \frac{\pi}{x\sinh(\pi x)}\Big|\Gamma\big(p + ix + \frac{1}{2}\big)\Big|^{-2}.$$

It is well known that, convolutions of integral transforms have a broad application in solving mathematical physics equations (see [18]), in evaluating various integrals and series (see [12]). On the other hand, the convolutions are also integral operators. They are studied in [8, 12]. Note that, integral equations of convolution type and their applications are widely investigated [12, 19, 23, 6, 7]

In 1967, Kakichev [9] gave a new definition of convolutions with (and without) weight-function:

A generalized convolution of functions f and g under three operators K, K_1, K_2 and with some weight-function γ is a function, denoted by the symbol $f * g$, such that the following factorization property holds

$$K(f \overset{\gamma}{*} g)(x) = \gamma(x)(K_1 f)(x)(K_2 g)(x).$$

An example of the generalized convolution was first introduced by Churchill

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)[g(|x-y|) - g(x+y)]dy$$

and the respective factorization property has the form

$$F_s(f * g)(x) = (F_s f)(x)(F_c g)(x)$$

where F_s, F_c are the Fourier sine and Fourier cosine transforms

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(y)\sin(xy)dy$$

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(y)\cos(xy)dy$$

(see [18]). Some authors have studied similar generalized convolutions for the transforms of Mellin type [22], the G-transforms [16], the H-transforms [25], transforms of Kontorovich-Lebedev type [24], the Fourier cosine and sine transforms [14], the Stieltjes-Hilbert and the Fourier cosine-sine transforms [13].

An H-transform is defined in [10, 25] as

$$(Hf)(x) = \frac{1}{2\pi i} \int_\sigma X^p_{\overline{m},\overline{a},\overline{\alpha}}(s) f^*(s) x^{-s} ds, \quad x > 0,$$

where

$$X^p_{\overline{m},\overline{a},\overline{\alpha}}(s) = \prod_{j=1}^{p} \Gamma^{m_j}(A_j + \alpha_j s), \quad p \in \mathbb{N},$$

$$A_j = \frac{1}{2} - \Big(a_j - \frac{1}{2}\Big)\mathrm{sign}\alpha_j,$$
$$\overline{m} = (m_1, m_2, \dots, m_p), \ \ m_j \in \mathbb{Z},$$
$$\overline{a} = (a_1, a_2, \dots, a_p), \ \ a_j \in \mathbb{C},$$
$$\overline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_p), \ \ \alpha_j \in \mathbb{R},$$
$$\alpha_{i+1} > 2(\mathrm{Re}\, a_i - 1)\mathrm{sign}\, \alpha_i,$$

and f^* is the Mellin transform [18] of function $f(x)$, $\sigma = \big\{s, \mathrm{Re}\, s = \frac{1}{2}\big\}$.

The aim of the present work is to investigate an I-transform, which generalizes the H-transforms [10, 25] and the G-transforms [16] and of the properties of a generalized convolution for this transform.

2. I-transform

Definition 1. The I-transform of a function f is defined as follows

$$F(x) \equiv (If)(x) = I^{p_i,r}_{\overline{m}_i,\overline{a}_i,\overline{\alpha}_i}(f)(x)$$
$$= \frac{1}{2\pi i}\int\limits_{\sigma} \Big(\sum_{i=1}^{r} X^{p_i}_{\overline{m}_i,\overline{a}_i,\overline{\alpha}_i}(s)\Big)^{-1} f^*(s)x^{-s}ds, \quad x > 0, \tag{1}$$

where (see [10])

$$X^{p_i}_{\overline{m}_i,\overline{a}_i,\overline{\alpha}_i}(s) = \prod_{j=1}^{p_i} \Gamma^{m_{ij}}(b_{ij} + \alpha_{ij}s), \quad m_{ij} \in \mathbb{Z},\ p_i \in \mathbb{N},$$
$$b_{ij} = \frac{1}{2} - \big(a_{ij} - \frac{1}{2}\big)\mathrm{sign}\, \alpha_{ij}, \quad a_{ij} \in \mathbb{C},\ \alpha_{ij} \in \mathbb{R}, \tag{2}$$
$$\overline{m}_i = (m_{i1}, m_{i2}, \dots, m_{ip_i}),\ \overline{a}_i = (a_{i1}, a_{i2}, \dots, a_{ip_i}),$$
$$\overline{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ip_i}),$$
$$\alpha_{ij} + 1 > (2\ \mathrm{Re} a_{ij} - 1)\mathrm{sign}\, \alpha_{ij}, \quad j = \overline{1, p_i},\ i = \overline{1, r},$$

$f^*(s)$ is the Mellins transform [15] of function $f(x)$, $\sigma = \big\{s, \mathrm{Re}\, s = \frac{1}{2}\big\}$. The parameters $\overline{a}_i$, $\overline{\alpha}_i$ are chosen so that

$$\sum_{i=1}^{r} X^{p_i}_{\overline{m}_i,\overline{a}_i,\overline{\alpha}_i}(s) \neq 0$$

on the contour σ.

A special case of the I-transform is the H-transform [10, 25]. Namely,

$$I^{p,1}_{\overline{m},\overline{a},\overline{\alpha}} = H^{p}_{-\overline{m},\overline{a},\overline{\alpha}}.$$

Definition 2 [17]. Let c, $\gamma \in \mathbb{R}$ and

$$2\ \mathrm{sign}\, c + \mathrm{sign}\, \gamma \geq 0. \tag{3}$$

Denote by $\mathfrak{M}^{-1}_{c,\gamma}(L)$ the space of functions given in the form

$$f(x) = \frac{1}{2\pi i}\int\limits_{\sigma} f^*(s)x^{-s}ds,$$

where $f^*(s)|s|^{\gamma}e^{\pi c|s|} \in L(\sigma)$.

Definition 3. As in [10, 25], let

$$c_i = \frac{1}{2}\sum_{j=1}^{p_i} m_{ij}|\alpha_{ij}|, \tag{4}$$

$$\gamma_i = \frac{1}{2}\sum_{j=1}^{p_i} m_{ij}(\operatorname{sign}\alpha_{ij} - \alpha_{ij}) - \operatorname{Re}\Big[\sum_{j=1}^{p_i}(1-a_{ij})\operatorname{sign}(m_{ij}\alpha_{ij})\Big].$$

We define a couple of characteristic numbers (c_0, γ_0) by setting

$$c_0 = \min_{i=\overline{1,r}}\{c_i\}, \quad \gamma_0 = \min_{i=\overline{1,r}}\{\gamma_i\}. \tag{5}$$

Using the method in [25] we have

Theorem 1. *I-transform* (1) *with the couple of characteristic numbers* (c_0, γ_0) *exists in the space* $\mathfrak{M}^{-1}_{c,\gamma}(L)$ *if and only if*

$$2\ \operatorname{sign}(c - c_0) + \operatorname{sign}(\gamma - \gamma_0) \geq 0. \tag{6}$$

If (6) *is fulfilled, then I-transform* (1) *maps homeomorphically the space* $\mathfrak{M}^{-1}_{c,\gamma}(L)$ *onto the space* $\mathfrak{M}^{-1}_{c-c_0,\gamma-\gamma_0}(L)$. *Under condition* (2), *its inverse transform has the form*

$$f(x) = (I^{-1}F)(x) = \sum_{i=1}^{r}\big(H^{p_i}_{\overline{m}_i,\overline{a}_i,\overline{\alpha}_i}F\big)(x), \tag{7}$$

where (see [10, 25]*)*

$$(H^{p_i}_{\overline{m}_i,\overline{a}_i,\overline{\alpha}_i}F)(x) = \frac{1}{2\pi i}\int\limits_{\sigma} X^{p_i}_{\overline{m}_i,\overline{a}_i,\overline{\alpha}_i}(s)F^*(s)x^{-s}ds, \quad x > 0.$$

Remark. Hereafter the Mellin-Parseval form of the I-transform and its inverse transform have the form

$$F(x) = (If)(x) = \int\limits_{0}^{+\infty} I\Big(\frac{x}{t}\big|r, p_i, \overline{m}_i, \overline{a}_i, \overline{\alpha}_i\Big)f(t)\frac{dt}{t},$$

provided

$$4\ \operatorname{sign} c_0 + 2\ \operatorname{sign}\gamma_0 + \operatorname{sign}|\delta_0| < 0, \quad f(x) \in \mathfrak{M}^{-1}_{c,\gamma}(L),$$

and

$$f(x)=(I^{-1}F)(x)=\sum_{i=1}^{r}\int\limits_{0}^{+\infty}H\Big(\frac{x}{t}|p_i,\overline{m}_i,\overline{a}_i,\overline{\alpha}_i\Big)F(t)\frac{dt}{t}\,, \tag{8}$$

if

$$-4\ \mathrm{sign}\,c_0-2\ \mathrm{sign}\,\gamma_0+\mathrm{sign}\,|\delta_0|>0,\quad F(x)\in\mathfrak{M}^{-1}_{c-c_0,\gamma-\gamma_0}(L).$$

Here $H(x|p_i,\overline{m}_i,\overline{a}_i,\overline{\alpha}_i)$ is Fox's H-function [5, 10], $I(x|r,p_i,\overline{m}_i,\overline{a}_i,\overline{\alpha}_i)$ is the I-function [20],

$$\delta_0=\min_{i=\overline{1,r}}\{\delta_i\},\quad \delta_i=\sum_{j=1}^{p_i}m_{ij}\alpha_{ij}.$$

3. A generalized convolution

Let us consider the I_k-transforms ($k=\overline{1,3}$), where

$$(I_kf)(x)=\frac{1}{2\pi i}\int\limits_{\sigma}\Big(\sum_{i=1}^{r_k}X^{p_{ki}}_{\overline{m}_{ki},\overline{a}_{ki},\overline{\alpha}_{ki}}(s)\Big)^{-1}f^*(s)x^{-s}ds,\quad x>0. \tag{9}$$

Using the standard definition of generalized convolution [11] in the original space $\mathfrak{M}^{-1}_{c,\gamma}(L)$ (resp., the image space $\mathfrak{M}^{-1}_{c-c_0,\gamma-\gamma_0}(L)$) we put

$$\begin{gathered}
f_i\overset{k}{*}g_j=I_k^{-1}\big((I_if_i)(I_jg_j)\big)=I_k^{-1}(F_i.G_j),\\
F_i=I_i(f_i),\quad G_j=I_j(g_j),\\
(F_i\overset{k}{*}G_j=I_k\big((I_i^{-1}F_i)(I_j^{-1}G_j)\big)=I_k(f_i.g_j)),
\end{gathered}$$

where $F_i.G_j$ ($f_i.g_j$) is a product of functions in $\mathfrak{M}^{-1}_{c-c_0,\gamma-\gamma_0}(L)$ (resp., in $\mathfrak{M}^{-1}_{c,\gamma}(L)$).

Executing commutation of integral order in $I_k^{-1}(I_if_i.I_jg_j)$, we have

$$\big(f_i\overset{k}{*}g_j\big)(x_k)=\int\limits_0^{+\infty}\int\limits_0^{+\infty}\frac{U_{1k}(x_k,x_i,x_j)}{x_ix_j}f_i(x_i)g_j(x_j)dx_idx_j \tag{10}$$

$$i,j,k=\overline{1,3},\ i\neq j,\quad k\neq j,\quad k\neq i,$$

where

$$\begin{aligned}U_{1k}(x_k,x_i,x_j)&=\sum_{\ell=1}^{r_k}\int_0^{+\infty}t^{-1}I_i\Big(\frac{t}{x_i}\Big)I_j\Big(\frac{t}{x_j}\Big)H\Big(\frac{x_k}{t}\big|p_{k\ell},\overline{m}_{k\ell},\overline{a}_{k\ell},\overline{\alpha}_{k\ell}\Big)dt\\&=\sum_{\ell=1}^{r_k}\int_0^{+\infty}t^{-1}I_i\Big(\frac{t}{x_i}\Big)I_j\Big(\frac{t}{x_j}\Big)H\Big(\frac{t}{x_k}\big|p_{k\ell},m_{k\ell},\overline{1-a_{k\ell}},\overline{\alpha}_{k\ell}\Big)dt\\&=\sum_{\ell=1}^{r_k}\frac{1}{(2\pi i)^2}\int_{\sigma_s}\int_{\sigma_u}\Theta_i(s)\Theta_j(u)\Big(\frac{x_k}{x_i}\Big)^s\Big(\frac{x_k}{x_j}\Big)^u\times\\&\quad\times\Big\{\int_0^{+\infty}t^{s+u-1}H\Big(t\big|p_{k\ell},\overline{m}_{k\ell},\overline{1-a_{k\ell}},\overline{\alpha}_{k\ell}\Big)dt\Big\}dsdu\\&=\sum_{\ell=1}^{r_k}\frac{1}{(2\pi i)^2}\int_{\sigma_s}\int_{\sigma_u}\Theta_i(s)\Theta_j(u)\Big(\frac{x_k}{x_i}\Big)^s\Big(\frac{x_k}{x_j}\Big)^u\times\\&\quad\times X^{p_{k\ell}}_{\overline{m}_{k\ell},\overline{1-a_{k\ell}},\overline{\alpha}_{k\ell}}(s+u)dsdu\\&=\sum_{\ell=1}^{r_k}I\left(\begin{matrix}x_k/x_i\\ \\ x_k/x_j\end{matrix}\;\middle|\;\begin{matrix}\Theta_i(s)\\ \Theta_j(u)\\ p_\ell,\overline{m}_{k\ell},\overline{a}_{k\ell},\overline{\alpha}_{k\ell}(u+s)\end{matrix}\right).\end{aligned}$$

Here

$$\Theta_i(u)=\Big(\sum_{h=1}^{r_i}X^{p_{ih}}_{\overline{m}_{ih},\overline{a}_{ih},\overline{\alpha}_{ih}}(u)\Big)^{-1},$$

and

$$\begin{aligned}&I\left(\begin{matrix}x_k/x_i\\ \\ x_k/x_j\end{matrix}\;\middle|\;\begin{matrix}\Theta_i(s)\\ \Theta_j(u)\\ p_\ell,\overline{m}_{k\ell},\overline{a}_{k\ell},\overline{\alpha}_{k\ell}(u+s)\end{matrix}\right)\\&=\frac{1}{(2\pi i)^2}\int_{\sigma_s}\int_{\sigma_u}\Theta_i(s)\Theta_j(u)X^{p_{k\ell}}_{\overline{m}_{k\ell},\overline{a}_{k\ell},\overline{\alpha}_{k\ell}}(u+s)\Big(\frac{x_k}{x_i}\Big)^s\Big(\frac{x_k}{x_j}\Big)^u dsdu.\end{aligned}$$

Analogously, the generalized convolution in the image space has the form

$$(F_i\overset{k}{*}G_j)(y_k)=\int_0^{+\infty}\int_0^{+\infty}\frac{U_{2k}(y_k,y_i,y_j)}{y_iy_j}F_i(y_i)G_j(y_j)dy_idy_j, \tag{11}$$

where $i, j, k = \overline{1,3}$, $i \neq j$, $k \neq j$, $i \neq k$,

$$U_{2k}(y_k, y_i, y_j) = \sum_{\substack{\xi = \overline{1, r_i} \\ \eta = \overline{1, r_j}}} I \left(\begin{matrix} y_k/y_i \\ \\ y_k/y_j \end{matrix} \middle| \begin{matrix} p_{i\xi}, \overline{m}_{i\xi}, \overline{a}_{i\xi}, \overline{\alpha}_{i\xi}(s) \\ p_{j\eta}, \overline{m}_{j\eta}, \overline{a}_{j\eta}, \overline{\alpha}_{j\eta}(u) \\ \Theta_k(u+s) \end{matrix} \right).$$

Using the method in [25], we can obtain Theorem 2 and Theorem 3 below.

Theorem 2. *The generalized convolution exists in the space* $\mathfrak{M}^{-1}_{c_k,\gamma_k}(L)$ *if and only if*

$$\begin{aligned}
&2\ \mathrm{sign}\,(c_k + c_{0k} - c_{0i}) + \mathrm{sign}\,(\gamma_k + \gamma_{0k} - \gamma_{0i} - \delta_{0k}) \geq 0, \\
&2\ \mathrm{sign}\,(c_k + c_{0k} - c_{0j}) + \mathrm{sign}\,(\gamma_k + \gamma_{0k} - \gamma_{0j} - \delta_{0k}) \geq 0, \\
&2\ \mathrm{sign}\,(2c_k - c_{0i} - c_{0j}) + \mathrm{sign}\,(2\gamma_k + 1 - \gamma_{0i} - \gamma_{0j}) \geq 0, \\
&\mathrm{sign}\,(c_k + c_{0k} - c_{0i}) + \mathrm{sign}\,(c_k + c_{0k} - c_{0j}) + \\
&+ \mathrm{sign}\,(2c_k - c_{0i} - c_{0j}) + 2\ \mathrm{sign}\,(\gamma_k + \gamma_{0k} - \gamma_{0i} - \gamma_{0j} - \delta_{0k}) \geq 0.
\end{aligned}$$

Under these conditions $(f_i \overset{k}{*} g_j) \in \mathfrak{M}^{-1}_{c'_k,\gamma'_k}(L)$, *where*

$$(c'_k, \gamma'_k) = \begin{cases} \left(\min \left\{ \begin{matrix} c_k - c_{0i} + c_{0k} \\ c_k - c_{0j} + c_{0k} \end{matrix} \right\}, \begin{matrix} \gamma_k - \gamma_{0i} + \gamma_{0k} - \delta_{0k} \\ \gamma_k - \gamma_{0j} + \gamma_{0k} - \delta_{0k} \end{matrix} \right), & \text{if } c_{0i} \neq c_{0j}, \\ (c_k - c_{0i} + c_{0k}, \min(\gamma_k - \gamma_{0i} + \gamma_{0k} - \delta_{0k}, \\ 2\gamma_k - \gamma_{0j} - \gamma_{0i} - \gamma_{0k} - \delta_{0k})) & \text{if } c_{0i} = c_{0j}, \end{cases}$$

and the following factorization property holds

$$I_k\big(f_i \overset{k}{*} g_j\big) = (I_i f_i)(I_j g_j),$$

$$i, j, k = \overline{1,3}, \quad i \neq j, \quad k \neq j, \quad i \neq k.$$

Besides, let the couple of characteristic numbers (c''_k, γ''_k) *be such that* $(f_i \overset{k}{*} g_j) \in \mathfrak{M}^{-1}_{c''_k,\gamma''_k}(L)$ *for functions* f_i, $g_j \in \mathfrak{M}^{-1}_{c_k,\gamma_k}(L)$, *then we have*

$$\mathfrak{M}^{-1}_{c''_k,\gamma''_k}(L) \supset \mathfrak{M}^{-1}_{c'_k,\gamma'_k}(L),$$

where (c_{0k}, γ_{0k}) *is the couple of characteristic numbers for the* I_k*-transforms.*

Corollary 1. *If* $r_i = 1$, $r_j = 1$, *then the kernel of the generalized convolution* (10) *is*

$$U_{1k}(x_k, x_i, x_j) = \sum_{t=1}^{r_k} H \left(\begin{matrix} x_k/x_i \\ \\ x_k/x_j \end{matrix} \middle| \begin{matrix} p_i, -\overline{m}_i, \overline{a}_i, \overline{\alpha}_i \\ p_j, -\overline{m}_j, \overline{a}_j, \overline{\alpha}_j \\ p_{kt}, \overline{m}_{kt}, \overline{a}_{kt}, \overline{\alpha}_{kt} \end{matrix} \right),$$

$$U_{1\ell}(x_\ell, x_k, x_\eta) = I\left(\begin{matrix} x_\ell/x_k \\ \\ x_\ell/x_\eta \end{matrix}\ \middle|\ \begin{matrix} \Theta_k(s) \\ p_\eta, -\overline{m}_\eta, \overline{a}_\eta, \overline{\alpha}_\eta \\ p_\ell, \overline{m}_\ell, \overline{a}_\ell, \overline{\alpha}_\ell \end{matrix}\right),$$

where $\ell, \eta = i, j, \quad \ell \neq \eta$ *and the function on the right hand side of the first formula is an* H*-function of two variables* [3].

Besides,

$$\begin{aligned} I_k\big(f_i \overset{k}{*} g_j\big) &= \big(H^{p_i}_{-\overline{m}_i,\overline{a}_i,\overline{\alpha}_i} f_i\big)\big(H^{p_j}_{-\overline{m}_j,\overline{a}_j,\overline{\alpha}_j} g_j\big), \\ H^{p_i}_{-\overline{m}_i,\overline{a}_i,\overline{\alpha}_i}\big(f_j \overset{i}{*} g_k\big) &= \big(H^{p_j}_{-\overline{m}_j,\overline{a}_j,\overline{\alpha}_j} f_j\big)(I_k g_k), \\ H^{p_j}_{-\overline{m}_j,\overline{a}_j,\overline{\alpha}_j}\big(f_k \overset{j}{*} g_i\big) &= (I_k f_k)\big(H^{p_i}_{-\overline{m}_i,\overline{a}_i,\overline{\alpha}_i} g_i\big). \end{aligned}$$

Theorem 3. *Generalized convolution* (11) *exists in the space* $\mathfrak{M}^{-1}_{c_k,\gamma_k}(L)$ *if and only if the following conditions are satisfied*

$$\begin{aligned} &2\ \mathrm{sign}(c_k - c_{0k} + c_{0i}) + \mathrm{sign}(\gamma_k - \gamma_{0k} + \gamma_{0i} + \delta_{0k}) \geq 0, \\ &2\ \mathrm{sign}(c_k - c_{0k} + c_{0j}) + \mathrm{sign}(\gamma_k - \gamma_{0k} + \gamma_{0j} + \delta_{0k}) \geq 0, \\ &2\ \mathrm{sign}(2c_k + c_{0i} + c_{0j}) + \mathrm{sign}(2\gamma_k + 1 + \gamma_{0i} + \gamma_{0j}) \geq 0, \\ &\mathrm{sign}(c_k - c_{0k} + c_{0i}) + \mathrm{sign}(c_k - c_{0k} + c_{0j}) + \\ &+ \mathrm{sign}(2c_k + c_{0i} + c_{0j}) + 2\mathrm{sign}(\gamma_k - \gamma_{0k} + \gamma_{0i} + \gamma_{0j} + \delta_{0k}) \geq 0. \end{aligned}$$

Then the generalized convolutions $(F_i \overset{k}{*} G_j)$ *belong to* $\mathfrak{M}^{-1}_{c'_k,\gamma'_k}(L)$*, where*

$$(c'_k, \gamma'_k) = \begin{cases} \left(\min\begin{Bmatrix} c_k + c_{0i} - c_{0k} \\ c_k + c_{0j} - c_{0k} \end{Bmatrix}, \begin{matrix} \gamma_k + \gamma_{0i} - \gamma_{0k} + \delta_{0k} \\ \gamma_k + \gamma_{0j} - \gamma_{0k} + \delta_{0k} \end{matrix}\right), & \text{if } c_{0i} \neq c_{0j}, \\ (c_k + c_{0i} - c_{0k}, \min(\gamma_k + \gamma_{0i} - \gamma_{0k} + \delta_{0k}, & \\ 2\gamma_k + \gamma_{0i} + \gamma_{0j} - \gamma_{0k} + \delta_{0k})) & \text{if } c_{0i} = c_{0j}, \end{cases}$$

and the following equalities hold

$$I_k^{-1}(F_i \overset{k}{*} G_j) = (I_i^{-1}F_i)(I_j^{-1}G_j), \tag{12}$$

$$i, j, k = \overline{1,3}, \quad i \neq j, \quad i \neq k, \quad j \neq k.$$

Besides, if the couple of characteristic numbers (c''_k, γ''_k) *is such that* $(F_i \overset{k}{*} G_j) \in \mathfrak{M}^{-1}_{c''_k,\gamma''_k}(L)$ *for functions* F_i, $G_j \in \mathfrak{M}^{-1}_{c_k,\gamma_k}(L)$ *then*

$$\mathfrak{M}^{-1}_{c''_k,\gamma''_k}(L) \supset \mathfrak{M}^{-1}_{c'_k,\gamma'_k}(L).$$

Corollary 2. *The equality* (12) *can be written in the following form*

$$\sum_{t=1}^{r_k} H^{p_{kt}}_{\overline{m}_{kt},\overline{a}_{kt},\overline{\alpha}_{kt}}(F_i \overset{k}{*} G_j) = \Big(\sum_{\ell=1}^{r_i} H^{p_{i\ell}}_{\overline{m}_{i\ell},\overline{a}_{i\ell},\overline{\alpha}_{i\ell}} F_i\Big)\Big(\sum_{\xi=1}^{r_j} H^{p_{j\xi}}_{\overline{m}_{j\xi},\overline{a}_{j\xi},\overline{\alpha}_{j\xi}} G_j\Big).$$

Corollary 3. *If $r_k = 1$, then the kernel of the generalized convolution is*

$$U_{2k}(x_k, x_i, x_j) = \sum_{\eta,\xi=\overline{1,r_i}} H\left(\begin{matrix} x_k/x_i \\ \\ x_k/x_j \end{matrix}\ \middle|\ \begin{matrix} p_{i\xi}, \overline{m}_{i\xi}, \overline{a}_{i\xi}, \overline{\alpha}_{i\xi} \\ p_{j\eta}, \overline{m}_{j\eta}, \overline{a}_{j\eta}, \overline{\alpha}_{j\eta} \\ p_k, -\overline{m}_k, \overline{a}_k, \overline{\alpha}_k \end{matrix}\right),$$

$$U_{2\ell}(x_\ell, x_k, x_\eta) = \sum_{t=1}^{r_\eta} I\left(\begin{matrix} x_\ell/x_k \\ \\ x_\ell/x_\eta \end{matrix}\ \middle|\ \begin{matrix} p_k, \overline{m}_k, \overline{a}_k, \overline{\alpha}_k \\ p_{\eta t}, \overline{m}_{\eta t}, \overline{a}_{\eta t}, \overline{\alpha}_{\eta t} \\ \Theta_\ell \end{matrix}\right),$$

where $\ell = i, j,\ \ \eta = i, j,\ \ \eta \neq \ell$, *and the following factorization properties hold*

$$H^{p_k}_{\overline{m}_k,\overline{a}_k,\overline{\alpha}_k}(F_i \overset{k}{*} G_j) = \Big(\sum_{t=1}^{r_i} H^{p_{it}}_{\overline{m}_{it},\overline{a}_{it},\overline{\alpha}_{it}} F_i\Big)\Big(\sum_{\eta=1}^{r_j} H^{p_{j\eta}}_{\overline{m}_{j\eta},\overline{a}_{j\eta},\overline{\alpha}_{j\eta}} G_j\Big),$$

$$\sum_{t=1}^{r_i} H^{p_{it}}_{\overline{m}_{it},\overline{a}_{it},\overline{\alpha}_{it}}(F_j \overset{i}{*} G_k) = \Big(\sum_{\eta=1}^{r_j} H^{p_{j\eta}}_{\overline{m}_{j\eta},\overline{a}_{j\eta},\overline{\alpha}_{j\eta}} F_j\Big)\big(H^{p_k}_{\overline{m}_k,\overline{a}_k,\overline{\alpha}_k} G_k\big),$$

$$\sum_{\eta=1}^{r_j} H^{p_{j\eta}}_{\overline{m}_{j\eta},\overline{a}_{j\eta},\overline{\alpha}_{j\eta}}(F_k \overset{j}{*} G_i) = \big(H^{p_k}_{\overline{m}_k,\overline{a}_k,\overline{\alpha}_k} F_k\big)\Big(\sum_{t=1}^{r_i} H^{p_{it}}_{\overline{m}_{it},\overline{a}_{it},\overline{\alpha}_{it}} G_i\Big).$$

For illustration we give an example.

Example. Examine the inverse I-transforms [4]

$$I_1^{-1} f = \Big(\frac{1}{\pi}\big\{\cos(2\sqrt{x})\big\} + \frac{x^{1/2}}{\sqrt{\pi}}\big\{\sin(2\sqrt{x})\big\} x^{-1/2}\Big) f,$$

$$I_2^{-1} g = \Big(\frac{1}{\sqrt{\pi}}\big\{\cos\big(\frac{2}{\sqrt{x}}\big)\big\} + \frac{x^{-1/2}}{\pi}\big\{\sin\big(\frac{2}{\sqrt{x}}\big)\big\} x^{1/2}\Big) g,$$

$$I_3^{-1} h = \Big(x^{1/2}\Lambda_+^{-1} x^{-1/2} + \frac{1}{\sqrt{\pi}}\big\{\cos\big(\frac{2}{\sqrt{x}}\big)\big\}\Big) h.$$

From formulae (13), (18), (19) (21), (22) in ([4] p.24-25) and Theorem 1 we have

$$(I_1 f)(x) = \frac{1}{2\pi i}\int_\sigma \frac{\Gamma(1/2 - s)}{\Gamma(s) + \Gamma(1+s)} f^*(s) x^{-s} ds,$$

$$(I_2 g)(x) = \frac{1}{2\pi i}\int_\sigma \frac{\Gamma(1/2 + s)}{\Gamma(-s) + \Gamma(1-s)} g^*(s) x^{-s} ds,$$

$$(I_3 h)(x) = \frac{1}{2\pi i}\int_\sigma \frac{\Gamma(1/2 + s)}{1 + \Gamma(-s)} h^*(s) x^{-s} ds.$$

The generalized convolution for I_i-transforms has the form:

$$(f_i \overset{k}{*} g_j)(x_k) = \int_0^{+\infty}\int_0^{+\infty} \frac{U_{1k}(x_k, x_i, x_j)}{x_i x_j} f_i(x_i) g_j(x_j) dx_i dx_j,$$

where

$$U_{11}(x_1, x_2, x_3) = I\left(\begin{matrix} x_1/x_2 \\ \\ x_1/x_3 \end{matrix} \left| \begin{matrix} \Theta_2(s) \\ \Theta_3(u) \\ 1,1,-1,1;1,-1,\frac{1}{2},-1 \end{matrix}\right.\right) + I\left(\begin{matrix} x_1/x_2 \\ \\ x_1/x_3 \end{matrix} \left| \begin{matrix} \Theta_2(s) \\ \Theta_3(u) \\ 1,1,0,1;1,-1,\frac{1}{2},-1 \end{matrix}\right.\right),$$

$$U_{12}(x_2, x_1, x_3) = I\left(\begin{matrix} x_2/x_1 \\ \\ x_2/x_3 \end{matrix} \left| \begin{matrix} \Theta_1(s) \\ \Theta_3(u) \\ 1,1,0,-1;1,-1,\frac{1}{2},1 \end{matrix}\right.\right) + I\left(\begin{matrix} x_2/x_1 \\ \\ x_2/x_3 \end{matrix} \left| \begin{matrix} \Theta_1(s) \\ \Theta_3(u) \\ 1,1,1,-1;1,-1,\frac{1}{2},1 \end{matrix}\right.\right),$$

$$U_{13}(x_3, x_1, x_2) = I\left(\begin{matrix} x_3/x_1 \\ \\ x_3/x_2 \end{matrix} \left| \begin{matrix} \Theta_1(s) \\ \Theta_2(u) \\ 1,1,1,0;1,-1,\frac{1}{2},1 \end{matrix}\right.\right) + I\left(\begin{matrix} x_3/x_1 \\ \\ x_3/x_2 \end{matrix} \left| \begin{matrix} \Theta_1(s) \\ \Theta_2(u) \\ 1,1,0,-1;1,-1,\frac{1}{2},1 \end{matrix}\right.\right),$$

$$\Theta_1(s) = \frac{\Gamma(1/2 - s)}{\Gamma(s) + \Gamma(1+s)}, \quad \Theta_2(s) = \frac{\Gamma(1/2 + s)}{\Gamma(-s) + \Gamma(1-s)},$$

$$\Theta_3(u+s) = \frac{\Gamma(1/2 + u + s)}{1 + \Gamma(-u-s)}\cdot$$

In addition,

$$I_k(f_i \overset{k}{*} g_j) = (I_i f_i)(I_j g_j),$$

$$i, j, k = \overline{1,3}, \quad i \neq j, \quad k \neq j, \quad i \neq k.$$

From (4) and Theorem 2 we have

$$c_{01} = c_{02} = c_{03} = 0, \quad \gamma_{01} = -\frac{1}{2}, \quad \gamma_{02} = \frac{3}{2}, \quad \gamma_{03} = \frac{1}{2},$$

$$\delta_{01} = 2, \quad \delta_{02} = -2, \quad \delta_{03} = -2.$$

It follows that

a) $(f_2 \overset{1}{*} g_3) \in \mathfrak{M}^{-1}_{c'_1,\gamma'_1}(L)$, $c'_1 = c_1$, $\gamma'_1 = \min(\gamma_1 - 4, 2\gamma_1 - 5/2)$, $c_1 > 0$, for all $\gamma_1 \in R$ and $c_1 = 0$, $\gamma_1 \geq -1$.

b) $(f_1 \overset{2}{*} g_3) \in \mathfrak{M}^{-1}_{c'_2,\gamma'_2}(L)$, $c'_2 = c_2$, $\gamma'_2 = \min(\gamma_2 + 4, 2\gamma_2 + 1/2)$, $c_2 > 0$, for all $\gamma_2 \in R$ and $c_2 = 0$, $\gamma_2 \geq -4$.

c) $(f_1 \overset{3}{*} g_2) \in \mathfrak{M}^{-1}_{c'_3,\gamma'_3}(L)$, $c'_3 = c_3$, $\gamma'_3 = \min(\gamma_3 + 3, 2\gamma_3 + 1/2)$, $c_3 > 0$, for all $\gamma_3 \in R$ and $c_3 = 0$, $\gamma_3 \geq -3$.

Generalized convolutions for I_i-transforms in the image space are

$$(F_i \overset{k}{*} G_j)(y_k) = \int_0^{+\infty}\int_0^{+\infty} \frac{U_{2k}(y_k, y_i, y_j)}{y_i y_j} F_i(y_i) G_j(y_j) dy_i dy_j,$$

where

$$U_{2k}(y_k, y_i, y_j) = \sum_{\substack{\xi = \overline{1, r_i} \\ \eta = \overline{1, r_j}}} I\left(\begin{matrix} y_k/y_i \\ \\ y_k/y_j \end{matrix} \;\middle|\; \begin{matrix} p_{i\xi}, \overline{m}_{i\xi}, \overline{a}_{i\xi}, \overline{\alpha}_{i\xi}(s) \\ p_{j\eta}, \overline{m}_{j\eta}, \overline{a}_{j\eta}, \overline{\alpha}_{j\eta}(u) \\ \Theta_k(u+s) \end{matrix} \right),$$

and the following factorization property is valid

$$I_k^{-1}(F_i \overset{k}{*} G_j) = (I_i^{-1} F_i)(I_j^{-1} G_j),$$

$$i, j, k = \overline{1,3}, \quad i \neq j, \quad k \neq j, \quad i \neq k.$$

Moreover,

a) $(F_2 \overset{1}{*} G_3) \in \mathfrak{M}^{-1}_{c'_1,\gamma'_1}(L)$, $c'_1 = c_1$, $\gamma'_1 = \min(\gamma_1 + 4, 2\gamma_1 + 9/2)$, $c_1 > 0$, for all $\gamma_1 \in R$ and $c_1 = 0$, $\gamma_1 \geq -9/2$.

b) $(F_1 \overset{2}{*} G_3) \in \mathfrak{M}^{-1}_{c'_2,\gamma'_2}(L)$, $c'_2 = c_2$, $\gamma'_2 = \min(\gamma_2 - 4, 2\gamma_2 - 7/2)$, $c_2 > 0$, for all $\gamma_2 \in R$ and $c_2 = 0$, $\gamma_2 \geq -1/2$.

c) $(F_1 \overset{3}{*} G_2) \in \mathfrak{M}^{-1}_{c'_3,\gamma'_3}(L)$, $c'_3 = c_3$, $\gamma'_3 = \min(\gamma_3 - 4, 2\gamma_3 - 3/2)$, $c_3 > 0$, for all $\gamma_3 \in R$ and $c_3 = 0$, $\gamma_3 \geq -1$.

Theorem 4. *Let generalized convolutions* $(f_1 \overset{3}{*} g_2)(a_{\ell ij} \pm 1)$, $\ell = \overline{1,3}$, $i = \overline{1, r_\ell}$, $j = \overline{1, p_i}$ *be obtained from (10), if in the right hand side the factor* $\Gamma^{m_{\ell ij}}(a_{\ell ij} +$

$\alpha_{\ell ij}\big((2-\ell)^2 s + \dfrac{(\ell-1)(4-\ell)}{2}t)\big)$ *is replaced by*

$$\Gamma^{sign(m_{\ell ij})}\Big(a_{\ell ij} \pm 1 + \alpha_{\ell ij}\big((2-\ell)^2 s + \frac{(\ell-1)(4-\ell)}{2}t)\big)\Big) \times$$
$$\times\, \Gamma^{m_{\ell ij}-sign(m_{\ell ij})}\Big(a_{\ell ij} + \alpha_{\ell ij}\big((2-\ell)^2 s + \frac{(\ell-1)(4-\ell)}{2}t)\big)\Big).$$

Moreover, assume that $a_{\ell ij_0} = a_{\ell j_0}$, $\alpha_{\ell ij_0} = \alpha_{\ell j_0}$, $a_{\ell ik_o} = a_{\ell k_0}$, $\alpha_{\ell ik_0} = \alpha_{\ell k_0}$, $i = \overline{1, r_\ell}$, $\ell = \overline{1, 3}$. *Then the following equalities hold*

a) $\alpha_{\ell ik_0}(f_1 \overset{3}{*} g_2)(a_{1ij_0}+1) + \alpha_{1ij_0}(f_1 \overset{3}{*} g_2)(a_{1ik_0}-1)$

$= (a_{1ij_0}\alpha_{1ik_0} + \alpha_{1ij_0} - \alpha_{1ij_0}a_{1ik_0})(f_1 \overset{3}{*} g_2)(x),$

$m_{1ij_0} < 0,\ \alpha_{1ij_0} < 0,\ m_{1ik_0} < 0,\ \alpha_{1ik_0} > 0,\ i = \overline{1, r_1},\ j_0, k_0 = \overline{1, p_1};$

b) $\alpha_{2ik_0}(f_1 \overset{3}{*} g_2)(a_{2ij_0}+1) - \alpha_{2ij_0}(f_1 \overset{3}{*} g_2)(a_{2ik_0}-1)$

$= (a_{2ij_0}\alpha_{2ik_0} - a_{2ik_0}\alpha_{2ij_0})(f_1 \overset{3}{*} g_2)(x),$

$m_{2ij_0} < 0,\ \alpha_{2ij_0} < 0,\ m_{2ik_0} < 0,\ \alpha_{2ik_0} < 0,\ i = \overline{1, r_2},\ j_0, k_0 = \overline{1.p_2};$

c) $\dfrac{\alpha_{3\eta t_0}}{\alpha_{1ij_0}}(f_1 \overset{3}{*} g_2)(a_{1ij_0}+1) + \dfrac{\alpha_{3\eta t_0}}{\alpha_{2\xi k_0}}(f_1 \overset{3}{*} g_2)(a_{2\xi k_0}+1) - (f_1 \overset{3}{*} g_2)(a_{3\eta t_0}+1)$

$= \Big(\dfrac{\alpha_{3\eta t_0}}{\alpha_{1ij_0}}a_{1ij_0} + \dfrac{\alpha_{3\eta t_0}}{\alpha_{2\xi k_0}}a_{2\xi k_0} - a_{3\eta t_0}\Big)(f_1 \overset{3}{*} g_2)(x),$

$m_{1ij_0} < 0,\ \alpha_{1ij_0} < 0,\ m_{2\xi k_0} < 0,\ \alpha_{2\xi k_0} < 0,\ m_{3\eta t_0} > 0,\ \alpha_{3\eta t_0} < 0,$

$i = \overline{1, r_1},\ j_0 = \overline{1, p_1},\ \xi = \overline{1, r_2},\ k_0 = \overline{1, p_2},\ \eta = \overline{1, r_3},\ t_0 = \overline{1, p_3};$

d) $\dfrac{\alpha_{3\eta t_0}}{\alpha_{1ij_0}}(f_1 \overset{3}{*} g_2)(a_{1i_0j}+1) - \dfrac{\alpha_{3\eta t_0}}{\alpha_{2\xi k_0}}(f_1 \overset{3}{*} g_2)(a_{2\xi k_0}-1) + (f_1 \overset{3}{*} g_2)(a_{3\eta t_0}-1)$

$= \Big[\dfrac{\alpha_{3\eta t_0}}{\alpha_{1ij_0}}a_{1ij_0} + \dfrac{\alpha_{3\eta t_0}}{\alpha_{2\xi k_0}}a_{2\xi k_0}(a_{2\xi k_0}-1) + 1 - a_{3\eta t_0}\Big](f_1 \overset{3}{*} g_2)(x),$

$m_{1ij_0} < 0,\ \alpha_{1ij_0} < 0,\ m_{2\xi k_0} < 0,\ \alpha_{2\xi k_0} > 0,\ m_{3\eta t_0} > 0,\ \alpha_{3\eta t_0} > 0.$

Proof. By virtue of formula (1) ([1], p.17) we have

$$(f_1 \overset{3}{*} g_2)(a_{1ij_0}+1)$$
$$= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{x_1x_2}\sum_{\ell=1}^{r_3} I\left(\begin{matrix} x/x_1 \\ \\ x/x_2 \end{matrix}\ \middle|\ \begin{matrix} (a_{1ij_0} - \alpha_{1ij_0}s)\Theta_1(s) \\ \Theta_2(u) \\ p_{3\ell}, \overline{m}_{3\ell}, \overline{a}_{3\ell}, \overline{\alpha}_{3l}(u+s) \end{matrix}\right) \times$$
$$\times\, f_1(x_1)g_2(x_2)dx_1dx_2,$$

$$(f_1 \overset{3}{*} g_2)(a_{2\xi k_0} - 1)$$
$$= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{x_1 x_2} \sum_{\ell=1}^{r_3} I\left(\begin{matrix} x/x_1 \\ \\ x/x_2 \end{matrix} \left| \begin{matrix} \Theta_1(s) \\ (1 - a_{2\xi k_0} + \alpha_{2\xi k_0} u)\Theta_2(u) \\ p_{3\ell}, \overline{m}_{3\ell}, \overline{a}_{3\ell}, \overline{\alpha}_{3\ell}(u+s) \end{matrix} \right. \right) \times$$
$$\times f_1(x_1) g_2(x_2) dx_1 dx_2,$$

$$(f_1 \overset{3}{*} g_2)(a_{3\eta t_0} - 1)$$
$$= \int_0^{+\infty}\int_0^{+\infty} \sum_{\ell=1}^{r_3} I\left(\begin{matrix} x/x_1 \\ \\ x/x_2 \end{matrix} \left| \begin{matrix} \Theta_1(s) \\ \Theta_2(u) \\ p_{3\ell}, \overline{m}_{3\ell}, \overline{a}_{3\ell}, \overline{\alpha}_{3\ell}; 1, 1 - a_{3\eta t_0}, \alpha_{3\eta t_0}(u+s) \end{matrix} \right. \right)$$
$$\times \frac{1}{x_1 x_2} f_1(x_1) g_2(x_2) dx_1 dx_2.$$

From this d) folows. The other three equalities are obtained by analogous arguments. □

Definition 4. Generalized convolution for I_k-transforms, $k = \overline{1,6}$, are defined as

$$(f_1 \overset{4}{*} g_2)(x) = \int_0^{+\infty}\int_0^{+\infty} \sum_{\ell=1}^{r_3}$$
$$I\left(\begin{matrix} x/x_1 \\ \\ x/x_2 \end{matrix} \left| \begin{matrix} \Theta_1(s) \\ \Theta_2(u) \\ p_{3\ell}, \overline{m}_{3\ell}, \overline{a}_{3\ell}, \overline{\alpha}_{3\ell}; 1, 1, \frac{\nu}{2}, \frac{1}{2}; 1, -1, 1 + \frac{\nu}{2}, -\frac{1}{2}(u+s) \end{matrix} \right. \right)$$
$$\times \frac{1}{x_1 x_2} f_1(x_1) g_2(x_2) dx_1 dx_2,$$

$$(f_1 \overset{5}{*} g_2)(x) = \int_0^{+\infty}\int_0^{+\infty} \sum_{\ell=1}^{r_3}$$
$$I\left(\begin{matrix} x/x_1 \\ \\ x/x_2 \end{matrix} \left| \begin{matrix} \Theta_1(s) \\ \Theta_2(u) \\ p_{3\ell}, \overline{m}_{3\ell}, \overline{a}_{3\ell}, \overline{\alpha}_{3\ell}; 1, 1, \frac{\nu}{2}, \frac{1}{2}; 1, 1, -\frac{\nu}{2}, \frac{1}{2}(u+s) \end{matrix} \right. \right)$$
$$\times \frac{1}{x_1 x_2} f_1(x_1) g_2(x_2) dx_1 dx_2,$$

$$(f_1 \overset{6}{*} g_2)(x) = \int_0^{+\infty}\int_0^{+\infty}\sum_{\ell=1}^{r_3}$$

$$I\left(\begin{matrix} x/x_1 \\ \\ x/x_2 \end{matrix}\ \middle|\ \begin{matrix} \Theta_1(s) \\ \Theta_2(u) \\ p_{3\ell}, \overline{m}_{3\ell}, \overline{a}_{3\ell}, \overline{\alpha}_{3\ell}; 1, 1, 0, 1; 1, -1, \frac{1+\nu}{2}, \frac{1}{2}(u+s) \end{matrix}\right)$$

$$\times \frac{1}{x_1 x_2} f_1(x_1) g_2(x_2) dx_1 dx_2.$$

Theorem 5. *The following equalities hold:*

a) $$\int_0^{+\infty} t^{-1} J_\nu(t)(f_1 \overset{3}{*} g_2)(tx)dt = \frac{1}{2}(f_1 \overset{4}{*} g_2)(2x),$$

where $J_\nu(.)$ *is the Bessel function of the first kind* [2], $Re\,\nu > -\dfrac{3}{2}$*;*

b) $$\int_0^{+\infty} t^{-1} K_\nu(t)(f_1 \overset{3}{*} g_2)(tx)dt = \frac{1}{4}(f_1 \overset{5}{*} g_2)(2x),\ -\frac{1}{2} < Re\,\nu < \frac{1}{2}.$$

where $K_\nu(.)$ *is the modified Bessel function of the third kind* [2]*;*

c) $$\int_0^{+\infty} t^{-1} e^{-\frac{1}{4}t^2} D_{-\nu}(t)(f_1 \overset{3}{*} g_2)(tx)dt = \sqrt{\frac{\pi}{2^\nu}}(f_1 \overset{6}{*} g_2)\Big(\frac{1}{\sqrt{2}}x\Big),$$

where $D_\nu(.)$ *is the parabolic cylinder function* [2], $\nu \in \mathbb{C}$*.*

Proof. From (10) and formula 1 in [2], (p. 286) we have

$$\int_0^{+\infty} t^{-1} J_\nu(t)(f_1 \overset{3}{*} g_2)(tx)dt$$

$$= \int_0^{+\infty}\int_0^{+\infty} f_1(u) g_2(v)\Big\{\int_0^{+\infty} t^{-1} U_{13}(tx, u, v) J_\nu(t)dt\Big\}\frac{du}{u}\frac{dv}{v}$$

$$= \int_0^{+\infty}\int_0^{+\infty} f_1(u) g_2(v)\Big\{\frac{1}{(2\pi\omega)^2}\sum_{\ell=1}^{r_3}\int_{\sigma_s}\int_{\sigma_y}\Theta_1(s)\Theta_2(y)\ \times$$

$$\times X^{p_{3\ell}}_{\overline{m}_{3\ell}, \overline{a}_{3\ell}, \overline{\alpha}_{3\ell}}(s+y)\Big(\frac{x}{u}\Big)^s\Big(\frac{x}{v}\Big)^y\Big(\int_0^{+\infty} t^{s+y-1} J_\nu(t)dt\Big)dsdy\Big\}\frac{du}{u}\frac{dv}{v}$$

$$= \int_0^{+\infty}\int_0^{+\infty} \frac{f_1(u)}{u}\frac{g_2(v)}{v}\Big\{\frac{1}{(2\pi\omega)^2}\sum_{\ell=1}^{r_3}\int_{\sigma_s}\int_{\sigma_y}\Theta_1(s)\Theta_2(y)\times$$

$$\times X^{p_{3\ell}}_{\overline{m}_{3\ell},\overline{a}_{3\ell},\overline{\alpha}_{3\ell}}(s+y)\frac{\Gamma\Big(\frac{\nu}{2}+\frac{s+y}{2}\Big)}{\Gamma\Big(1+\frac{\nu}{2}-\frac{s+y}{2}\Big)}2^{s+y-1}\Big(\frac{x}{u}\Big)^s\Big(\frac{x}{v}\Big)^y\Big\}dudv$$

$$= \frac{1}{2}(f_1 \overset{4}{*} g_2)(2x).$$

Thus the first equality is proved. By the same way, from formulas 26 in [2], (p. 289) and 1 in [2] (p. 294) we can verify other equalities □

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