ACTA MATHEMATICA VIETNAMICA
Volume 28, Number 2, 2003, pp. 147-157

ON NON-LINEAR APPROXIMATIONS OF PERIODIC FUNCTIONS OF BESOV CLASSES USING WAVELET DECOMPOSITIONS

MAI XUAN THAO

ABSTRACT. In the present paper, we extend results of Dinh Dung [5] on non-linear n-term L_q-approximation and non-linear widths to the Besov class $SB^{\omega}_{p,\theta}$ where $1 \le p,\ q \le \infty$, $0 < \theta \le \infty$, and ω is a given function of modulus of smoothness type.

1. Introduction

Recently it has been of great interest in non-linear n-term approximations. Among many papers on this topic we would like to mention [6], [7], [8] and [10] which are related to our paper. For brief surveys in non-linear n-term approximations and relevant problems the reader can see [4], [6].

Let X be a quasi-normed linear space and $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ a family of elements in X. Denote by $M_n(\Phi)$ the set of all linear combinations φ of n free terms of the form

$$\varphi = \sum_{k \in Q} a_k \varphi_k,$$

where Q is a set of natural numbers having n elements. We also put $M_0(\Phi) = \{0\}$. Obviously, $M_n(\Phi)$ is a non-linear set. The approximation to an element $f \in X$ by elements of $M_n(\Phi)$ is called the n-term approximation to f with regard to the family Φ.

The error of this approximation is measured by

$$\sigma_n(f, \Phi, X) := \inf_{\varphi \in M_n(\Phi)} \|f - \varphi\|. \tag{1}$$

Let W be a subset in X. Then the worst case error of n-term approximation to the elements in W with regard to the family Φ, is given by

$$\sigma_n(W, \Phi, X) := \sup_{f \in W} \sigma_n(f, \Phi, X). \tag{2}$$

Received December 20, 2001; in revised form August 5, 2002.
1991 *Mathematics Subject Classification.* 41A25, 41A46, 42C15.
Key words and phrases. Non-linear n-term approximation, non-linear n-width, wavelet decomposition, Besov class.

An algorithm of n-term approximation with regard to Φ can be represented as a mapping S from W into $M_n(\Phi)$. If S is continuous, then the algorithm is called continuous. Denote by $\mathcal{F}(X)$ the set of all bounded Φ such that for any finite-dimetional subspace $L \subset X$, the set $\Phi \cap L$ is finite. We will restrict the approximation by elements of $M_n(\Phi)$ only to those using continuous algorithms and in addition only for families Φ from $\mathcal{F}(X)$.

The n-term approximation with these restrictions leads to the non-linear n-width $\tau_n(W, X)$ which is given by

$$\tau_n(W, X) := \inf_{\Phi, S} \sup_{f \in W} \|f - S(f)\|, \tag{3}$$

where the infimum is taken over all continuous mappings from W into $M_n(\Phi)$ and all families $\Phi \in \mathcal{F}(X)$.

The non-linear n-width $\tau_n'(W, X)$ is defined similarly to $\tau_n(W, X)$, but the infimum is taken over all continuous mappings S from W into a finite-dimensional subset of $M_n(\Phi)$ or equivalently, over all continuous mappings S from W into $M_n(\Phi)$ and all finite families Φ in X.

Let l_∞ be the normed space of all bounded sequences of numbers $x = \{x_k\}_{k=1}^\infty$ equipped with the norm

$$\|x\|_\infty := \sup_{1 \le k < \infty} |x_k|.$$

Denote by M_n the subset in l_∞ of all $x \in l_\infty$ for which $x_k = 0$, $k \notin Q$. Consider the mapping R_Φ from the metric space M_n into $M_n(\Phi)$ which is defined as follows

$$R_\Phi(x) := \sum_{k \in Q} x_k \varphi_k$$

if $x = \{x_k\}_{k=1}^\infty$ and $x_k = 0$, $k \notin Q$.

Notice that $M_n(\Phi) = R_\Phi(M_n)$ and if the family Φ is bounded, then R_Φ is a continuous mapping. Any algorithm S of n-term approximation to f with regard to Φ, can be treated as a composition $S = R_\Phi \circ G$ for some mapping G from W into M_n. Therefore, if G is required to be continuous, then the algorithm S will also be continuous. These preliminary remarks are a basis for the notion of the non-linear n-width $\alpha_n(W, X)$ which is given by

$$\alpha_n(W, X) := \inf_{\Phi, G} \sup_{f \in W} \|f - R_\Phi(G(f))\|, \tag{4}$$

where the infimum is taken over all continuous mappings G from W into M_n and all bounded families Φ in X.

The non-linear widths τ_n, τ_n', α_n were introduced by Dinh Dung [5]. There are other non-linear n-widths which are based on continuous algorithms of non-linear approximations, but different from n-term approximation. They are the Alexandroff n-width $a_n(W, X)$, the non-linear manifold n-width $\delta_n(W, X)$, introduced by DeVore, Howard and Micchelli [2], the non-linear n-width $\beta_n(W, X)$ (see [5] for its definition). All these non-linear n-widths are different. However, they possess some common properties and are closely related (see [6] for details).

We now give a definition of Besov spaces. Let $1 \le q \le \infty$ and $\mathbf{T} := [-\pi, \pi]$ be the torus. Denote by $L_q = L_q(\mathbf{T})$ the normed space of functions on $\mathbf{T}$, equipped with the usual p-integral norm. Let

$$\omega_l(f,t)_q := \sup_{|h|<t} \left\| \Delta_h^l f \right\|_{L_q}$$

be the l-th modulus of smoothness of f, where the l-th difference $\Delta_h^l f$ is defined inductively by

$$\Delta_h^l := \Delta_h^1 \Delta_h^{l-1}$$

starting from

$$\Delta_h^1 f := f(\cdot + h/2) - f(\cdot - h/2).$$

The class MS_l of functions ω of modulus of smoothness type is defined as follows. It consists of all non-negative ω on $[0, \infty)$ such that

(i) $\omega(0) = 0$.

(ii) $\omega(t) \le \omega(t')$ if $t \le t'$.

(iii) $\omega(kt) \le k^l \omega(t)$ for $k = 1, 2, 3, \ldots$

(iv) ω satisfies Condition Z_l, that is, there exist a positive number $a < l$ and positive constant C_l such that

$$\omega(t)t^{-a} \ge C_l \omega(h) h^{-a}, \quad 0 \le t \le h.$$

(v) ω satisfies Condition BS, that is, there exist a positive number b and positive constant C such that

$$\omega(t)t^{-b} \le C\omega(h)h^{-b}, \quad 0 \le t \le h \le 1.$$

Let $\omega \in MS_l$, $1 \le p \le \infty$, $0 < \theta \le \infty$. Denote by $B^\omega_{p,\theta}$ the space of all functions $f \in L_p$ for which the Besov semi-quasi norm

$$|f|_{B^\omega_{p,\theta}} := \begin{cases} \left(\int\limits_0^\infty \left\{ \omega_l(f,t)_p / \omega(t) \right\}^\theta dt/t \right)^{1/\theta} & \text{for } \theta < \infty \\ \sup\limits_{t>0} \left\{ \omega_l(f,t)_p / \omega(t) \right\} & \text{for } \theta = \infty \end{cases} \tag{5}$$

is finite.

The Besov quasi-norm is defined by

$$\|f\|_{B^\omega_{p,\theta}} := \|f\|_p + |f|_{B^\omega_{p,\theta}}. \tag{6}$$

For $1 \le p \le \infty$, the definition of $B^\omega_{p,\theta}$ does not depend on l, it means for a given ω, (5) and (6) determine equivalent quasi-norms for all l such that $\omega \in MS_l$ (see [4]). Denote by $SB^\omega_{p,\theta}$ the unit ball of the Besov space $B^\omega_{p,\theta}$.

The trigometric polynomial

$$V_m(t) := \frac{1}{3m^2} \sum_{k=m}^{2m-1} D_k(t) = \frac{\sin \dfrac{mt}{2} \sin \dfrac{3mt}{2}}{3m^2 \sin^2 \left(\dfrac{t}{2} \right)}$$

is called the de la Vallée Poussin kernel of order m, where $D_m(t) = \sum_{|k|\le m} e^{ikt}$ is the Dirichlet kernel of order m.

We let

$$v_{k,s} := v_k\Big(. - \frac{2\pi s}{2^k}\Big), \quad s = 0, 1, \dots, 2^k - 1$$

be the integer translates of the dyadic scaling functions

$$v_0 := 1, \quad v_k := V_{2^k - 1}; \quad k = 1, 2, \dots$$

Each function $f \in L_q$ has a wavelet decomposition

$$f = \sum_{k=0}^{\infty} \sum_{s=0}^{2^k-1} \lambda_{k,s} v_{k,s} \tag{7}$$

with the convergence in L_q, where $\lambda_{k,s}$ are certain coefficient functionals of f (see [4] for details).

Let V_k be the span of the functions $v_{k,s}$, $s = 0, 1, \dots, 2^{k-1} - 1$. Then the family $\{V_k\}_{k=0}^{\infty}$ forms a multiresolution of L_q with the following properties:

MR1. $V_k \subset V_{k'}$, for $k < k'$.

MR2. $\bigcup_{k\in Z} V_k$ is dense in L_p.

MR3. For $k = 0, 1, ...$ dim $V_k = 2^k$ and the functions $v_{k,s} := v_k\big(. - 2\pi s/2^k\big)$, $s = 0, 1, \dots, 2^{k-1} - 1$, form a Riesz basis for V_k, it means there are positive constants C_q and C'_q such that

$$C_q 2^{-k/q} \big\|\{a_s\}\big\|_q \le \Big\| \sum_{s=0}^{2^k-1} a_s v_k(. - s) \Big\|_q \le C'_q 2^{-k/q} \big\|\{a_s\}\big\|_q$$

for all $\{a_s\}_{s=0}^{2^k-1} \in l_q^{2^k}$ (see [4]).

Let us give a wavelet decomposition and discrete characterization for the Besov space $B^{\omega}_{p,\theta}$ of functions on $\mathbf{T}$. Let $1 \le p < \infty$ and $0 < \theta \le \infty$. A function $f \in L_p$ belongs to the Besov space on $B^{\omega}_{p,\theta}$ if f has a wavelet decomposition (7) and in addition the quasi-norm of the Besov space $B^{\omega}_{p,\theta}$ given in (6) is equivalent to the discrete quasi-norm

$$\|f\|_{B^{\omega}_{p,\theta}} \asymp \Big(\sum_{k=0}^{\infty} \big(\big\|\{\lambda_{k,s}\}\big\|_p / 2^{k/p} \omega(2^{-k})\big)^{\theta} \Big)^{1/\theta} \tag{8}$$

(the sum is changed to the supremum when $\theta = \infty$).

For the space $B^{r}_{p,\theta}$, $r > 0$, a proof of the equivalence of quasi-norms and a contruction of continuous coefficient functionals $\lambda_{k,s}$ were given in [5]. In the general case they can be obtained similarly.

For n-term approximation of the functions from $SB^{\omega}_{p,\theta}$, we take the family of wavelets

$$V := \big\{v_{k,s} \,:\, s = 0, 1, \dots, 2^k - 1; \quad k = 0, 1, 2, \dots\big\}.$$

Denote by γ_n any one of the non-linear n-widths τ_n, τ'_n, α_n, β_n, a_n and δ_n. We use the notations $a_+ := \max\{a, 0\}$; $A \asymp B$ if $A \ll B$ and $B \ll A$; and $A \ll B$ if $A \le cB$ with c an absolute constant. We say that ω satisfies Condition $R(p,q)$ if $\omega(t)t^{-\big(1/p-1/q\big)_+}$ satisfies Condition BS.

The main result of the present paper is the following

Theorem 1. *Let $1 \le p,\ q \le \infty$, $0 < \theta \le \infty$ and ω satisfy Condition $R(p,q)$. Then we have*

$$\sigma_n\big(SB^{\omega}_{p,\theta}, V, L_q\big) \asymp \gamma_n\big(SB^{\omega}_{p,\theta}, L_q\big) \asymp \omega\big(1/n\big). \tag{9}$$

The case $\omega(t) = t^{\alpha}$, $\alpha > 0$, of Theorem 1 was proved in [5]. To prove Theorem 1 we develop further the method of [5]. However, because the smoothness of the class $B^{\omega}_{p,\theta}$ is complicated, we have to overcome certain difficulties.

2. Auxiliary results

In this section we give necessary auxiliaries for proving Theorem 1. For $0 < p \le \infty$, denote by l^m_p the space of all sequence $x = \big\{x_k\big\}^m_{k=1}$ of numbers, equipped with the quasi-norm

$$\big\|\{x_k\}\big\|_{l^m_p} = \big\|x\big\|_{l^m_p} := \Big(\sum_{k=1}^{m} |x_k|^p\Big)^{1/p}$$

(the sum is changed to max when $p = \infty$).

Let $\mathcal{E} = \big\{e_k\big\}^m_{k=1}$ be the canonical basis in l^m_p. It means that $x = \sum\limits_{k=1}^{m} x_k e_k$ for $x = \big\{x_k\big\}^m_{k=1} \in l^m_p$. We let the set $\big\{k_j\big\}^m_{j=1}$ be ordered so that

$$|x_{k_1}| \ge |x_{k_2}| \ge \cdots \ge |x_{k_j}| \ge \cdots \ge |x_{k_n}| \ge \cdots \ge |x_{k_m}|.$$

The greedy algorithm G_n for the n-term approximation with regard to $\mathcal{E}$ is defined by

$$G_n(x) := \sum_{j=1}^{n} x_{k_j} e_{k_j}.$$

Clearly, G_n is not continuous. However, the mapping

$$G^C_n(x) := \begin{cases} \sum\limits_{j=1}^{n} \big(x_{k_j} - |x_{k_{n+1}}|\text{sign } x_{k_j}\big) e_{k_j}, & \text{for} \quad p < q \\ \sum\limits_{k=1}^{n} x_k e_k & \text{for} \quad p \ge q, \end{cases}$$

defines a continuous algorithm of n-term approximation.

Denote by B^m_p the unit ball in l^m_p.

Lemma 1. *Let $0 < p, q \le \infty$. Then we have for any positive integer $n < m$*

$$\sup_{x \in B_p^m} \big\|x - G_n(x)\big\|_{l_q^m} \le \sup_{x \in B_p^m} \big\|x - G_n^C(x)\big\|_{l_q^m} \le A_{p,q}(m,n),$$

where

$$A_{p,q}(m,n) := \begin{cases} n^{1/q-1/p} & \text{for} \quad p < q \\ (m-n)^{1/q-1/p} & \text{for} \quad p \ge q. \end{cases}$$

Lemma 1 and the following two lemmas were proved in [7].

Lemma 2. *Let $0 < q \le \infty$ and L be a s-dimensional linear subspace in l_q^m $(s \le m)$. Then we have for any positive integer $n < s$,*

$$\sigma_n\big(B_\infty^m \cap L, \mathcal{E}, l_\infty^m\big) = 1$$

and for any positive integer $n < s - 1$,

$$\sigma_n\big(B_\infty^m \cap L, \mathcal{E}, l_q^m\big) \ge (m - n - 1)^{1/q}.$$

Lemma 3. *Let $0 < q \le \infty$ and $n < s \le m$. Let L be a s-dimensional linear subspace in l_q^m and $P : l_q^m \longrightarrow L$ is a linear projector in l_q^m. Then we have*

$$a_n\big(B_\infty^m \cap L, l_q^m\big) \ge \|P\|^{-1}(m-n)^{1/q}.$$

3. Upper bounds

To prove the upper bound of $\sigma_n(SB_{p,\theta}^\omega, V, L_q)$, we explicitly construct a finite subset V^* of V and a positive homogeneous mapping $G^* : B_{p,\theta}^\omega \longrightarrow M_n$ such that

$$\sup_{f \in SB_{p,\theta}^\omega} \big\|f - S_n^*(f)\big\|_q \ll \omega\big(1/n\big), \tag{10}$$

where $S_n^* := R_{V^*} \circ G^*$. This means that the algorithm S^* of n-term approximation with regard to V is asymptotically optimal for σ_n.

Because $\|\,.\,\|_{B_{p,\infty}^\omega} \le C\|\,.\,\|_{B_{p,\theta}^\omega}$ (for $0 < \theta < \infty$), the space $B_{p,\theta}^\omega$ can be considered as a subspace of the largest space $B_{p,\infty}^\omega$. Hence, it is sufficient to construct S_n^* for $H := SB_{p,\infty}^\omega$.

For each function

$$g = \sum_{s=0}^{2^k-1} a_s v_{k,s} \tag{11}$$

belonging to V_k, we have by MR3

$$\|g\|_q \asymp 2^{-k/q}\big\|\{a_s\}\big\|_q. \tag{12}$$

Using the equivalence of quasi-norms (8) for H, from (7) we find that a function $f \in L_p$ belongs to H if f can be decomposed into the functions f_k by a series

$$f = \sum_{k=0}^{\infty} f_k, \tag{13}$$

where the functions

$$f_k = \sum_{s=0}^{2^k-1} \lambda_{k,s} v_{k,s}$$

are from V_k and satisfy the condition

$$\|f_k\|_p \asymp 2^{-k/p} \big\| \{\lambda_{k,s}\} \big\|_{l_p^{2^k}} \le C\omega(2^{-k}), \quad k = 0, 1, 2, \ldots \tag{14}$$

(see [4]).

For a non-negative number n, let $\{n_k\}$ be a sequence of non-negative integers such that

$$\sum_{k=0}^{\infty} n_k \le n. \tag{15}$$

Let $\mathcal{E} = \{e_s\}_{s=0}^{2^k-1}$ be the canonical basis in $l_q^{2^k}$. For $a = \sum\limits_{s=0}^{2^k-1} a_s e_s \in l_q^{2^k}$, we let the set $\{s_j\}_{j=0}^{2^k-1}$ be ordered so that

$$|a_{s_0}| \ge |a_{s_1}| \ge \cdots \ge |a_{s_{n_k-1}}| \ge \cdots \ge |a_{s_{2^k-1}}|.$$

Then, the greedy algorithm G_{n_k} for the n_k-term approximation with regard to $\mathcal{E}$ is

$$G_{n_k}(a) := \sum_{j=0}^{n_k-1} a_{s_j} e_{s_j}. \tag{16}$$

For any positive integer $n_k < 2^k$ and all $a \in B_q^{2^k}$, by Lemma 1 we have

$$\big\| a - G_{n_k}(a) \big\|_{l_q^{2^k}} \le A_{p,q}(2^k, n_k), \tag{17}$$

where

$$A_{p,q}(2^k, n_k) := \begin{cases} n_k^{1/q-1/p} & \text{for} \quad p < q \\ (2^k - n_k)^{1/q-1/p} & \text{for} \quad p \ge q. \end{cases}$$

Observe that the greedy algorithm G_{n_k} in $l_q^{2^k}$ corresponds to the greedy algorithm G'_{n_k} of n_k-term approximation in V_k which is given by

$$G'_{n_k}(g) := \sum_{j=0}^{n_k-1} a_{s_j} v_{k,s_j} \tag{18}$$

for a function represented as in (11). Because of the norm equivalence (12) for each function $g \in V_k$, we have

$$\big\| g - G'_{n_k}(g) \big\|_q \asymp 2^{-k/q} \big\| \{a_s\} - G_{n_k}\big(\{a_s\}\big) \big\|_{l_q^{2^k}}. \tag{19}$$

For each function $f \in H$ represented as in (13), from (19) we obtain

$$\begin{aligned}\big\|f_k - G'_{n_k}(f_k)\big\|_q &\asymp 2^{-k/q}\big\|\{\lambda_{k,s}\} - G_{n_k}(\{\lambda_{k,s}\})\big\|_{l_q^{2^k}}\\ &\le C.2^{k/p-k/q}\omega(2^{-k})\big\|\{\lambda^*_{k,s}\} - G_{n_k}\big(\{\lambda^*_{k,s}\}\big)\big\|_{l_q^{2^k}}\\ &\le C.2^{k\big(1/p-1/q\big)}\omega(2^{-k})A_{p,q}(2^k, n_k),\end{aligned}$$

where

$$\lambda^*_{k,s} = \frac{\lambda_{k,s}}{C.2^{k/p}\omega(2^{-k})} \quad \text{and} \quad \{\lambda^*_{k,s}\} \in B_p^{2^k}. \tag{20}$$

Because ω satisfies Condition $R(p,q)$, there exist $C_1 > 0$ and $\delta > 0$ such that for $k \ge k'$

$$\omega(2^{-k})(2^{-k})^{-\big(1/p-1/q\big)-\delta} \le C_1\omega(2^{-k'})(2^{-k'})^{-\big(1/p-1/q\big)-\delta}. \tag{21}$$

Let us now select a sequence $\{n_k\}_{k=0}^{\infty}$ satisfying the condition (15). For simplicity we consider the case $p < q$ (the other cases can be treated similarly). Fix a number ε so that $0 < \varepsilon < \delta\big/(1/p - 1/q)$. For a given natural number n, let the integer r be defined from the conditions $2^{r+2} \le n < 2^{r+3}$. Then an appropriate selection of $\{n_k\}_{k=0}^{\infty}$ is given by

$$n_k = \begin{cases} 2^k & \text{for} \quad k \le r \\ \big[an2^{-\varepsilon(k-r)}\big] & \text{for} \quad k > r, \end{cases} \tag{22}$$

where $a = \dfrac{2^\varepsilon - 1}{2}$ and $[t]$ denotes the integer part of t. Then we have

$$\sum_{k=0}^{\infty} n_k \le \sum_{k=0}^{r} 2^k + \sum_{k=r+1}^{\infty} an2^{-\varepsilon(k-r)} = (2^{r+1} - 1) + \frac{an}{2^\varepsilon - 1} \le \frac{n}{2} + \frac{n}{2} = n.$$

This means that (15) is satisfied. We take a positive constant λ so that

$$\frac{1+\varepsilon}{\varepsilon} > \lambda > \frac{1/p - 1/q + \delta}{\delta}$$

and put $k^* = [\lambda r]$.

We construct a mapping $S_k : H \longrightarrow M_{n_k}(V)$ as follows

$$S_k(f) := \begin{cases} G'_{n_k}(f_k) & \text{for} \quad k \le k^* \\ 0 & \text{for} \quad k > k^*. \end{cases}$$

Notice that for $k \le r$, we have $S_k(f) = f_k$ and therefore,

$$\|f_k - S_k(f)\|_q = 0. \tag{23}$$

Next, for $r < k \le k^*$, from (20) we have

$$\big\|f_k - S_k(f)\big\|_q \le C2^{k\big(1/p-1/q\big)}\omega(2^{-k})A_{p,q}(2^k, n_k), \tag{24}$$

and for $k > k^*$, we have

$$\big\|f_k - S_k(f)\big\|_q = \|f_k\|_q \le C' 2^{k\left(1/p-1/q\right)}\omega(2^{-k}). \tag{25}$$

Put

$$S_n^*(f) := \sum_{k=0}^{\infty} S_k(f) \quad \text{for} \quad f = \sum_{k=0}^{\infty} f_k.$$

Then by (21), (23), (24) and (25) we get

$$\big\|f - S_n^*(f)\big\|_q \le \sum_{k=r+1}^{\infty} \big\|f_k - S_k(f)\big\|_q \le C^*\omega(2^{-k_0}) \asymp C^*\omega\big(1/n\big). \tag{26}$$

Put

$$G^*(f) := \big\{G'_{n_k}(f_k)\big\}_{k\le k^*} \quad \text{for} \quad f = \sum_{k\le k^*} f_k \in H.$$

Then G^* is a positive homogeneous mapping H into M_n, and $S_n^* = R_{V^*} \circ G^*$, where $V^* := \{v_{k,s} \;:\; s = 0, 1, \dots, 2^k - 1; \;\; k \le k^*\}$. From (26) we obtain (10). This also proves the upper bound of $\sigma_n\big(SB^{\omega}_{p,\theta}, V, L_q\big)$.

We now prove the upper bound

$$\gamma_n\big(SB^{\omega}_{p,\theta}, L_q\big) \ll \omega\big(1/n\big). \tag{27}$$

Using inequalities between α_n, τ_n, τ'_n, δ_n, β_n, and a_n (see [6]), we prove only for one of them, namely for α_n. If in (26), G'_{n_k} are replaced by $G^{c'}_{n_k}$, then S_n is a continuous algorithm of n-term approximation, which satisfy (14). Hence, we prove the upper bound of $\alpha_n\big(SB^{\omega}_{p,\theta}, L_q\big)$ and we receive (27). The upper bounds of (9) in Theorem 1 are proved.

4. Lower bounds

We first prove the lower bound for σ_n:

$$\sigma_n\big(SB^{\omega}_{p,\theta}, V, L_q\big) \gg \omega\big(1/n\big). \tag{28}$$

Because of the inequality $\|\,.\,\|_\infty \ge c\|\,.\,\|_p$ for $1 \le p < \infty$, it is sufficient to prove (28) for the case $p = \infty$. For a positive integer k, denote by $B(k)$ the space of all trigonometric polynomials f of the form

$$f = \sum_{s=0}^{2^k-1} \lambda_{k,s} v_{k,s},$$

and for $1 \le \eta \le \infty$, denote by $B(k)_\eta$ the subspace in L_η, which consists of all $f \in B(k)$. For $SB(k)_\infty$ the unit ball in $B(k)_\infty$, by (8) we have $\omega(2^{-k})SB(k)_\infty \subset aSB^{\omega}_{\infty,\theta}$ with some $a > 0$. Hence

$$\sigma_n(SB^{\omega}_{\infty,\theta}, V, L_q) \gg \omega(2^{-k})\sigma_n(SB(k)_\infty, V, L_q). \tag{29}$$

Let X be a normed space and Y a subspace of X, $W \subset X$, and let Φ be a family in X. If $P : X \longrightarrow Y$ is a linear projection such that $\|P(f)\| \le \|f\|$ for every $f \in X$, then $\sigma_n(W, \Phi, X) \geq \sigma_n(W, P(\Phi), Y)$. Applying this inequality to the linear projection

$$P(k, f) = \sum_{s=0}^{2^k-1} \lambda_{k,s} v_{k,s}$$

in the space L_q, gives

$$\sigma_n(SB(k)_\infty, V, L_q) \geq \sigma_n\big(SB(k)_\infty, V', B(k)q\big), \tag{30}$$

where $V' = P(k, V)$ (see [4]). From (29) and (30) we have

$$\sigma_n(SB^\omega_{\infty,\theta}, V, L_q) \gg \omega(2^{-k})\sigma_n\big(SB(k)_\infty, V', B(k)_q\big). \tag{31}$$

Let us give a lower bound for $\sigma_n(SB(k)_\infty, V', B(k)_q)$.

Define $k = k(n)$ from the conditions

$$n \asymp 2^k \asymp \dim\ B(k) > 2n. \tag{32}$$

From (8) we have

$$\|f\|_{B(k)_\infty} \asymp \|J(f)\|_{l_\infty^{2^k}}, \quad \|f\|_{B(k)_q} \asymp 2^{-k/q}\big\|J(f)\big\|_{l_q^{2^k}}, \tag{33}$$

where J is the positive homogeneous continuous mapping from $B(k)_q$ into $l_\infty^{2^k}$, given by

$$J(f) := \big\{\lambda_{k,s}\big\}_{s=0}^{2^k-1} \quad \text{for} \quad f = \sum_{s=0}^{2^k-1} \lambda_{k,s} v_{k,s}.$$

Clearly, $J(V') = \mathcal{E}'$ and $J\big(B(k)_q\big) = l_q^{2^k}$, where $\mathcal{E}'$ is the canonical basis in $l_q^{2^k}$ (see [4]).

Also, if S is an algorithm of n-term approximation with regard to V' in $B(k)_q$, then $J \circ S$ will be an algorithm of n-term approximation with regard to $\mathcal{E}'$ in $l_q^{2^k}$. Therefore, by (32), (33) and Lemma 2, we obtain

$$\begin{aligned} \sigma_n\big(SB(k)_\infty, V', B(k)_q\big) &\asymp 2^{-k/q}\sigma_n\big(B_\infty^{2^k}, \mathcal{E}', l_q^{2^k}\big) \\ &\geq 2^{-k/q}(m-n-1)^{1/q} \gg 1. \end{aligned} \tag{34}$$

where $m \asymp \dim\ B(k) \asymp 2^k$. From (34) and (31) we obtain (28).

Because of inequalities between $\alpha_n, \tau_n, \tau'_n, \beta_n, a_n$, and δ_n (see [6]), it is enough to prove $a_n(SB^\omega_{p,\theta}, L_q) \gg \omega\big(1/n\big)$. It can be proved in the same way as the proof of the lower bound for $\sigma_n(SB^\omega_{p,\theta}, V, L_q)$, but by using Lemma 1 and Lemma 3. Thus, we have completed the proof of Theorem 1.

References

[1] R. DeVore, *Nonlinear approximation*, Acta Numerica **7** (1998), 51–150.

[2] R. DeVore, R. Howard, and C. Micchelli, *Optimal non-linear approximation*, Manuscripta Math. **63** (1989), 469–478.

[3] R. DeVore, G. Lorentz, *Constructive Approximation*, Springer-Verlag, 1993.

[4] Dinh Dung, *Non-linear approximations using wavelet decompositions*, Vietnam J. Math. **29** (2001), 197–224.

[5] Dinh Dung, *On non-linear n-widths and n-term approximation*, Vietnam J. Math. **26** (1998), 165–176.

[6] Dinh Dung, *Continuous algorithms in n-term approximation and non-linear n-widths*, J. Approx. Theory **102** (2000), 217–242.

[7] Dinh Dung, *Asymptotic orders of optimal non-linear approximations*, East J. on Approx. **7** (2001), 55–76.

[8] B. Kashin and V. Temlyakov, *On best m-term approximation and the entropy of sets in the space L^1*, Math. Notes **36** (1994), 1137–1157.

[9] V. Temlyakov, *Approximation of periodic functions*, Nova Science Publishers, New York, 1993.

[10] V. Temlyakov, *Greedy algorithms with regard to the multivariate systems with a special structure*, Constr. Approx. **16** (2000), 339–425.

Hong Duc University,
Thanh Hoa, Vietnam