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LYAPUNOV'S INEQUALITY FOR LINEAR DIFFERENTIAL ALGEBRAIC EQUATION

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Abstract. We introduce a concept of Lyapunov exponents and Lyapunov spectrum of a linear differential algebraic equation (DAE) and derive Lyapunov's inequality for a DAE of index 1. We derive some estimates for sum of Lyapunov exponents from a fundamental solution matrix of a DAE by means of its coefficients.

1. Introduction

In science and practical application there are numerous problems such as the problem of description of dynamic systems, electric circuit systems or problems in cybernetics etc... requiring investigation of solutions of differential equations of the type $Ax' + Bx = 0$, where A and B are constant or continuous time-dependent matrices of order m with $\det A = 0$. Such equations are called *differential algebraic equations* (DAEs).

Investigation of DAEs was carried out intensively by a group of researchers from Humboldt University of Berlin (see [3, 4, 5, 6, 7]) and by Russian mathematicians (see [8] and the references therein). Many results on stability properties of DAEs were obtained such as asymptotic and exponential stability of DAEs which are of index 1 and 2 [4,7], a criterion for stability of a DAE of index 1 [6], stability of periodic DAEs [5]. The method used in the above papers is based on reduction of investigation of a DAE to investigation of the corresponding ODE.

For a DAE under certain conditions, we are able to transform it into a system consisting of a system of ordinary differential equations (ODEs) and a system of algebraic equations so that we can use methods and results of the theory of ordinary differential equations. For studying the stability and asymptotic behaviour of solutions of ODEs, the key tools are two methods by Lyapunov: the method of Lyapunov exponents and the method of Lyapunov's functions [1,2]. Although the theory of Lyapunov exponents for an ODE has been well developed, a concept of Lyapunov exponents of a DAE has still not been discussed in the literature. In this paper we develop a concept of Lyapunov exponents and obtain for DAEs Lyapunov's inequality which is an analogue of the one from the qualitative theory of ODEs.

Now we recall some basic notions of the theory of Lyapunov exponents [1, 2].

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Definition 1.1. For a real function $f : \mathbb{R} \to \mathbb{R}$, the number (or $\pm\infty$)

$$\lambda(f) := \overline{\lim_{t\to\infty}} \frac{1}{t} \ln |f(t)|$$

is called *Lyapunov exponent of f*.

The Lyapunov exponent of a matrix function $F(t) = \left[f_{jk}(t)\right]$ $(j, k = 1, 2, \ldots, m)$ is defined by

$$\lambda(F(t)) := \max_{i,k} \lambda\big(f_{jk}(t)\big).$$

Note that one can define Lyapunov exponent for a function defined on half line or on the set of positive integers.

Lyapunov exponent has two principal properties:

1) $\lambda\Big(\sum_{k=1}^{m} f_k(t)\Big) \le \max_{1\le k\le m} \lambda\big(f_k(t)\big)$,

2) $\lambda\Big(\prod_{k=1}^{m} f_k(t)\Big) \le \sum_{k=1}^{m} \lambda\big(f_k(t)\big)$.

In this article, we define Lyapunov spectrum of a DAE and derive Lyapunov's inequality in case the equation is transferable.

2. Lyapunov exponents of a DAE

Let $\mathcal{G}$ be an open connected set in $\mathcal{R} := \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$. We consider a differential equation

$$f(x'(t), x(t), t) = 0 \tag{2.1}$$

where $f : \mathcal{G} \longrightarrow \mathbb{R}^m$, $(y, x, t) \longmapsto f(y, x, t)$, is continuous in $\mathcal{G}$ and has continuous partial derivatives with respect to y and x in $\mathcal{G}$. Furthermore, we assume that for each $(y, x, t) \in \mathcal{G}$ any triple $(\widehat{y}, x, t)$ with

$$\widehat{y} - y \in \text{ Ker } (f'_y(y, x, t))$$

belongs to $\mathcal{G}$.

A function $x(t)$ is called a *classical solution* of (2.1) on the interval $[t_0, T]$ if the following three conditions are satisfied:

1. $x(t)$ is differentiable on $[t_0, T]$,
2. $\big(x'(t), x(t), t\big) \in \mathcal{G}$ for all $t \in [t_0, T]$,
3. $f(x'(t), x(t), t) = 0$ for all $t \in [t_0, T]$.

It turns out that this classical notion of solution, while being natural for ODEs, is too narrow for DAEs. For example let us regard the DAE

$$\begin{cases} x_1'(t) = g(x_1(t), x_2(t), t) \\ x_2(t) = h(x_1(t), t) \end{cases}$$

as an implicit system $f(x'(t),x(t),t)=0$ with

$$x=\begin{pmatrix}x_1\\x_2\end{pmatrix},\quad f(y,x,t)=\begin{pmatrix}y_1-g(x_1,x_2,t)\\x_2-h(x_1,t)\end{pmatrix}.$$

Then we obtain for a function h not being partially differentiable with respect to t a non-differentiable component x_2 of the solution x.

Let $N:[t_0,T]\to\mathbb{R}^m$ be a smooth subspace-valued function with constant $\dim N(t)$, and $Q(\cdot)=I-P(\cdot)\in C^1$ be a projector function onto N. We denote by $C^1_N[t_0,T]$ the function space

$$C^1_N[t_0,T]:=\{x\in C[t_0,T]\,:\,Px\in C^1[t_0,T]\}$$

equipped with the norm $\|x\|:=\|x\|_\infty+\|(Px)'\|_\infty$, where

$$\|x\|_\infty:=\sup_{t_0\le t\le T}\|x(t)\|.$$

One can show that this definition is independent of the choice of the projector function $Q\in C^1$, namely two projectors generate equivalent norms on the same function space (see Griepentrog and März [3, p.29]).

Now we turn to the case of a linear DAE

$$A(t)x'+B(t)x=0,\quad t\in J, \tag{2.2}$$

where A, B are continuous $m\times m$-matrix functions, $\operatorname{rank}A(t)=r(r<m)$, $N(t):=\ker A(t)$ is of the same dimension $m-r$ for all $t\in J$. The region $\mathcal{G}$ of definition of (2.2) is assumed equal $\mathbb{R}^m\times\mathbb{R}^m\times J$, where J is the time-interval of (2.2). Sometime one may consider the case of finite time interval $J=[t_0,T]$. However, for our aim of defining Lyapunov spectrum of DAEs we shall consider only the case of infinite time interval $J=[t_0,\infty)$. Furthermore, we take for the solution spaces of the DAEs (2.1) the space

$$C^1_N[t_0,\infty):=\bigcap_{T>t_0}C^1_N[t_0,T].$$

Definition 2.1. Assume that $N(t)$ is smooth, i.e. there exists a differentiable projector function $Q\in C^1$ onto $N(t)$, $P=I-Q$. A function $x\in C^1_N[t_0,\infty)$ is said to be *a solution of* (2.2) on J if the identity

$$Ax'+Bx=A\big[(Px)'-P'x\big]+Bx=0$$

is satisfied for all $t\in J$.

Definition 2.2. The linear DAE (2.2) is called *transferable* on $\mathcal{G}$ if there exists a smooth projector function $Q=I-P\in C^1$ onto $N(t)$ and the matrix $G(t)=A(t)+B(t)Q(t)$ has bounded inverse $G^{-1}(t)$ on each interval $[t_0,T]\subset J$.

The following proposition on the existence and uniqueness of the solution of an initial value problem (IVP) for transferable equation was proved in Griepentrog and März [3, p.36].

Proposition 2.1. *Suppose that the DAE* (2.2) *is transferable on* $\mathcal{G}$. *Then for each given* $x^0 \in \mathbb{R}^m$ *the IVP*

$$A(t)x' + B(t)x = 0, \quad x(t_0) - x^0 \in N(t_0), \tag{2.3}$$

is uniquely solvable on each interval $[t_0, T] \subset J$. *The solution is defined by the state variable system*

$$u'(t) = P'(t)u(t) - P(t)(I + P'(t))G^{-1}(t)B(t)u(t), \quad u(t_0) = P(t_0)x^0, \tag{2.4}$$

$$x(t) = u(t) - Q(t)G^{-1}(t)B(t)u(t). \tag{2.5}$$

Furthermore, $u(t) = P(t)x(t)$.

Note that using the projector $P_s(t) = I - Q(t)G^{-1}(t)B(t)$ onto

$$S(t) := \{x \in \mathbb{R}^m \, : \, B(t)x \in \text{ im } A(t)\},$$

the formulas (2.4), (2.5) can be rewritten:

$$u'(t) = \big[P'(t)P_s(t) - P(t)G^{-1}(t)B(t)\big]u(t), \tag{2.6}$$

$$x(t) = P_s(t)u(t). \tag{2.7}$$

Definition 2.3. The equation (2.2) is called of *index 1* if for all $t \in [t_0, \infty)$

$$N(t) \oplus S(t) = \mathbb{R}^m. \tag{2.8}$$

Note that, according to Theorem A13 of Griepentrog and März [3], expression (2.8) is equivalent to the condition that the matrix

$$G(t) := A(t) + B(t)Q(t)$$

is nonsingular for all $t \in [t_0, \infty)$. If the equation (2.2) is of index 1, then it is equivalent to the system

$$u'(t) = \big[P'(t)P_s(t) - P(t)G^{-1}(t)B(t)\big]u(t), \quad t \in [t_0, \infty), \tag{2.9}$$

$$v(t) + Q(t)G^{-1}(t)B(t)u(t) = 0, \tag{2.10}$$

where $u = Px$ and $v = Qx$. Moreover if $u(t_0) = u_0 \in \text{im}P(t_0)$, then $u(t) \in \text{im}\, P(t)$ for all $t \in J$.

Definition 2.4. In case (2.2) is of index 1, (2.6) is called the *corresponding (under* P*) ordinary differential equation* (ODE) of (2.2).

For transferable DAE (2.3) with $t \in [t_0, \infty)$, Proposition 2.1 provides its unique solvability on the infinite interval $[t_0, \infty)$.

If $x(t)$ is a solution of equation (2.3) with $t \in J$, then for all $t \in J$, $x(t)$ belongs to subspace $S(t)$.

Theorem 2.1. *Suppose that the linear DAE* (2.2) *is transferable and its coefficient matrices* $A(t)$, $B(t)$ *and* $G^{-1}(t)$ *are bounded on* J. *Then the Lyapunov exponent of any nontrivial solution* $x(t)$ *of* (2.2) *equals the Lyapunov exponent of the corresponding solution of the corresponding ODE of* (2.2).

Proof. Since $u(t_0) = u_0 \in \text{ im } P(t_0)$, we have $u(t) \in \text{ im } P(t)$ for all $t \in J$. By (2.7), a solution of (2.2) has the form

$$x(t) = P_s(t)u(t)$$

where $P_s = I - Q_s$, $Q_s = QG^{-1}B$, $u(t)$ is the solution of the corresponding ODE with initial condition $u(t_0) = P(t_0)x^0$.

By the assumption of the theorem we have $\|G^{-1}\| \leq k$, $\|B\| \leq b$, $\|A\| \leq a$ for some positive constants k, b, a. Since $P = G^{-1}A$ we have

$$\|P\| = \|G^{-1}A\| \leq ka,$$
$$\|Q\| = \|I - P\| \leq 1 + ka.$$

Therefore

$$\|Q_s\| = \|QG^{-1}B\| \leq (1 + ka)kb,$$
$$\|P_s\| = \|I - Q_s\| \leq 1 + (1 + ka)kb.$$

Consequently, P_s and P are bounded on J, hence

$$\lambda(P_s) \leq 0 \quad \text{and} \quad \lambda(P) \leq 0.$$

This implies that

$$\lambda(x) = \lambda(P_s u) \leq \lambda(P_s) + \lambda(u) \leq \lambda(u),$$
$$\lambda(u) = \lambda(Px) \leq \lambda(P) + \lambda(x) \leq \lambda(x),$$

consequently $\lambda(x) = \lambda(u)$. □

3. Lyapunov's inequality for DAEs

In this section we introduce the notion of Lyapunov spectrum of a DAE and derive Lyapunov's inequality for a DAE by using Lyapunov's inequality of the corresponding ODE. For doing this, we need a concept of a fundamental solution matrix of a DAE.

Suppose we are given a transferable DAE with continuous coefficients

$$A(t)x' + B(t)x = 0, \quad t \in J, \tag{3.1}$$

and an initial condition

$$P(t_0)\big(x(t_0) - x^0\big) = 0, \tag{3.2}$$

where $P(t) = I - Q(t)$, $Q(t) \in C^1$ is a projector function onto $N(t) := \operatorname{Ker} A(t)$. Let $\operatorname{rank} A(t) = r = \text{constant}$, $r < m$.

Definition 3.1. A square matrix $X(t)$ of order m is called a *fundamental solution matrix* (FSM) of (3.1) if its first r vector-columns are linearly independent solutions of (3.1) and the last $m - r$ vector-columns of $X(t)$ are zero.

Note that any solution $x(t)$ of (3.1) belongs to a subspace $S(t)$ of dimension r, so that we have at most r linearly independent solutions. Hence the set of all solutions of (3.1) is a linear subspace of dimension $\leq r$. Moreover, it is known (see [3, p.40]) that if p_j $(j = 1, 2, ..., r)$ are r linearly independent vector-columns of im $P(t_0)$ and the vectors $u_j(t)$, $x_j(t)$ are derived from the linear state variable system

$$\begin{aligned} u'(t) &= \big[P'(t)P_s(t) - P(t)G^{-1}(t)B(t)\big]u(t), \\ x(t) &= P_s(t)u(t), \end{aligned}$$

with the initial vectors $u_j(t_0) = p_j$, then vectors $x_1(t), ..., x_r(t)$ are linearly independent and

$$\begin{aligned} \text{im } P(t) &= \text{span } \big(u_1(t), ..., u_r(t)\big), \\ \text{im } S(t) &= \text{span } \big(x_1(t), ..., x_r(t)\big). \end{aligned}$$

Therefore, the set of all solutions of (3.1) is a linear subspace of dimension r, which we denote by $\mathcal{R}^r$. Any FSM has the form

$$X(t) = \big(x_1(t), x_2(t), ..., x_r(t), 0, 0, ..., 0\big).$$

For simplicity of notation, in what follows we shall write a FSM shortly as

$$X_r(t) = \big(x_1(t), x_2(t), ..., x_r(t)\big).$$

Theorem 3.1. *Suppose that the coefficients $A(t)$, $B(t)$ of (3.1) are bounded. Assume further that (3.1) is transferable on J with a projector $Q(t) = I - P(t) \in C^1(J)$ onto $N(t)$ such that $Q'(t)$ and $G^{-1}(t) = \big(A(t) + B(t)Q(t)\big)^{-1}$ are bounded on J. Then any nontrivial solution $x = x(t)$ of (3.1) has finite Lyapunov exponent.*

Proof. From Theorem 2.1 we have $\lambda(x) = \lambda(u)$, where $x(t)$ is a solution of (3.1), $u(t)$ is the corresponding solution of the corresponding ODE

$$u'(t) = \big[P'(t)P_s(t) - P(t)G^{-1}(t)B(t)\big]u(t). \tag{3.3}$$

Since (3.1) is transferable and $P'(t)$, $A(t)$, $B(t)$, $G^{-1}(t)$ are bounded, the functions $P(t)$, $P_s(t)$, are bounded too. This implies that $\|P'(t)P_s(t) - P(t)G^{-1}(t)B(t)\| < \infty$. Thus (3.3) is a linear ODE with bounded coefficients, hence any nontrivial solution $u(t)$ of (3.3) has finite Lyapunov exponent. Consequently, any nontrivial solution of (3.1) has finite Lyapunov exponent. □

Definition 3.2. The set of all finite Lyapunov exponents of all solutions of a DAE is called *Lyapunov spectrum* of this DAE.

Note that since the DAE (3.1) has at most r linear independent solutions, its Lyapunov spectrum consists of at most r distinct numbers, which we may order by increasing values

$$\lambda_1(A, B) < \lambda_2(A, B) < ... < \lambda_d(A, B), \quad d \leq r.$$

Definition 3.3. A FSM $X_r(t) = \big(x_1(t), ..., x_r(t)\big)$ of (3.1) is called *normal* if the expression

$$\sigma_{X_r} := \sum_{i=1}^{r} \lambda(x_i(t))$$

attains its minimum in the set of all FSMs of (3.1).

We denote by $n_s (s = 1, \ldots, d)$ the maximum number of linearly independent solutions of (3.1) with Lyapunov exponent equal $\lambda_s(A, B)$. Put

$$\mathcal{R}^s := \big\{x(t) \,:\, x(t) \text{ is a solution of (3.1) and } \lambda\big(x(t)\big) \leq \lambda_s(A, B)\big\},$$

then $\mathcal{R}^s$ is a linear subspace of the solution space $\mathcal{R}^r$ of (3.1) and $\dim\mathcal{R}^s = N_s$, where

$$\begin{aligned} &N_k = n_1 + ... + n_k, \quad k = 1, 2, ..., d,\\ &N_1 < N_2 < ... < N_d = r. \end{aligned}$$

Definition 3.4. We say that a system of non-zero vector-functions $x_1(t), ..., x_k(t)$ has the *property of incompressibility* if for any linear combination

$$y(t) = \sum_{i=1}^{k} c_i x_i(t)$$

we have

$$\lambda(y) = \max_{\substack{i \in \{1, ..., k\} \\ c_i \neq 0}} \lambda(x_i).$$

Note that, any set of vector-functions with different Lyapunov exponents obviously has the property of the incompressibility.

Similar to the theory of ODEs, we can easily prove the following result.

Theorem 3.2. *A FSM $X_r = (x_1(t), ..., x_r(t))$ of* (3.1) *is normal if and only if the system $x_1(t), ..., x_r(t)$ has the property of incompressibility.*

Note that in all normal FSMs the number n_s of the solutions with Lyapunov exponent equal $\lambda_s(A, B)$ $(s = 1, ..., d)$ is identical. This number n_s is called *multiplicity* of the exponent $\lambda_s(A, B)$ of (3.1). Furthermore, any normal FSM $X_r(t)$ of (3.1) realizes the Lyapunov spectrum of (3.1): each $\lambda_i(A, B)$ $(i = 1, ..., d)$ equals $\lambda(x_k)$ for some $k = 1, 2, ..., r$.

It is not difficult to prove (like in the theory of ODEs) that for any FSM X_r of (3.1) we have

$$\sigma_{X_r} \geq \sum_{i=1}^{d} n_i \lambda_i(A, B),$$

and X_r is normal if and only if

$$\sigma_{X_r} = \sum_{i=1}^{d} n_i \lambda_i(A, B).$$

Now we come to the main result of this paper on *Lyapunov's inequality* for a DAE. Note that in case of DAEs, besides the coefficient matrices, projector functions appear in the formula of the Lyapunov's inequality. The presence of projectors is natural since they have appeared already in the definition of a solution of a DAE. In some cases of the next section we may be able to get Lyapunov's inequality without the presence of projectors.

Theorem 3.3. *Assume that (3.1) is transferable and the coefficient matrices $A(t), B(t)$ and matrix $G^{-1}(t)$ are bounded on J. Assume further that the nullspace $N(t)$ of $A(t)$ does not depend on t. Then we have the following* Lyapunov's inequality

$$\sum_{i=1}^{d} n_i \lambda_i(A, B) \geq \overline{\lim_{t\to\infty}} \frac{1}{t} \int_{t_0}^{t} \operatorname{tr}(P' P_s - P G^{-1} B)(t_1) dt_1. \tag{3.4}$$

Proof. Let

$$X_r(t) = \big(x_1(t), ..., x_r(t)\big)$$

be a normal FSM of (3.1), and

$$U(t) = \big(u_1(t), ..., u_r(t),\ \ u_{r+1}(t), ..., u_m(t)\big)$$

be a corresponding FSM of the corresponding ODE (2.6), i.e. we have $x_i(t) = P_s(t) u_i(t)$ for $i = 1, ..., r$. Note that the correspondence between $X_r(t)$ and $U_t(t)$ is restricted to the first r vector-functions and implies

$$u_i(t) \in \text{ im } P(t) \quad \text{for} \quad i = 1, ..., r.$$

Since im $P(t_0) \oplus$ Ker $P(t_0) = \mathbb{R}^m$, we can choose

$$u_i(t_0) \in \text{ im } P(t_0) \quad \text{for} \quad i = 1, ..., r,$$
$$u_j(t_0) \in \text{ Ker } P(t_0) \quad \text{for} \quad j = r+1, ..., m.$$

We show that with this choice of initial values the solution $u_j(t)$, $j = r+1, \ldots, m$, are constant.

We have

$$G^{-1}A = G^{-1}A(I-Q) = G^{-1}(A+BQ)(I-Q) = I - Q,$$
$$G^{-1}BQ = G^{-1}(A + BQ - A) = I - G^{-1}A = I - (I-Q) = Q,$$
$$Q = G^{-1}B(I-P) = G^{-1}B - G^{-1}BP,$$

hence $G^{-1}B = Q + G^{-1}BP$, which implies, for $j \in \{r+1, ..., m\}$,

$$\begin{aligned}(Pu_j)' &= P'u_j + Pu_j' = P'u_j + P(P'P_s - PG^{-1}B)u_j \\ &= (P' + PP'P_s - P'P_s)u_j + (P'P_s - PG^{-1}B)u_j \\ &= (P' - QP'P_s)u_j + (P'P_s - PG^{-1}B)u_j \\ &= (P' - P'PP_s)u_j + \big[P'P_sP - P(Q + G^{-1}BP)\big]u_j \\ &= (P' - P'P)u_j + (P'P_s - PG^{-1}B)Pu_j \\ &= P'Qu_j + (P'P_s - PG^{-1}B)Pu_j.\end{aligned}$$

Let $Q_\perp = I - P_\perp$ be the orthogonal projector onto the nullspace of $A(t)$. Then $P_\perp$ is independent of t and therefore, we have

$$P'Q = (PP_\perp)'(Q_\perp Q) = P'P_\perp Q_\perp Q = 0.$$

Consequently,

$$(Pu_j)' = (P'P_s - PG^{-1}B)Pu_j,$$

and since $P(t_0)u_j(t_0) = 0$, we have $Pu_j \equiv 0$, $t \in J$ for $j = r+1, ..., m$. On the other hand, for $j = r+1, ..., m$, we have

$$\begin{aligned}(Qu_j)' &= Q'u_j + Qu_j' = Q'u_j + Q(P'P_s - PG^{-1}B)u_j \\ &= Q'u_j + QP'P_su_j \\ &= Q'u_j + P'PP_su_j \\ &= (Q' + P'P)u_j = (Q' - Q'P)u_j = Q'Qu_j = -P'Qu_j = 0,\end{aligned}$$

hence $Qu_j = c_j$ for all $t \in J$.

Thus, $u_j(t) = P(t)u_j(t) + Q(t)u_j(t) = c_j = \text{const}$ for $j = r+1, ..., m$, which implies that

$$U(t) = \big(u_1(t), ..., u_r(t), c_{r+1}, ..., c_m\big).$$

By Theorem 2.1, $\lambda(x_i) = \lambda(u_i)$ for $i = 1, ..., r$. On the other hand, $\lambda(u_j) = \lambda(c_j) = 0$ for $j = r+1, ..., m$. Consequently, since $X_r(t)$ is normal, by using the Lyapunov's inequality for the ODE (2.6) we get

$$\begin{aligned}\sum_{i=1}^{d} n_i\lambda_i(A,B) = \sum_{i=1}^{r}\lambda(x_i) &= \sum_{i=1}^{m}\lambda(u_i) \\ &\geq \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t} \text{tr } (P'P_s - PG^{-1}B)(t_1)dt_1.\end{aligned}$$

□

Corollary 3.1. *If the coefficients of* (3.1) *have the form*

$$A(t) = \begin{pmatrix} W(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix},$$

where $W(t)$ is a nonsingular square matrix of order r, $B_{22}(t)$ is an invertible matrix of order $m-r$ and $B_{22}^{-1}(t)$, $B(t)$, $W^{-1}(t)$ are bounded on J, then Lyapunov's inequality of (3.1) *is of the following form*

$$\sum_{i=1}^{d} n_i\lambda_i(A,B) \geq \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t} \operatorname{tr}\,(W^{-1}B_{12}B_{22}^{-1}B_{21} - W^{-1}B_{11})(t_1)dt_1.$$

Proof. Clearly, $N(t) := \operatorname{Ker} A(t) = \left\{ \begin{pmatrix} 0 \\ z_2 \end{pmatrix} : z_2 \in \mathbb{R}^{m-r} \right\}$ is independent of t, and $Q(t) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. We have

$$G(t) = A(t) + B(t)Q(t) = \begin{pmatrix} W(t) & B_{12}(t) \\ 0 & B_{22}(t) \end{pmatrix},$$
$$G^{-1}(t) = \begin{pmatrix} W^{-1}(t) & -W^{-1}(t)B_{12}(t)B_{22}^{-1}(t) \\ 0 & B_{22}^{-1}(t) \end{pmatrix},$$

$$\begin{aligned} P'(t)P_s(t) - P(t)G^{-1}(t)B(t) &= -P(t)G^{-1}(t)B(t) \\ &= \begin{pmatrix} -W^{-1}(t)B_{11}(t) + W^{-1}(t)B_{12}(t)B_{22}^{-1}(t)B_{21}(t) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

By Theorem 3.3, we have

$$\sum_{i=1}^{d} n_i\lambda_i(A,B) \geq \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t} \operatorname{tr}\,(W^{-1}B_{12}B_{22}^{-1}B_{21} - W^{-1}B_{11})(t_1)dt_1.$$

□

Remark 3.1

(i) From the proof of Theorem 3.3 it is clear that Theorem 3.3 remains true if we replace the condition that the nullspace $N(t)$ is independent of t by the (weaker) condition that $P'Q = 0$.

(ii) If (3.1) has the Kronecker normal form with index 1, i.e.,

$$A(t) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_{11}(t) & 0 \\ 0 & I_{m-r} \end{pmatrix},$$

then the Lyapunov's inequality of (3.1) has the form

$$\sum_{i=1}^{d} n_i\lambda_i(A,B) \geq \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t} -\operatorname{tr}\,B_{11}(t_1)dt_1.$$

(iii) If the assumptions of Theorem 3.3 hold with $Q = Q_\perp$ (the orthogonal projector), then the Lyapunov's inequality of (3.1) has the form

$$\sum_{i=1}^{d} n_i\lambda_i(A,B) \geq \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t} -\mathrm{tr}(PG^{-1}B)(t_1)dt_1.$$

4. Some explicit estimates of sum of Lyapunov exponents of a DAE

In this section, using Lyapunov's inequality we shall derive some estimates for sum of Lyapunov exponents of a DAE, which can be computed explicitely via coefficients of the DAE (no projector presented).

Theorem 4.1. *Suppose we are given a transferable DAE on J*

$$A(t)x' + B(t)x = 0, \tag{4.1}$$

where the coefficients $A(t)$, $B(t) \in C^1(J)$ have the following block form

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix},$$

here $A_{11}(t)$ and $B_{11}(t)$ are square matrices of order r = rank $A(t)$, A_{11} and $B_{22} - A_{21}A_{11}^{-1}B_{12}$ are nonsingular matrices. Assume further that the nullspace $N(t)$ of $A(t)$ is independent of t and the canonical projector $Q_s(t)$ onto $N(t)$ along $S(t)$ is bounded on J. Then, Lyapunov's inequality for the DAE (4.1) *has the form*

$$\sum_{i=1}^{d} n_i\lambda_i(A,B) \geq \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t} tr(-C(t_1)dt_1, \tag{4.2}$$

where

$$C := \left[B_{11} - B_{12}\left(B_{22} - A_{21}A_{11}^{-1}B_{12}\right)^{-1}\left(B_{21} - A_{21}A_{11}^{-1}B_{11}\right)\right]$$
$$\times \left[A_{11} - A_{12}\left(B_{22} - A_{21}A_{11}^{-1}B_{12}\right)^{-1}\left(B_{21} - A_{21}A_{11}^{-1}B_{11}\right)\right]^{-1}.$$

Proof. First we will transfer our system to a standard Kronecker form, and then apply the arguments of the proof of Theorem 3.3.

Put

$$L_1 = \begin{pmatrix} I_r & 0 \\ -A_{21}A_{11}^{-1} & I_{m-r} \end{pmatrix}.$$

Note that $\overline{A}_{22}(t) := A_{22}(t) - A_{21}(t)A_{11}^{-1}(t)A_{12}(t) = 0$ (see [8, p.33]). (This is because, if conversely $\overline{A}_{22}(t_0) \neq 0$ for some $t_0 \in J$, then rank $L(t_0)A(t_0) > r$ = rank $(A(t))$. Therefore,

$$L_1A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}, \quad L_1B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} - A_{21}A_{11}^{-1}B_{11} & B_{22} - A_{21}A_{11}^{-1}B_{12} \end{pmatrix}.$$

Put $H_{11} = B_{11}$, $H_{12} = B_{12}$, $H_{21} = B_{21} - A_{21}A_{11}^{-1}B_{11}, H_{22} = B_{22} - A_{21}A_{11}^{-1}B_{12}$. By assumption of the theorem, $H_{22} = B_{22} - A_{21}A_{11}^{-1}B_{12}$ is nonsingular, hence we can define

$$R_1 = \begin{pmatrix} I & 0 \\ -H_{22}^{-1}H_{21} & H_{22}^{-1} \end{pmatrix}.$$

We have

$$L_1AR_1 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -H_{22}^{-1}H_{21} & H_{22}^{-1} \end{pmatrix} = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & 0 \end{pmatrix},$$
$$L_1BR_1 = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -H_{22}^{-1}H_{21} & H_{22}^{-1} \end{pmatrix} = \begin{pmatrix} \overline{B}_{11} & \overline{B}_{12} \\ 0 & I \end{pmatrix},$$

where

$$\overline{A}_{11} = A_{11} - A_{12}H_{22}^{-1}H_{21}, \quad \overline{A}_{12} = A_{12}H_{22}^{-1},$$
$$\overline{B}_{11} = H_{11} - H_{12}H_{22}^{-1}H_{21}, \quad \overline{B}_{12} = H_{12}H_{22}^{-1}.$$

Now put

$$L_0 := \begin{pmatrix} I & -\overline{B}_{12} \\ 0 & I \end{pmatrix},$$

and $W_1 := L_0L_1$. Then we have

$$W_1AR_1 = L_0L_1AR_1 = \begin{pmatrix} I & -\overline{B}_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & 0 \end{pmatrix},$$
$$W_1BR_1 = L_0L_1BR_1 = \begin{pmatrix} I & -\overline{B}_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} \overline{B}_{11} & \overline{B}_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} \overline{B}_{11} & 0 \\ 0 & I \end{pmatrix}.$$

Since (4.1) is a transferable DAE on J, the matrix pencil $\{A, B\}$ is regular with index 1 (see [3, p. 198] and [8. p. 53]), i.e.

$$\deg\big(\det(\lambda A + B)\big) = \operatorname{rank} A = r.$$

On the other hand,

$$\det(\lambda A + B) = \det\big(W_1^{-1}R_1^{-1}\big)\det(\lambda\overline{A}_{11} + \overline{B}_{11}).$$

This implies $\det \overline{A}_{11} \neq 0$ for all $t \in J$.

Now put

$$L_2 = \begin{pmatrix} I & \overline{B}_{11}\overline{A}_{11}^{-1}\overline{A}_{12} \\ 0 & I \end{pmatrix}, \quad R_2 = \begin{pmatrix} \overline{A}_{11}^{-1} & -\overline{A}_{11}^{-1}\overline{A}_{12} \\ 0 & I \end{pmatrix}.$$

We have

$$L_2W_1AR_1R_2 = \begin{pmatrix} I & \overline{B}_{11}\overline{A}_{11}^{-1}\overline{A}_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} \overline{A}_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{A}_{11}^{-1} & -A_{11}^{-1}\overline{A}_{12} \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

$$L_2W_1BR_1R_2 = \begin{pmatrix} I & \overline{B}_{11}\overline{A}_{11}^{-1}\overline{A}_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} \overline{B}_{11} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \overline{A}_{11}^{-1} & -\overline{A}_{11}^{-1}\overline{A}_{12} \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} \overline{B}_{11}\overline{A}_{11}^{-1} & 0 \\ 0 & I \end{pmatrix}.$$

Hence, with the notation

$$W := L_2W_1 = L_2L_0L_1 = \begin{pmatrix} I & \overline{B}_{11}\overline{A}_{11}^{-1}\overline{A}_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -\overline{B}_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}$$

$$R := R_1R_2 = \begin{pmatrix} I & 0 \\ -H_{22}^{-1}H_{21} & H_{22}^{-1} \end{pmatrix} \begin{pmatrix} \overline{A}_{11}^{-1} & -\overline{A}_{11}^{-1}\overline{A}_{12} \\ 0 & I \end{pmatrix}$$

$$\begin{aligned} C := \overline{B}_{11}\overline{A}_{11}^{-1} &= (H_{11} - H_{12}H_{22}^{-1}H_{21})(A_{11} - A_{12}H_{22}^{-1}H_{21})^{-1} \\ &= \left[B_{11} - B_{12}(B_{22} - A_{21}A_{11}^{-1}B_{12})^{-1}(B_{21} - A_{21}A_{11}^{-1}B_{11})\right] \times \\ &\quad \times \left[A_{11} - A_{12}(B_{22} - A_{21}A_{11}^{-1}B_{12})^{-1}(B_{21} - A_{21}A_{11}^{-1}B_{11})\right]^{-1}, \end{aligned}$$

we have

$$A = W^{-1}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} R^{-1}, \quad B = W^{-1}\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} R^{-1}.$$

Consider the projector function

$$Q(t) = R(t)\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} R^{-1}(t)$$

onto $N(t) := \ker A(t)$. It is evident $Q(t) \in C^1(J)$ since $A, B \in C^1(J)$.

We have

$$\begin{aligned} P = I - Q &= R\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} R^{-1} \\ G = A + BQ & \\ &= W^{-1}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} R^{-1} + W^{-1}\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} R^{-1}R\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} R^{-1} \\ &= (RW)^{-1} \\ G^{-1} &= RW, \\ Q_s = QG^{-1}B &= R\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} R^{-1}RWW^{-1}\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} R^{-1} \\ &= R\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} R^{-1} = Q. \end{aligned}$$

Therefore

$$
\begin{aligned}
&P'P_s - PG^{-1}B = \\
&= \left[R\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1}\right]' R\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1} - R\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1}RWW^{-1}\begin{pmatrix} C & 0\\ 0 & I\end{pmatrix}R^{-1}\\
&= R'\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1}R\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1} - R\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1}R'R^{-1}R\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1}\\
&\quad - R\begin{pmatrix} C & 0\\ 0 & 0\end{pmatrix}R^{-1}\\
&= R'\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1} - R\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1}R'\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1} - R\begin{pmatrix} C & 0\\ 0 & 0\end{pmatrix}R^{-1}\\
&= \left[I - R\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1}\right]R'\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}R^{-1} - R\begin{pmatrix} C & 0\\ 0 & 0\end{pmatrix}R^{-1}\\
&= R\begin{pmatrix} 0 & 0\\ M & 0\end{pmatrix}R^{-1} - R\begin{pmatrix} C & 0\\ 0 & 0\end{pmatrix}R^{-1}\\
&= R\begin{pmatrix} -C & 0\\ M & 0\end{pmatrix}R^{-1},
\end{aligned}
$$

where $\begin{pmatrix} 0 & 0\\ 0 & I\end{pmatrix}R^{-1}R'\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & 0\\ M & 0\end{pmatrix}$.

On the other hand,

$$
\operatorname{tr}\left[R\begin{pmatrix} -C & 0\\ M & 0\end{pmatrix}R^{-1}\right] = \operatorname{tr}\left[RR^{-1}\begin{pmatrix} -C & 0\\ M & 0\end{pmatrix}\right] = \operatorname{tr}\begin{pmatrix} -C & 0\\ M & 0\end{pmatrix} = \operatorname{tr}(-C).
$$

By assumption of the theorem, we have $\lambda(Q) = \lambda(Q_s) = 0$. Therefore, the Lyapunov exponent of any nontrivial solution $x(t)$ of (4.1) equals the Lyapunov exponent of the corresponding solution of the corresponding ODE of (4.1). Now we note that the condition on boundedness of A, B, G^{-1} in Theorem 3.3 is needed for proving equality of the Lyapunov exponents of $x(t)$ and of the corresponding solution of the corresponding ODE, and here we was able to prove the equality directly from the assumption of our theorem. Because the nullspace $N(t)$ of $A(t)$ is independent of t, using arguments similar to those of the proof of Theorem 3.3 we get

$$
\begin{aligned}
\sum_{i=1}^{d} n_i\lambda_i(A,B) &\geq \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t} \operatorname{tr}(P'P_s - PG^{-1}B)(t_1)dt_1\\
&= \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t} \operatorname{tr}(-C(t_1))dt_1.
\end{aligned}
$$

□

Theorem 4.2. *Suppose we are given a transferable DAE*

$$A(t)x' + B(t)x = 0, \quad t \in J, \tag{4.3}$$

where the coefficients $A(t), B(t) \in C^1(J)$ are bounded on J and have the block form

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix},$$

with $A_{11}(t)$ being an invertible square matrix of order r=rank $A(t)$ and the matrices A_{11}^{-1}, $A', \overline{B}_{22}^{-1}$ are bounded on J. Then

$$\begin{aligned}\sum_{i=1}^{d} n_i\lambda_i(A,B) \geq \overline{\lim_{t\to\infty}}\frac{1}{t}\int_{t_0}^{t} \Big\{ tr\Big[&\big(B_{12} - B_{11}A_{11}^{-1}A_{12} + A'_{11}A_{11}^{-1}A_{12} - A'_{12}\big) \\ &\big(B_{22} - A_{21}A_{11}^{-1}A_{12} - B_{21}A_{11}^{-1}A_{12} + A_{21}A_{11}^{-1}B_{11}A_{11}^{-1}A_{12}\big)^{-1} \\ &\big(B_{21}A_{11}^{-1} - A_{21}A_{11}^{-1}B_{11}A_{11}^{-1}\big)\Big] \\ &- \ tr\big(B_{11}A_{11}^{-1} - A'_{11}A_{11}^{-1}\big)\Big\}(t_1)dt_1,\end{aligned} \tag{4.4}$$

where $\overline{B}_{22} = B_{22} - A_{21}A_{11}^{-1}A_{12} - B_{21}A_{11}^{-1}A_{12} + A_{21}A_{11}^{-1}B_{11}A_{11}^{-1}A_{12}$.

Proof. Multiplying both parts of equation (4.3) from the left by

$$w(t) = \begin{pmatrix} I_r & o \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}$$

we get

$$w(t)A(t)x' + w(t)B(t)x = 0. \tag{4.5}$$

Put $x(t) = R(t)y(t)$, where

$$R := \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & I \end{pmatrix}.$$

Then (4.5) becomes

$$w(t)A(t)R(t)y' + \big[w(t)B(t)R(t) + w(t)A(t)R'(t)\big]y = 0.$$

Since r = rank $A(t)$ = rank $A_{11}(t)$ and A_{11} is nonsingular, we have $A_{22} - A_{21}A_{11}^{-1}A_{12} = 0$ (see [8, p.53]). Therefore, the last equation is equivalent to the system

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} y' + \begin{pmatrix} \overline{B}_{11}(t) & \overline{B}_{12}(t) \\ \overline{B}_{21}(t) & \overline{B}_{22}(t) \end{pmatrix} y = 0, \tag{4.6}$$

where

$$\begin{aligned}
\overline{B}_{11} &= B_{11}A_{11}^{-1} - A'_{11}A_{11}^{-1},\\
\overline{B}_{12} &= B_{12} - B_{11}A_{11}^{-1}A_{12} + A'_{11}A_{11}^{-1}A_{12} - A'_{12},\\
\overline{B}_{21} &= B_{21}A_{11}^{-1} - A_{21}A_{11}^{-1}B_{11}A_{11}^{-1},\\
\overline{B}_{22} &= B_{22} - A_{21}A_{11}^{-1}A_{12} - B_{21}A_{11}^{-1}A_{12} + A_{21}A_{11}^{-1}B_{11}A_{11}^{-1}A_{12}.
\end{aligned}$$

Since, this equation has index 1, the matrix $\overline{B}_{22}$ must be invertible on J.

Because $A(t)$, $B(t)$, $A'(t)$, $A_{11}^{-1}(t)$ are bounded on J, $R(t)$, $R^{-1}(t)$ and the matrices $\overline{B}_{11}$, $\overline{B}_{12}, \overline{B}_{21}$, $\overline{B}_{12}$ are bounded on J, hence $\lambda(x) = \lambda(y)$.

Therefore, from Corollary 3.1 it follows

$$\sum_{i=1}^{r} \lambda(x_i) = \sum_{i=1}^{r} \lambda(y_i) \geq \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{tr}\big(\overline{B}_{12}\overline{B}_{22}^{-1}\overline{B}_{21} - \overline{B}_{11}\big)(t_1)dt_1.$$

Inequality (4.4) follows immediately from this inequality. □

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