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DIRECTIONAL KUHN-TUCKER CONDITION AND DUALITY FOR QUASIDIFFERENTIABLE PROGRAMS

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Dedicated to Professor Pham Huu Sach on the occasion of his sixtieth birthday.

Abstract. In this paper a notion called "directional Kuhn-Tucker condition" for quasidifferentiable programs with inequality constraints is introduced. This is a version of the Lagrange multiplier rule where the Lagrange multipliers depend on the directions. It is proved that this condition is a necessary condition for optimality. Under the assumption that the problem is directionally η-invex, it is also a sufficent condition for optimality. Some results on duality of the class of problems are obtained.

1. Introduction

Quasidifferentiable functions whose directional derivatives are representable as the difference of two sublinear functions, were introduced by V. F. Demyanov and A. M. Rubinov in 1980 [5]. Since then the quasidifferential calculus has been developed (see [4], [6], [10], [14], [15], [18], [19],...) and various optimality conditions for unconstrained and constrained quasidifferentiable problems have been obtained. Most of the optimality conditions are of geometric forms (see [7], [9], [12], [19], [21],...).

The aim of this paper is to study a so-called "directional Kuhn-Tucker condition" which is a version of the Lagrange multiplier rule where the Lagrange multipliers depend on the directions. It is shown that this is not only a necessary but also a sufficient condition for optimality in the case where the problem is directionally η-invex. We will also present some duality results.

The paper is organized as follows. In the remainder of this section we will recall the notion of quasidifferentiable function and present the problem that we will deal with throughout the paper. In Section 2, a directional Kuhn-Tucker condition is introduced. It is proved that the condition is weaker than the usual Kuhn-Tucker condition but stronger than the generalized Kuhn-Tucker condition proposed in [12], [21]. In Section 3, it is proved that the directional Kuhn-Tucker

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condition is a necessary and also a sufficient (under an invexity assumption) optimality condition for quasidifferentiable problems with inequality constraints. As a consequence, we prove that for convex (as well as differentiable, or locally Lipschitz and regular) problems, the Langrange multipliers can be chosen to be constants. This shows that the directional Kuhn-Tucker condition can be considered as a generalization of the Lagrange multiplier rule to nonconvex problems. An example is given to show that in general, the dependence of the Lagrange multiplires on the directions can not be ignored (i.e., the multipliers can not be chosen to be constants). Section 4 is devoted to the dual problems constructed by means of directional Kuhn-Tucker condition and duality theorems.

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be quasidifferentiable at $x_0 \in \mathbb{R}^n$ if the directional derivative

$$f'(x_0, d) := \lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda}$$

is well defined for all $d \in \mathbb{R}^n$ and there exists a pair of convex, compact subsets $\underline{\partial} f(x_0)$ and $\overline{\partial} f(x_0)$ of $\mathbb{R}^n$ such that

$$f'(x_0, d) = \max_{\xi \in \underline{\partial} f(x_0)} \langle d, \xi \rangle + \min_{\xi \in \overline{\partial} f(x_0)} \langle d, \xi \rangle, \quad \forall d \in \mathbb{R}^n.$$

From the previous equality it follows that $f'(x_0, .)$ is represented as the difference of two sublinear functions. The pair

$$Df(x_0) := [\underline{\partial} f(x_0), \overline{\partial} f(x_0)]$$

is called a quasidifferential of f at x_0. The sets $\underline{\partial} f(x_0)$ and $\overline{\partial} f(x_0)$ are called the subdifferential and the superdifferential, respectively. The class of quasidifferentiable functions contains convex, concave, differentiable, DC-functions. It even contains functions which are not locally Lipschitz (see [9], [15]).

Let f, g_i $(i = 1, 2, \ldots, m)$ be functions defined on $\mathbb{R}^n$. Consider the following mathematical programming problem (P)

$$\min f(x) \tag{1.1}$$

subject to

$$g_i(x) \leq 0, \quad i = 1, 2, \ldots, m. \tag{1.2}$$

Let S be the set of all the feasible points of (P) (i.e. points of $\mathbb{R}^n$ that satisfy (1.2)) and x_0 be a point of S. Set $I(x_0) = \{i \mid g_i(x_0) = 0\}$.

From now on we tacitly assume that f and g_i, $i = 1, 2, \ldots, m$, are quasidifferentiable at x_0, and g_i is continuous at x_0 for all $i \notin I(x_0)$.

2. Directional Kuhn-Tucker condition for (P)

In this section we introduce a directional version of the Kuhn-Tucker condition and discuss its relation with other types of the Kuhn-Tucker condition proposed in the literature.

Definition 2.1. A point $x_0 \in S$ is said to be a *directional Kuhn-Tucker point* of (P) (or Problem (P) satisfies the *directional Kuhn-Tucker condition* at x_0) if for every $r \in \mathbb{R}^n$ there exists $\lambda(r) \in \mathbb{R}^m_+$, $\lambda(r) = (\lambda_1(r), \lambda_2(r), \ldots, \lambda_m(r))$, such that the following conditions hold

$$f'(x_0, r) + \sum_{i=1}^{m} \lambda_i(r) g_i'(x_0, r) \geq 0, \tag{2.1a}$$

$$\lambda_i(r) g_i(x_0) = 0 \quad \text{for all} \quad i = 1, 2, \ldots, m. \tag{2.1b}$$

Definition 2.2. ([12], [17], [21]) A point $x_0 \in S$ is said to be a *Kuhn-Tucker point* of (P) if

$$-\overline{\partial} f(x_0) \subset \bigcap_{\substack{\overline{w}_i \in \overline{\partial} g_i(x_0) \\ i \in I(x_0)}} \Big[\underline{\partial} f(x_0) + \sum_{i \in I(x_0)} \text{cone}\, \big(\underline{\partial} g_i(x_0) + \overline{w}_i \big) \Big], \tag{2.2}$$

where cone $A = \{tv \mid v \in A,\ t \in \mathbb{R}_+\}$.

Definition 2.3. ([12], [17], [21]) A point $x_0 \in S$ is said to be a *generalized Kuhn-Tucker point* of (P) if

$$-\overline{\partial} f(x_0) \subset \bigcap_{\substack{\overline{w}_i \in \overline{\partial} g_i(x_0) \\ i \in I(x_0)}} \left[\underline{\partial} f(x_0) + \text{ cl} \left\{ \sum_{i \in I(x_0)} \overline{\text{cone}} \big(\underline{\partial} g_i(x_0) + \overline{w}_i \big) \right\} \right], \tag{2.3}$$

where $\overline{\text{cone}}\, A$ denotes the topological closure of cone A and $\text{cl}\, \Omega$ denotes the topological closure of a set $\Omega \subset \mathbb{R}^n$.

Relations between the three types of Kuhn-Tucker points will be discussed in the following theorem.

Theorem 2.1. (i) *If x_0 is a Kuhn-Tucker point of* (P) *then it is a directional Kuhn-Tucker point,*

(ii) *If x_0 is a directional Kuhn-Tucker point then it is also a generalized Kuhn-Tucker point.*

Proof. (i) Suppose that x_0 is a Kuhn-Tucker point, that is (2.2) holds, and $r \in \mathbb{R}^n$ is an arbitrary point. We will search for a vector $\lambda(r) = (\lambda_1(r), \lambda_2(r), ..., \lambda_m(r)) \in \mathbb{R}^m_+$ satisfying (2.1a) and (2.1b).

Since $\overline{\partial} f(x_0)$, $\overline{\partial} g_i(x_0)$, $i \in I(x_0)$, are compact and nonempty, we can find

$$\overline{v} \in \operatorname*{argmin}_{\xi \in \overline{\partial} f(x_0)} \langle r, \xi \rangle,$$

$$\overline{v}_i \in \operatorname*{argmin}_{\xi_i \in \overline{\partial} g_i(x_0)} \langle r, \xi_i \rangle, \quad i \in I(x_0).$$

It follows from (2.2) that

$$0 \in \underline{\partial} f(x_0) + \overline{v} + \sum_{i \in I(x_0)} \text{cone } \big(\underline{\partial} g_i(x_0) + \overline{v}_i\big).$$

This implies that there exist $a \in \underline{\partial} f(x_0)$, $b_i \in \underline{\partial} g_i(x_0)$, and $\mu_i \geq 0$, $i \in I(x_0)$, such that

$$0 = a + \overline{v} + \sum_{i \in I(x_0)} \mu_i (b_i + \overline{v}_i).$$

Therefore

$$\begin{aligned}
f'(x_0, r) + \sum_{i=1}^{m} \mu_i g_i'(x_0, r) &= \max_{v \in \underline{\partial} f(x_0)} \langle r, v \rangle + \min_{w \in \overline{\partial} f(x_0)} \langle r, w \rangle + \\
&\quad + \sum_{i \in I(x_0)} \mu_i \big[\max_{\xi_i \in \underline{\partial} g_i(x_0)} \langle r, \xi_i \rangle + \min_{\eta_i \in \overline{\partial} g_i(x_0)} \langle r, \eta_i \rangle \big] \\
&= \max_{v \in \underline{\partial} f(x_0)} \langle r, v \rangle + \langle r, \overline{v} \rangle + \\
&\quad + \sum_{i \in I(x_0)} \mu_i \big[\max_{\xi_i \in \underline{\partial} g_i(x_0)} \langle r, \xi_i \rangle + \langle r, \overline{v}_i \rangle \big] \\
&\geq \langle r, a \rangle + \langle r, \overline{v} \rangle + \sum_{i \in I(x_0)} \mu_i \big[\langle r, b_i \rangle + \langle r, \overline{v}_i \rangle \big] \\
&\geq \langle r, a + \overline{v} + \sum_{i \in I(x_0)} \mu_i (b_i + \overline{v}_i) \rangle \\
&\geq 0.
\end{aligned}$$

Set $\lambda_i(r) = \mu_i$ for $i \in I(x_0)$ and $\lambda_j(r) = 0$ for $j \notin I(x_0)$. Then $\lambda(r) = (\lambda_1(r), ..., \lambda_m(r))$ satisfies (2.1a) and (2.1b), which proves that x_0 is a directional Kuhn-Tucker point of (P).

(ii) Assume that x_0 is a directional Kuhn-Tucker point of (P). We will prove that for all $\overline{w}_i \in \overline{\partial} g_i(x_0)$, $i \in I(x_0)$, the following inclusion holds

$$-\overline{\partial} f(x_0) \subset \underline{\partial} f(x_0) + \text{ cl } \Big\{ \sum_{i \in I(x_0)} \overline{\text{cone}} \big(\underline{\partial} g_i(x_0) + \overline{w}_i\big) \Big\}. \tag{2.4}$$

To obtain a contradiction, suppose that there exists $a \in \overline{\partial} f(x_0)$ such that

$$-a \notin \underline{\partial} f(x_0) + \text{ cl } \Big\{ \sum_{i \in I(x_0)} \overline{\text{cone}} \big(\underline{\partial} g_i(x_0) + \overline{w}_i\big) \Big\},$$

or equivalently,

$$0 \notin \underline{\partial} f(x_0) + a + \text{ cl } \Big\{ \sum_{i \in I(x_0)} \overline{\text{cone}} \big(\underline{\partial} g_i(x_0) + \overline{w}_i\big) \Big\} =: M. \tag{2.5}$$

Since $\underline{\partial} f(x_0) + a$ is a nonempty convex, compact set and

$$\mathrm{cl}\left\{ \sum_{i \in I(x_0)} \overline{\mathrm{cone}}\big(\underline{\partial} g_i(x_0) + \overline{w}_i\big)\right\}$$

is closed and convex, the set M in the right hand side of (2.5) is closed and convex. It follows from (2.5) and the separation theorem that there exists $\overline{r} \in \mathbb{R}^n$ such that

$$0 > \langle \overline{r}, c\rangle \quad \text{for all } c \in M,$$

hence

$$0 > \langle \overline{r}, u + a + \sum_{i \in I(x_0)} \mu_i (v_i + \overline{w}_i)\rangle \tag{2.6}$$

for all $u \in \underline{\partial} f(x_0)$, $v_i \in \underline{\partial} g_i(x_0)$, $i \in I(x_0)$, and for all $\mu = (\mu_i)_{i \in I(x_0)} \geq 0$.

Let us take

$$\underline{u} \in \operatorname*{argmax}_{\xi \in \underline{\partial} f(x_0)} \langle \overline{r}, \xi\rangle, \quad \underline{v}_i \in \operatorname*{argmax}_{\eta_i \in \underline{\partial} g_i(x_0)} \langle \overline{r}, \eta_i\rangle, \quad i \in I(x_0).$$

Since $\lambda(\overline{r}) = (\lambda_1(\overline{r}), \lambda_2(\overline{r}), \dots, \lambda_m(\overline{r})) \in \mathbb{R}^m_+$, it follows from (2.6) that

$$\begin{aligned}
0 &> \langle \overline{r}, \underline{u} + a + \sum_{i \in I(x_0)} \lambda_i(\overline{r})(\underline{v}_i + \overline{w}_i)\rangle = \\
&> \langle \overline{r}, \underline{u}\rangle + \langle \overline{r}, a\rangle + \sum_{i \in I(x_0)} \lambda_i(\overline{r})\big[\langle \overline{r}, \underline{v}_i\rangle + \langle \overline{r}, \overline{w}_i\rangle\big] \\
&\geq \max_{\xi \in \underline{\partial} f(x_0)} \langle \overline{r}, \xi\rangle + \min_{\xi \in \overline{\partial} f(x_0)} \langle \overline{r}, \xi\rangle + \sum_{i \in I(x_0)} \lambda_i(\overline{r})\Big[\max_{\eta_i \in \underline{\partial} g_i(x_0)} \langle \overline{r}, \eta_i\rangle + \min_{w_i \in \overline{\partial} g_i(x_0)} \langle \overline{r}, w_i\rangle\Big] \\
&\geq f'(x_0, \overline{r}) + \sum_{i \in I(x_0)} \lambda_i(\overline{r}) g_i'(x_0, \overline{r}) = \\
&\geq f'(x_0, \overline{r}) + \sum_{i=1}^{m} \lambda_i(\overline{r}) g_i'(x_0, \overline{r}),
\end{aligned}$$

which contradicts (2.1a). Thus (2.4) is proved and the proof is complete. □

Remark 2.1.

(i) Note that in Definition 2.1 the *Lagrange multiplier* $\lambda(.)$ depends on $r \in \mathbb{R}^n$. As we can see in the proof (part (i)) of Theorem 2.1, the reason for this is the existence of nonzero vectors in $\overline{\partial} f(x_0)$ and $\overline{\partial} g_i(x_0)$, $i \in I(x_0)$. In the case where these sets reduce to $\{0\}$ (then the directional derivatives turn to be convex with respect to the directions), $\lambda(.)$ can be chosen as a constant function (see Corollaries 3.2, 3.3). It is the situation where f, g_i are differentiable or convex or locally Lipschitz or, more general, subdifferentiable at x_0 in the sense of Pshenichnyi (see [18]). So the "directional Kuhn-Tucker condition" can be considered as a generalization of the Lagrange multiplier rule to nonconvex problems. The fact that the Langange multipliers can be not constant was noticed by H. Frankowska.

The Lagrange multipliers depend on the specific choice of each nonzero vector $w = (w_0, w_1, \dots, w_m)$ with $w_0 \in \overline{\partial} f(x_0)$ and $w_i \in \overline{\partial} g_i(x_0)$, $i = 1, 2, 3, \dots, m$ (see [8], [12]). It is shown in Example 3.2 (see also the Remarks 3.1, 3.2, and 3.3) that the notion of directional Kuhn-Tucker point is really weaker than that of Kuhn-Tucker point and that for some problem (even very simple), the multiplier $\lambda(.)$ can not be chosen to be constant.

(ii) To verify whether a point $x_0 \in S$ is a directional Kuhn-Tucker or not we have to consider the following system of linear inequalities of variables λ_i, $i = 1, 2, \dots, m$, ($r \in \mathbb{R}^n$ is fixed):

$$(*) \qquad \begin{cases} f'(x_0, r) + \sum_{i=1}^{m} \lambda_i g_i'(x_0, r) \geq 0, \\ \lambda_i \geq 0, \\ \lambda_i \cdot g_i(x_0) = 0 \ \text{ for all } i = 1, 2, \dots, m. \end{cases}$$

If for each $r \in \mathbb{R}^n$ the system $(*)$ has at least a solution $\lambda = (\lambda_1, \dots, \lambda_m)$, then x_0 is a directional Kuhn-Tucker point of (P). On the contrary, if there is $r \in \mathbb{R}^n$ such that the system $(*)$ has no solution then x_0 is not a directional Kuhn-Tucker point of (P) (see Remark 3.3 for an application of the idea).

In the remainder of this section we shall prove that under some constraint qualification the three types of Kuhn-Tucker points coincide.

Let $J := \{1, 2.., p\}$. Given nonempty compact convex sets B_i $(i \in I)$ in $\mathbb{R}^n$, we define

$$\varphi_i(\xi) := \max_{b_i \in B_i} \langle \xi, b_i \rangle, \quad i \in J$$

and consider the following system of inequalities of variable $\xi \in \mathbb{R}^n$

$$(2.7) \qquad \varphi_i(\xi) < 0, \quad i \in J.$$

The following lemma is a special case of Proposition 2.2 in [20].

Lemma 2.1. [20] *System* (2.7) *has a solution if and only if*

$$(2.8) \qquad 0 \notin \ co \bigcup_{i \in J} B_i,$$

where $co\,\Omega$ *denotes the convex hull of a set* $\Omega \subset \mathbb{R}^n$.

Definition 2.4. [17] Problem (P)is said to be *regular* at x_0 if for all $v_i \in \overline{\partial} g_i(x_0)$, $i \in I(x_0)$, one has

$$(2.9) \qquad 0 \notin \ \text{co} \bigcup_{i \in I(x_0)} (\underline{\partial} g_i(x_0) + v_i).$$

Lemma 2.2. *Suppose that Problem* (P) *is regular at* x_0 *and* $\overline{w}_i \in \overline{\partial} g_i(x_0)$, $i \in I(x_0)$. *Then the set*

$$Q := \sum_{i \in I(x_0)} cone\ \left(\underline{\partial} g_i(x_0) + \overline{w}_i\right)$$

is closed. Consequently,

$$Q = \sum_{i \in I(x_0)} \overline{cone}\ \big(\underline{\partial} g_i(x_0) + \overline{w}_i\big).$$

Proof. Let $\overline{w}_i \in \overline{\partial} g_i(x_0)$, $i \in I(x_0)$. Assume that $(c_k)_k \subset Q$ and $c_k \to c$ as k tends to infinity. We will prove that $c \in Q$.

Since $c_k \in Q$ for all $k \in \mathbf{N}$, one has

$$c_k = \sum_{i \in I(x_0)} \mu_i^k (v_i^k + \overline{w}_i), \tag{2.10}$$

where $\mu_i^k \geq 0,\ \ v_i^k \in \underline{\partial} g_i(x_0)$, $k \in \mathbf{N}$, $i \in I(x_0)$.

We first claim that the sequence $(\mu^k)_k$ (where $\mu^k = (\mu_i^k)_{i \in I(x_0)})$ is bounded. In fact, if this is not true then, without loss of generality, we can assume that $\|\mu^k\| \to \infty$. Setting $\gamma_k = \sum\limits_{i \in I(x_0)} \mu_i^k > 0$ and dividing (2.10) by γ_k we get

$$\frac{c_k}{\gamma_k} \in \ \text{co} \bigcup_{i \in I(x_0)} \big(\underline{\partial} g_i(x_0) + \overline{w}_i\big).$$

Letting $k \to \infty$ and taking into account the fact that the set in the right hand side of the previous inclusion is compact we get

$$0 \in \ \text{co} \bigcup_{i \in I(x_0)} \big(\underline{\partial} g_i(x_0) + \overline{w}_i\big),$$

which conflicts with the regularity of (P) at x_0.

Therefore, we can assume that

$$\mu_i^k \to \mu_i, \quad v_i^k \to v_i \in \underline{\partial} g_i(x_0), \quad \forall i \in I(x_0),\ k \to \infty$$

(note that for each $i \in I(x_0)$ the set $\underline{\partial} g_i(x_0)$ is compact). Together with (2.10), this implies

$$c = \sum_{i \in I(x_0)} \mu_i (v_i + \overline{w}_i) \in Q.$$

The proof is complete. □

The following fact is a direct consequence of Theorem 2.1 and Lemma 2.2.

Corollary 2.1. *If Problem* (P) *is regular at* x_0 *then the three types of Kuhn-Tucker points in Definitions 2.1 - 2.3 are the same.*

3. Necessary and sufficient conditions for optimality

In this section we will show that if (P) is regular then the directional Kuhn-Tucker condition is necessary for optimality and under some generalized convexity condition, it is also sufficient.

3.1. **Necessary conditions for optimality.**

Theorem 3.1. *If $x_0 \in S$ is a local solution of* (P) *then for each $\xi \in I\!R^n$ there exists $\lambda(\xi) = (\lambda_0(\xi), \lambda_1(\xi), \ldots, \lambda_m(\xi)) \neq 0$, $\lambda_i(\xi) \geq 0$ for all $i = 1, 2, 3, \ldots, m$, such that the following conditions hold:*

$$\lambda_0(\xi) f'(x_0, \xi) + \sum_{i=1}^{m} \lambda_i(\xi) g_i'(x_0, \xi) \geq 0, \tag{3.1}$$

$$\lambda_i(\xi) \cdot g_i(x_0) = 0 \quad \textit{for all } i = 1, 2, \ldots, m. \tag{3.2}$$

Proof. Observe that the optimality of x_0 implies the inconsistency of the following system of variable $\xi \in \mathbb{R}^n$:

$$\begin{cases} f'(x_0, \xi) < 0, \\ g_i'(x_0, \xi) < 0, \ i \in I(x_0). \end{cases} \tag{3.3}$$

In fact, if $\bar{x} \in \mathbb{R}^n$ is a solution of (3.3) then by the definition of directional derivatives and the continuity of g_i for all $i \notin I(x_0)$, we can find $\lambda \in \mathbb{R}$, $\lambda > 0$ satisfying

$$g_i(x_0 + \lambda\bar{x}) < g_i(x_0) = 0 \quad \text{for all } i \in I(x_0),$$
$$g_i(x_0 + \lambda\bar{x}) < g_i(x_0) < 0 \quad \text{for all } i \notin I(x_0),$$

and

$$f(x_0 + \lambda\bar{x}) < f(x_0).$$

These conflict with the optimality of x_0.

For any $\xi \in \mathbb{R}^n$ (fixed), select $\overline{v} \in \overline{\partial} f(x_0)$, $\overline{v}_i \in \overline{\partial} g_i(x_0)$ such that

$$\langle \xi, \overline{v} \rangle = \min_{v \in \overline{\partial} f(x_0)} \langle \xi, v \rangle, \quad \langle \xi, \overline{v}_i \rangle = \min_{v_i \in \overline{\partial} g_i(x_0)} \langle \xi, v_i \rangle, \quad i \in I(x_0),$$

and set

$$\Phi(x) := \max_{v \in \underline{\partial} f(x_0)} \langle x, v \rangle + \langle x, \overline{v} \rangle, \quad x \in \mathbb{R}^n,$$
$$\Psi_i(x) := \max_{v \in \underline{\partial} g_i(x_0)} \langle x, v \rangle + \langle x, \overline{v}_i \rangle, \quad x \in \mathbb{R}^n.$$

It is clear that $\Phi(\xi) = f'(x_0, \xi)$ and $\Psi_i(\xi) = g_i'(x_0, \xi)$, $i \in I(x_0)$.

Note that the inconsistency of the system (3.3) implies the inconsistency of the following system of variable $x \in \mathbb{R}^n$:

$$\begin{cases} \Phi(x) < 0, \\ \Psi_i(x) < 0, \quad i \in I(x_0). \end{cases} \tag{3.4}$$

Indeed, if ξ_0 is a solution of (3.4) then we have

$$\begin{cases} f'(x_0, \xi_0) \leq \Phi(\xi_0) < 0, \\ g_i'(x_0, \xi_0) \leq \Psi_i(\xi_0) < 0, \quad i \in I(x_0) \end{cases}$$

which conflicts with the inconsistency of (3.3).

It is clear that the functions Φ, Ψ_i where $i \in I(x_0)$, are convex. By the Gordan alternative theorem (see [13]), the inconsistency of (3.4) implies the existence of $\lambda_0(\xi) \geq 0$, $\lambda_i(\xi) \geq 0$, $i \in I(x_0)$, not all zero, satisfying the following inequality

$$\lambda_0(\xi)\Phi(x) + \sum_{i \in I(x_0)} \lambda_i(\xi)\Psi_i(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{3.5}$$

In particular,

$$\lambda_0(\xi)\Phi(\xi) + \sum_{i \in I(x_0)} \lambda_i(\xi)\Psi_i(\xi) \geq 0,$$

hence

$$\lambda_0(\xi)f'(x_0, \xi) + \sum_{i \in I(x_0)} \lambda_i(\xi)g_i'(x_0, \xi) \geq 0.$$

Let us set $\lambda_i(\xi) = 0$ for all $i \notin I(x_0)$. Then the vector $\lambda(\xi) := (\lambda_0(\xi), \lambda_1(\xi), \ldots, \lambda_m(\xi)) \neq 0$ satisfies the desired conditions (3.1), (3.2). The proof is complete. $\square$

It is possible to give a simpler proof for Theorem 3.1 (see [2]). However, we prefer the previous one since it serves as the first part for the proof of Theorem 3.2 below.

Theorem 3.2. *Suppose that* (P) *is regular at* x_0. *If* x_0 *is a (local) solution of* (P) *then* x_0 *is a directional Kuhn-Tucker point of* (P).

We need the following lemma for the proof of Theorem 3.2.

Lemma 3.1. *Problem* (P) *is regular at* x_0 *if and only if the following* (RC) *regular condition holds:*

(RC) *There exists* $\overline{x} \in \mathbb{R}^n$ *such that*

$$\max_{v_i \in \underline{\partial} g_i(x_0)} \langle \overline{x}, v_i \rangle + \max_{w_i \in \overline{\partial} g_i(x_0)} \langle \overline{x}, w_i \rangle < 0, \quad \forall i \in I(x_0). \tag{3.6}$$

The regular condition (RC) was introduced in [12] for (P) with $m = 1$.

Proof of Lemma 3.1. (*Necessity*) Suppose that (P) is regular at x_0 then if $\overline{v}_i \in \overline{\partial} g_i(x_0)$, $i \in I(x_0)$, one has

$$0 \notin \operatorname{co} \bigcup_{i \in I(x_0)} (\underline{\partial} g_i(x_0) + \overline{v}_i).$$

We will prove that

$$0 \notin \operatorname{co} \bigcup_{i \in I(x_0)} (\underline{\partial} g_i(x_0) + \overline{\partial} g_i(x_0)). \tag{3.7}$$

Assume to the contrary that

$$0 \in \operatorname{co} \bigcup_{i \in I(x_0)} (\underline{\partial} g_i(x_0) + \overline{\partial} g_i(x_0)).$$

Then there exist $\underline{u}_i \in \underline{\partial} g_i(x_0)$, $\overline{u}_i \in \overline{\partial} g_i(x_0)$, and $\lambda_i \geq 0$, where $i \in I(x_0)$, such that

$$1 = \sum_{i \in I(x_0)} \lambda_i,$$
$$0 = \sum_{i \in I(x_0)} \lambda_i(\underline{u}_i + \overline{u}_i) \in \sum_{i \in I(x_0)} \lambda_i(\underline{\partial} g_i(x_0) + \overline{u}_i),$$

or equivalently,

$$0 \in \ \text{co} \bigcup_{i \in I(x_0)} (\underline{\partial} g_i(x_0) + \overline{u}_i), \quad \overline{u}_i \in \overline{\partial} g_i(x_0), \quad i \in I(x_0),$$

which contradicts the assumption. Hence

$$0 \notin \ \text{co} \bigcup_{i \in I(x_0)} (\underline{\partial} g_i(x_0) + \overline{\partial} g_i(x_0)).$$

It follows from (3.7) and Lemma 2.1 that there exists $\overline{\xi} \in \mathbb{R}^n$ such that

$$\max_{a_i \in \underline{\partial} g_i(x_0)} \langle \overline{\xi}, a_i \rangle + \max_{b_i \in \overline{\partial} g_i(x_0)} \langle \overline{\xi}, b_i \rangle = \max_{c_i \in \underline{\partial} g_i(x_0) + \overline{\partial} g_i(x_0)} \langle \overline{\xi}, c_i \rangle \ < 0, \quad i \in I(x_0).$$

Thus (3.6) holds.

(*Sufficiency*) Assume that (3.6) holds for some $\overline{x} \in \mathbb{R}^n$ but (P) is not regular at x_0. Then there exist $\overline{v}_i \in \overline{\partial} g_i(x_0)$, $i \in I(x_0)$, such that

$$0 \in \ \text{co} \bigcup_{i \in I(x_0)} (\underline{\partial} g_i(x_0) + \overline{v}_i). \tag{3.8}$$

It follows from (3.8) and the Lemma 2.1 that the following system of inequalities (of variable $\xi \in \mathbb{R}^n$) is inconsistent

$$\varphi_i(\xi) = \max_{b_i \in \underline{\partial} g_i(x_0) + \overline{v}_i} \langle \xi, b_i \rangle \ < 0, \quad i \in I(x_0),$$

or equivalently, the following system of inequalities of variable $\xi \in \mathbb{R}^n$ is inconsistent

$$\max_{u_i \in \underline{\partial} g_i(x_0)} \langle \xi, u_i \rangle + \langle \xi, \overline{v}_i \rangle \ < 0, \quad i \in I(x_0),$$

which contradicts (3.6).

The lemma has been proved. □

Proof of Theorem 3.2. It suffices to prove that under the regularity of (P) at x_0 we have $\lambda_0(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$.

Assume to the contrary that there exists $\xi^* \in \mathbb{R}^n$ such that $\lambda_0(\xi^*) = 0$. Then it follows from (3.5) (see the proof of Theorem 3.1) that

$$\sum_{i \in I(x_0)} \lambda_i(\xi^*) \Psi_i(x) \geq 0 \quad \text{for all } \ x \in \mathbb{R}^n. \tag{3.9}$$

By the regularity of (P) at x_0, Lemma 3.1, and the definition of the functions Ψ_i, there exists $\overline{x} \in \mathbb{R}^n$ such that $\Psi_i(\overline{x}) < 0$ for all $i \in I(x_0)$. Since $\lambda_i(\xi^*) \geq 0$ and $\lambda_i(\xi^*)$ are not all zero for all $i \in I(x_0)$, we get

$$\sum_{i \in I(x_0)} \lambda_i(\xi^*)\Psi_i(\overline{x}) < 0,$$

which contradicts (3.9). The theorem is thus proved. $\square$

3.2. **Sufficient conditions for optimality.** We now prove that the directional Kuhn-Tucker condition is also a sufficient condition for optimality for (P) whenever (P) is directionally η-invex.

Definition 3.1. ([1], [17]) We say that the functions f, g_i, $i \in I(x_0)$, are *directionally η-invex*[1] at x_0 on S if there is a function $\eta : S \longrightarrow \mathbb{R}^n$ such that, for all $x \in S$,

$$\begin{aligned} f(x) - f(x_0) &\geq f'(x_0, \eta(x)), \\ g_i(x) - g_i(x_0) &\geq g_i'(x_0, \eta(x)) \quad \text{for all} \;\; i \in I(x_0). \end{aligned}$$

The following lemma is useful but its simple proof will be omitted.

Lemma 3.2. *If f, g_i, $i \in I(x_0)$, are directionally η-invex at x_0 on S then for all $\lambda = (\lambda_i)_{i \in I(x_0)}$ with $\lambda_i \geq 0$ the function*

$$\Phi_\lambda := f + \sum_{i \in I(x_0)} \lambda_i g_i$$

is directionally η-invex at x_0 on S (with the same function η), that is,

$$\Phi_\lambda(x) - \Phi_\lambda(x_0) \geq \Phi_\lambda'(x_0, \eta(x)) = f'(x_0, \eta(x)) + \sum_{i \in I(x_0)} \lambda_i g_i'(x_0, \eta(x)).$$

Theorem 3.3. *Suppose that f, g_i, $i \in I(x_0)$, are directionally η-invex at x_0 on S. If x_0 is a directional Kuhn-Tucker point of* (P) *then x_0 is a (global) minimizer of* (P).

Proof. Let x be an arbitrary point of S. Since x_0 is a directional Kuhn-Tucker point of (P), if we take $\xi = \eta(x)$, there exists $\lambda = \big(\lambda_1(\eta(x)), \lambda_2(\eta(x)), \ldots, \lambda_m(\eta(x))\big)$ such that

$$f'(x_0, \eta(x)) + \sum_{i \in I(x_0)} \lambda_i(\eta(x)) g_i'(x_0, \eta(x)) = \Phi_\lambda'(x_0, \eta(x)) \geq 0. \tag{3.10}$$

By Lemma 3.2, the function

$$\Phi_\lambda := f + \sum_{i \in I(x_0)} \lambda_i(\eta(x)) g_i$$

[1] It is "η-invex" in [1] and [17]. The terminology "directionally η-invex" was proposed by one of the unknown referees

is directionally η-invex at x_0 on S. One has

$$\Phi_\lambda(x) - \Phi_\lambda(x_0) \geq \Phi'_\lambda(x_0, \eta(x)), \quad \forall x \in S.$$

Together with (3.10) this gives

$$\Phi_\lambda(x) - \Phi_\lambda(x_0) \geq \Phi'_\lambda(x_0, \eta(x)) \geq 0.$$

Hence $\Phi_\lambda(x) \geq \Phi_\lambda(x_0)$, or equivalently,

$$f(x) + \sum_{i \in I(x_0)} \lambda_i(\eta(x)) g_i(x) \geq f(x_0) + \sum_{i \in I(x_0)} \lambda_i(\eta(x)) g_i(x_0). \tag{3.11}$$

Since $\lambda_i(\eta(x))g_i(x_0) = 0$ for all $i \in I(x_0)$, $\lambda_i(\eta(x)) \geq 0$, and $g_i(x) \leq 0$ for all i, from (3.11) we get

$$f(x) \geq f(x_0),$$

which proves that x_0 is a global minimizer of (P) since x is an arbitrary point of S. □

The following corollary is immediate from Theorem 3.3 and Theorem 2.1.

Corollary 3.1. *Suppose that f, g_i, $i \in I(x_0)$, are directionally η-invex at x_0 on S. If x_0 is a Kuhn-Tucker point of* (P) *then x_0 is a minimizer of* (P).

The two corolaries below show that for the problems where f, g_i are convex, the Lagrange multipliers can be chosen to be constants.

Corollary 3.2. [11] *Suppose that the functions f, g_i $(i \in I)$ are proper and convex and g_i is continuous at x_0 for all $i \notin I(x_0)$. If x_0 is a minimizer of* (P) *then there exist $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, such that $\lambda_i g_i(x_0) = 0$, for all $i \in I$ and*

$$\lambda_0 f'(x_0, \xi) + \sum_{i \in I} \lambda_i g'_i(x_0, \xi) \geq 0, \quad \forall \xi \in I\!R^n. \tag{3.12}$$

Besides, if there exists $\tilde{x} \in S$ such that $g'_i(x_0, \tilde{x}) < 0$ for all $i \in I(x_0)$ then $\lambda_0 \neq 0$.

Conversely, if $x_0 \in S$ satisfies (3.12) *for some $\lambda_0 > 0$ $\lambda_i \geq 0$, $i \in I$, then x_0 is a global solution of* (P).

Proof. Note that under the assumption of the corollary, $f'(x_0, .)$ and $g'_i(x_0, .)$ $(i \in I(x_0))$ are convex. The proof of the first conclusion follows directly from the Gordan theorem (see [13]) and the inconsistency of the convex system (3.3). Note also that if there is $\tilde{x} \in S$ such that $g'_i(x_0, \tilde{x}) < 0$ for all $i \in I(x_0)$, then the condition (RC) holds, and as a consequence of Theorem 3.2, $\lambda_0 \neq 0$.

If f, g_i, $i \in I$ are convex then (P) is directionally η-invex at x_0 on S with $\eta(x) = x - x_0$. The sufficient condition follows from Theorem 3.3. □

Corollary 3.3. [16] *Suppose that the functions f, g_i, $i \in I$ are convex and continuous at x_0. If x_0 is a solution of* (P) *then there exist $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i \in I$, not all zero, such that*

$$0 \in \lambda_0 \partial f(x_0) + \sum_{i \in I} \lambda_i \partial g_i(x_0), \quad \lambda_i g_i(x_0) = 0, \quad \forall i \in I.$$

Moreover, if (P) *satisfies the Slater condition, that is, $g_i(\bar{x}) < 0$ for all $i \in I(x_0)$, and for some $\bar{x} \in S$, then the following is necessary and sufficient for $x_0 \in S$ to be a (global) solution of* (P)*:*

There exists $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in I\!R^m_+$ such that

$$0 \in \partial f(x_0) + \sum_{i \in I} \lambda_i \partial g_i(x_0), \quad \lambda_i g_i(x_0) = 0, \quad \forall i \in I.$$

Corollary 3.3 is a direct consequence of Corollary 3.2 and the separation theorem. It is possible to establish analogous results for differentiable problems as well as for the problems in which the functions are locally Lipschitz and regular in the sense of Clarke (see [2]).

Example 3.1. Consider the following problem (P1)

$$\min f(x)$$

subject to

$$g(x) \leq 0, \quad x \in \mathrm{I\!R}^2$$

where the functions $f, g : \mathrm{I\!R}^2 \longrightarrow \mathrm{I\!R}$ are defined by

$$\begin{aligned} &f(x) := -x_2, \\ &g(x) := |\,|x_1| + x_2|, \quad x = (x_1, x_2) \in \mathrm{I\!R}^2. \end{aligned}$$

The functions f and g are quasidifferentiable at $x_0 = (0, 0) \in \mathrm{I\!R}^2$. Namely, we can choose

$$\begin{aligned} &\underline{\partial} f(x_0) = \{(0, -1)\}, \quad \overline{\partial} f(x_0) = \{(0, 0)\} \quad \text{and} \\ &\underline{\partial} g(x_0) = \text{ co } \{(0, 0), (2, 2), (-2, 2)\}, \quad \overline{\partial} g(x_0) = \text{ co } \{(1, -1), (-1, -1)\}. \end{aligned}$$

Note that if $r = (r_1, r_2) \in \mathrm{I\!R}^2$ and $r_2 > 0$ then $g'(x_0, r) > 0$. In fact, it is clear that for such r one has

$$\min_{w \in \overline{\partial} g(x_0)} \langle r, w \rangle < 0.$$

Let

$$\overline{w} \in \operatorname*{argmin}_{w \in \overline{\partial} g(x_0)} \langle r, w \rangle \subset \text{ co } \{(1, -1), (-1, -1)\}.$$

Then

$$-2\overline{w} \in \text{ co } \{(2, 2), (-2, 2)\} \subset \underline{\partial} g(x_0).$$

Since $\langle r, \overline{w}\rangle < 0$, we have

$$\begin{aligned} g'(x_0, r) &= \max_{w \in \underline{\partial} g(x_0)} \langle r, w\rangle + \min_{w \in \overline{\partial} g(x_0)} \langle r, w\rangle \\ &\geq \langle r, -2\overline{w}\rangle + \langle r, \overline{w}\rangle \\ &= 2\langle r, -\overline{w}\rangle + \langle r, \overline{w}\rangle \\ &> \langle r, -\overline{w}\rangle + \langle r, \overline{w}\rangle \\ &= 0. \end{aligned}$$

Let $\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ be defined by setting

$$\lambda(r) := \begin{cases} \dfrac{r_2}{g'(x_0, r)} & \text{if } r_2 > 0 \\ 0 & \text{if } r_2 \leq 0 \end{cases}$$

where $r = (r_1, r_2)$. We claim that the inequality

$$f'(x_0, r) + \lambda(r) g'(x_0, r) \geq 0$$

holds for all $r = (r_1, r_2) \in \mathbb{R}^2$. In fact, if $r_2 \leq 0$ then

$$f'(x_0, r) + \lambda(r) g'(x_0, r) = -r_2 \geq 0.$$

If $r_2 > 0$ then

$$f'(x_0, r) + \lambda(r) g'(x_0, r) = -r_2 + \frac{r_2}{g'(x_0, r)} g'(x_0, r) = 0.$$

Therefore $x_0 = (0, 0) \in \mathbb{R}^2$ is a directional Kuhn-Tucker point of (P1).

On the other hand, if we set $\eta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

$$\eta(x) := \begin{cases} (0, 0) & \text{if } x_2 \leq 0 \\ (0, x_2) & \text{if } x_2 > 0 \end{cases}$$

where $x = (x_1, x_2)$, then f, g are directionally η-invex at $x_0 = (0, 0)$ on $\mathbb{R}^2$. In fact, if $x = (x_1, x_2) \in \mathbb{R}^2$ and $x_2 \leq 0$ then

$$\begin{aligned} f(x) - f(x_0) &= -x_2 \geq 0 = f'(x_0, 0) = f'(x_0, \eta(x)), \\ g(x) - g(x_0) &= |\, |x_1| + x_2| \geq 0 = g'(x_0, \eta(x)). \end{aligned}$$

If $x = (x_1, x_2) \in \mathbb{R}^2$, where $x_2 > 0$, then

$$\begin{aligned} f(x) - f(x_0) = -x_2 = \langle (0, -1), (0, x_2)\rangle &= \langle (0, -1), \eta(x)\rangle \\ &= f'(x_0, \eta(x)), \end{aligned}$$

and

$$\begin{aligned} g(x) - g(x_0) = |\, |x_1| + x_2| \geq x_2 &= x_2 . g'(x_0, e_2) \\ &= g'(x_0, \eta(x)), \end{aligned}$$

where $e_2 = (0, 1)$.

We have just proved that f, g are directionally η-invex at $x_0 = (0, 0)$ and x_0 is a directional Kuhn-Tucker point of $(P1)$. By Theorem 3.3, x_0 is a minimizer of $(P1)$.

Example 3.2. Consider the following problem (P2)

$$\min f(x)$$
$$\text{subject to}$$
$$g(x) \le 0, \quad x \in \mathbb{R}^2$$

where $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are the functions defined by

$$f(x) := x_2,$$
$$g(x) := \begin{cases} x_1 + (x_1^2 + x_2^2)^{\frac{1}{2}} - x_2 & \text{if } \; x_2 \ge 0 \\ x_1 + (x_1^2 + x_2^2)^{\frac{1}{2}} & \text{if } \; x_2 < 0, \end{cases}$$
$$x = (x_1, x_2) \in \mathbb{R}^2$$

(see [22, Example 3.2], [12, Example 3]).

Take $x_0 = (0,0)$ and note that $x_0 \in \overline{\partial} g(x_0) \subset \mathbb{R}^2$.

In [12], it is shown that

$$-\overline{\partial} f(x_0) \not\subset \underline{\partial} f(x_0) + \text{ cone } (\, \underline{\partial} g(x_0) + x_0), \quad x_0 = (0,0) \in \overline{\partial} g(x_0) \subset \mathbb{R}^2,$$

i.e., $x_0 = (0,0)$ is not a Kuhn-Tucker point of (P2).

We now prove that $x_0 = (0,0) \in \mathbb{R}^2$ is a directional Kuhn-Tucker point of (P2). Moreover, the functions f and g are directionally η-invex at x_0. Hence, by Theorem 3.3, x_0 is a minimizer of (P2). Meanwhile, (P2) is irregular at x_0 (see Corollary 2.1).

On one hand, it is clear that if $\lambda > 0$ and $x \in \mathbb{R}^2$ then

$$f(\lambda x) = \lambda f(x)$$
$$g(\lambda x) = \lambda g(x).$$

Therefore, for all $r = (r_1, r_2) \in \mathbb{R}^2$,

$$f'(x_0, r) = \lim_{\lambda \downarrow 0} \frac{f(\lambda r) - f(x_0)}{\lambda} = \lim_{\lambda \downarrow 0} \frac{f(\lambda r)}{\lambda} = f(r), \tag{3.13a}$$

$$g'(x_0, r) = \lim_{\lambda \downarrow 0} \frac{g(\lambda r) - g(x_0)}{\lambda} = \lim_{\lambda \downarrow 0} \frac{g(\lambda r)}{\lambda} = g(r). \tag{3.13b}$$

On the other hand, for each $r = (r_1, r_2) \in \mathbb{R}^2$ where $r_2 < 0$, we have

$$g(r) = r_1 + (r_1^2 + r_2^2)^{\frac{1}{2}} > r_1 + (r_1^2)^{\frac{1}{2}} = r_1 + |r_1| \ge 0.$$

Let $\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ be defined by setting

$$\lambda(r) = \begin{cases} -\dfrac{r_2}{g(r)} & \text{if } r_2 < 0 \\ 0 & \text{if } r_2 \ge 0 \end{cases}$$

for all $r = (r_1, r_2) \in \mathbb{R}^2$. We claim that

$$f'(x_0, r) + \lambda(r) g'(x_0, r) \ge 0$$

for all $r = (r_1, r_2) \in \mathbb{R}^2$. In fact, if $r_2 < 0$ then $f'(x_0, r) + \lambda(r)g'(x_0, r) = r_2 + \Big(- \dfrac{r_2}{g(r)} \Big) \cdot g(r) = 0$. If $r_2 \geq 0$ then $f'(x_0, r) + \lambda(r)g'(x_0, r) = r_2 \geq 0$. Consequently, $x_0 = (0, 0) \in \mathbb{R}^2$ is a directional Kuhn-Tucker point of (P2).

Since $f(x_0) = 0$ and $g(x_0) = 0$, it follows from (3.13) that the functions f, g are directionally η-invex on $\mathbb{R}^2$ at x_0 where $\eta(x) = x$ for all $x \in \mathbb{R}^2$.

Remark 3.1. Example 3.2 shows that the directional Kuhn-Tucker condition is weaker than the Kuhn-Tucker condition introduced in [12] and [21].

Remark 3.2. Note that for (P2), with $r = (r_1, r_2) \in \mathbb{R}^2$, the inequality

$$f'(x_0, r) + \lambda(r)g'(x_0, r) \geq 0 \tag{3.14}$$

is equivalent to

$$r_2 + \lambda(r)g(r) \geq 0. \tag{3.14'}$$

Consider $r = (r_1, r_2) \in \mathbb{R}^2$ with $r_2 < 0$. Then $g(r) > 0$ (see Example 3.2). Therefore $\lambda(r)$ satisfies (3.14) (or (3.14')) if and only if $\lambda(r) \in \Big[- \dfrac{r_2}{g(r)}, +\infty \Big)$. The multilpier $\lambda(r) := -\dfrac{r_2}{g(r)}$ chosen in Example 3.2 (when $r_2 < 0$) is the smallest number possible such that (3.14) still holds.

We now consider a sequence of directions $(r_k)_k \subset \mathbb{R}^2$ satisfying

$$r_k = (r_{1k}, r_{2k}), \quad r_{2k} = -1 \text{ for all } k \in \mathbf{N}, \text{ and } r_{1k} \to -\infty \text{ as } k \to +\infty.$$

Then

$$\lambda(r_k) := -\frac{r_{2k}}{g(r_k)} = \frac{1}{r_{1k} + \sqrt{1 + r_{1k}^2}} = \sqrt{1 + r_{1k}^2} - r_{1k} \to +\infty \text{ as } k \to +\infty.$$

This shows that there exists a sequence of directions such that the sequence of the corresponding multipliers tends to infinity. In other words, in the case of Problem (P2) the multiplier $\lambda(.)$ can not be a constant.

Remark 3.3. We consider Problem (P2) and suppose that we have not known the candidate for minimizer (that is x_0). We shall use the idea given in Remark 2.1 (ii) to search for such a point for (P2).

Recal that $x_0 \in \mathbb{R}^2$ is a directional Kuhn-Tucker point of (P2) if and only if for all $r = (r_1, r_2) \in \mathbb{R}^2$, the following system of linear inequalities of variable $\lambda \in \mathbb{R}$ possesses at least one solution:

$$\begin{cases} f'(x_0, r) + \lambda g'(x_0, r) \geq 0 \\ \lambda \geq 0 \\ \lambda g(x_0) = 0. \end{cases} \tag{3.15}$$

We first observe that for $x^* = (x_1, x_2) \in \mathbb{R}^2$, $g(x^*) = 0$ if anf only if

$$x_1 = 0, \ x_2 \geq 0 \quad \text{or} \quad x_1 \leq 0, \ x_2 = 0. \tag{3.16}$$

(α) If $x^* = (x_1, x_2) \in \mathbb{R}^2$ and $g(x^*) \neq 0$ then system (3.15) is equivalent to

$$\begin{cases} \lambda \geq 0 \\ r_2 \geq 0 \end{cases}$$

which has no solution if we take $r = (r_1, r_2) \in \mathbb{R}^2$ with $r_2 < 0$. Thus x^* is not a directional Kuhn-Tucker point of (P2).

(β) If $x^* = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 = 0$, $x_2 > 0$ then $g(x^*) = 0$ and an easy calculation shows that $f'(x^*, r) = r_2$ and $g'(x^*, r) = r_1$. Hence (3.15) is equivalent to the system

$$\begin{cases} r_2 + \lambda r_1 \geq 0 \\ \lambda \geq 0 \end{cases}$$

which has no solution if $r = (r_1, r_2)$ when $r_1 < 0$ and $r_2 < 0$. This proves that x^* is not a directional Kuhn-Tucker point of (P2).

(γ) If $x^* = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 < 0$, $x_2 = 0$ then $g(x^*) = 0$ and with $r = (r_1, r_2) \in \mathbb{R}^2$, $r_2 < 0$, we get $f'(x^*, r) = r_2$ and $g'(x^*, r) = 0$. This time system (3.15) is equivalent to

$$\begin{cases} r_2 + \lambda .0 \geq 0 \\ \lambda \geq 0 \end{cases}$$

which has no solution if $r = (r_1, r_2)$ with $r_1 < 0$ and $r_2 < 0$. This means that x^* is not a directional Kuhn-Tucker point of (P2).

Therefore, every $x \in \mathbb{R}^2$, except for $x_0 = (0, 0)$, is not a directional Kuhn-Tucker point of (P2). As it is shown in Example 3.2, $x_0 = (0, 0)$ is directional Kuhn-Tucker point of (P2) and it is actually the unique solution of (P2).

4. Duality

4.1. **Mond-Weir dual problem of (P).** Consider the Mond-Weir dual problem (MWD) of (P):

$$\max f(\xi) \tag{4.1}$$

subject to

$$(\xi, \lambda) \in Y. \tag{4.2}$$

Here Y is the set of all pairs (ξ, λ) with $\xi \in \mathbb{R}^n$ and $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m_+$, $\lambda(r) = (\lambda_1(r), \lambda_2(r), \ldots, \lambda_m(r))$, satisfying the following conditions for all $r \in \mathbb{R}^n$:

$$f'(\xi, r) + \sum_{i=1}^{m} \lambda_i(r) g_i'(\xi, r) \geq 0, \tag{4.3a}$$

$$\sum_{i=1}^{m} \lambda_i(r) g_i(\xi) \geq 0. \tag{4.3b}$$

Theorem 4.1. *Assume that $\overline{x} \in S$ and $(\overline{\xi}, \overline{\lambda}) \in Y$. If f, g_i $(i = 1, 2, \ldots, m)$ are quasidifferentiable at $\overline{\xi}$ and directionally η-invex at $\overline{\xi}$ on S then*

$$f(\overline{x}) \geq f(\overline{\xi}).$$

Proof. Since $(\overline{\xi}, \overline{\lambda}) \in Y$, we have

$$\overline{\lambda}(\eta(\overline{x})) = \big((\overline{\lambda}_1(\eta(\overline{x})), \overline{\lambda}_2(\eta(\overline{x})), \ldots, \overline{\lambda}_m(\eta(\overline{x}))\big) \geq 0,$$

$$f'(\xi, \eta(\overline{x})) + \sum_{i=1}^{m} \lambda_i(\eta(\overline{x})) g_i'(\xi, \eta(\overline{x})) \geq 0. \tag{4.4}$$

It follows from Lemma 3.2 that the function

$$\Phi_{\overline{\lambda}} := f + \sum_{i=1}^{m} \overline{\lambda}_i(\eta(\overline{x})) g_i$$

is directionally η-invex at x_0 on S. That is, for all $x \in S$,

$$\Phi_{\overline{\lambda}}(x) - \Phi_{\overline{\lambda}}(\overline{\xi}) \geq \Phi'_{\overline{\lambda}}(\overline{\xi}, \eta(x)).$$

In particular, due to (4.4),

$$\begin{aligned} \Phi_{\overline{\lambda}}(\overline{x}) - \Phi_{\overline{\lambda}}(\overline{\xi}) \geq \Phi'_{\overline{\lambda}}(\overline{\xi}, \eta(\overline{x})) = \\ \geq f'(\xi, \eta(\overline{x})) + \sum_{i=1}^{m} \overline{\lambda}_i(\eta(\overline{x})) g_i'(\xi, \eta(\overline{x})) \geq 0, \end{aligned}$$

or equivalently,

$$f(\overline{x}) + \sum_{i=1}^{m} \overline{\lambda}_i(\eta(\overline{x})) g_i(\overline{x}) \geq f(\overline{\xi}) + \sum_{i=1}^{m} \overline{\lambda}_i(\eta(\overline{x})) g_i(\overline{\xi}). \tag{4.5}$$

Since $g_i(\overline{x}) \leq 0$ and $\overline{\lambda}_i(\eta(\overline{x})) \geq 0$ for all $i = 1, 2, ..., m$, we have

$$\sum_{i=1}^{m} \overline{\lambda}_i(\eta(\overline{x})) g_i(\overline{x}) \leq 0.$$

On the other hand, by (4.3b),

$$\sum_{i=1}^{m} \overline{\lambda}_i(\eta(\overline{x})) g_i(\overline{\xi}) \geq 0.$$

Combining these with (4.5) we get

$$f(\overline{x}) \geq f(\overline{\xi}),$$

as desired. □

Theorem 4.2. *Assume that x_0 is a point in S,* (P) *is regular at x_0. Assume further that for all $(\xi, \lambda) \in Y$ the functions f, g_i, $i = 1, 2, \ldots, m$ are directionally η-invex at ξ on S. If x_0 is a minimizer of* (P) *then x_0 is also a maximizer of* (MWD).

Proof. As a consequence of Theorem 4.1 we get

$$f(x_0) \geq \sup\{f(\xi) \mid (\xi, \lambda) \in Y\}. \tag{4.6}$$

On the other hand, by Theorem 3.2, x_0 is a directional Kuhn-Tucker point of (P). Hence there exists a function $\overline{\lambda} : \mathbb{R}^n \longrightarrow \mathbb{R}^m_+$ such that

$$f'(x_0, r) + \sum_{i=1}^{m} \overline{\lambda}_i(r) g_i'(x_0, r) \geq 0, \quad \text{for all } r \in \mathbb{R}^n,$$

$$\overline{\lambda}_i(r).g_i(x_0) = 0, \quad i = 1, 2, \ldots, m.$$

This ensures that $(x_0, \overline{\lambda})$ is a feasible point of (MWD). Together with (4.6) we get

$$f(x_0) \geq f(\xi) \quad \text{for all } (\xi, \lambda) \in Y,$$

which completes the proof. ☐

4.2. **Wolfe dual problem of(P).** Let Y_1 be the set of all pairs (ξ, λ) satisfying (4.3a), i.e.,

$$Y_1 = \{(\xi, \lambda) \mid \xi \in \mathbb{R}^n,\ \lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m_+,\ \lambda(r) = (\lambda_1(r), .., \lambda_m(r)) \text{ satisfying}$$
$$f'(\xi; r) + \sum_{i=1}^{m} \lambda_i(r) g_i'(\xi; r) \geq 0 \text{ for all } r \in \mathbb{R}^n\}.$$

For the sake of convenience, let us set

$$g = (g_1, g_2, \ldots, g_m), \quad g'(x_0, r) = (g_1'(x_0, r), g_2'(x_0, r), \ldots, g_m'(x_0, r)),$$
$$\langle \lambda(r), g(z) \rangle = \sum_{i=1}^{m} \lambda_i(r) g_i(z),$$

where $r, z \in \mathbb{R}^n$.

Define $\Psi : Y_1 \longrightarrow \mathbb{R}$ by

$$\Psi(\xi, \lambda) := f(\xi) + \inf_{r \in \mathbb{R}^n} \langle \lambda(r), g(\xi) \rangle, \quad (\xi, \lambda) \in Y_1.$$

By Wolfe dual problem of (P) we mean the following problem (WD):

$$\begin{aligned} &\max \Psi(\xi, \lambda) \\ &(\xi, \lambda) \in Y_1. \end{aligned} \tag{4.7}$$

Theorem 4.3. *Let* $\overline{x} \in S$, $(\overline{\xi}, \overline{\lambda}) \in Y_1$. *If* f, g_i $(i = 1, 2, \ldots, m)$ *are quasidifferentiable and directionally* η*-invex at* $\overline{\xi}$ *on* S*, then*

$$f(\overline{x}) \geq \Psi(\overline{\xi}, \overline{\lambda}).$$

Proof. Following the same way as in the proof of Theorem 4.1 we arrive at

$$\begin{aligned} f(\overline{x}) &\geq f(\overline{\xi}) + \langle \overline{\lambda}(\eta(\overline{x})), g(\overline{\xi})\rangle \\ &\geq f(\overline{\xi}) + \inf_{r\in\mathbb{R}^n} \langle \overline{\lambda}(r), g(\overline{\xi})\rangle \\ &\geq \Psi(\overline{\xi}, \overline{\lambda}). \end{aligned}$$

□

As a direct consequence of Theorem 4.3 we get

Corollary 4.1. *If for all $(\xi, \lambda) \in Y_1$ the functions f, g_i $(i = 1, 2, \dots, m)$ are quasidifferentiable and directionally η-invex at ξ then*

$$\inf_{x\in S} f(x) \geq \sup_{(\xi,\lambda)\in Y_1} \Psi(\xi, \lambda).$$

Theorem 4.4. *Suppose that $\overline{x}$ is a minimizer of* (P)*, the functions f, g_i $(i = 1, 2, \dots, m)$ are quasidifferentiable and directionally η-invex at $\overline{x}$ on S. Suppose furthermore that* (P) *is regular at $\overline{x}$. Then there exists a function $\overline{\lambda} : \mathbb{R}^n \longrightarrow \mathbb{R}^m_+$ such that $(\overline{x}, \overline{\lambda}) \in Y_1$ and*

$$f(\overline{x}) = \Psi(\overline{x}, \overline{\lambda}).$$

Besides, if for every $(\xi, \lambda) \in Y_1$ the functions f, g_i $(i = 1, 2, \dots, m)$ are quasidifferentiable and directionally η-invex at ξ on S then $(\overline{x}, \overline{\lambda})$ is a solution of (WD)*.*

Proof. Since $\overline{x}$ is a minimizer of (P), by Theorem 3.2 there exists a function $\overline{\lambda} : \mathbb{R}^n \longrightarrow \mathbb{R}^m_+$ such that

$$\begin{aligned} &f'(\overline{x}, r) + \langle \overline{\lambda}(r), g'(\overline{x}, r)\rangle \geq 0, \quad \forall r \in \mathbb{R}^n, \\ &\overline{\lambda}_i(r).g_i(\overline{x}) = 0, \quad i = 1, 2, \dots, m, \ \forall r \in \mathbb{R}^n. \end{aligned}$$

Hence $(\overline{x}, \overline{\lambda}) \in Y_1$. Taking Theorem 4.3 into account, we have

$$f(\overline{x}) \geq \Psi(\overline{x}, \overline{\lambda}) = f(\overline{x}) + \inf_{r\in\mathbb{R}^n} \langle \overline{\lambda}(r), g(\overline{x})\rangle = f(\overline{x}).$$

That is, $f(\overline{x}) = \Psi(\overline{x}, \overline{\lambda})$. The last assertion of the theorem follows immediately from this and Corollary 4.1. □

Remark 4.1. It is possible to extend all the previous results to quasidifferentiable problems with the presence of equality constraints (see [3]). Also, a more general approach which is applicable to larger classes of problems (than that of the quasidifferentiable ones) is introduced in [2].

Remark 4.2. When this paper is in the process of publishing, the authors receive paper [1] from Professor B. D. Craven. It turns out that the idea that the Lagrange multipliers depend on the directions (in nonconvex cases) was found by B. D. Craven in the year 2000. The necessary condition of Fritz-John type as in Theorem 3.1 was established in [1] for the feasible directions from x_0 (this causes a little difficulty in application since one has to solve the problem of finding the feasible directions first). No necessary condition of Kuhn-Tucker type (as in our Theorem 3.2) was found in [1]. But a sufficient condition of the same form as in Theorem 3.3 (once again, for the feasible directions) was established with the same definition of directional invexity. Also, in [1], a special case where the Lagrange multiplier is constant, was pointed out, and for this case, a strong dual theorem was proved.

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