ACTA MATHEMATICA VIETNAMICA 11
Volume 28, Number 1, 2003, pp. 11-15

HYPER-GROUPS OF ORTHOGONAL POLYNOMIALS

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Dedicated to Professor Nguyen Duy Tien on his 60th birth day

Abstract. In the present paper we give a new condition for existence of dual weak hypergroups of hypergroups generated by orthogonal polynomials. In the case of Jacobi polynomials we prove a representation theorem for Lévy type processes in terms of their infinitesimal operators.

1. Notations and Preliminaries

Throughout the paper we shall preserve the notation and terminology in Lasser [9, 10, 11] and Thu [15]. In particular, given a locally compact Hausdorff totological space E let $P(E)$ denote the class of p.m.'s on E with the weak convergence. Let $C_b(E)$ denote the Banach space of all bounded continuous complex valued functions on E with the usual supremum norm.

Let τ^x, $x \in E$, denote a generalized translation operator on $C_b(E)$ as defined in [15] (see also Levitan [12]).

Let $\circ$ be a stochastic convolution on $P(E)$ in the sense of Vol'kovich [18] such that the pair $(P(E), \circ)$ stands for a *commutative hypergroup* (cf. Lasser [8, 9, 10, 11], Heyer [7], Thu [15], Vol'kovich [18] for the concept of hypergroup).

Suppose that $a_n, b_n, c_n,\ n \in N$, are real numbers satisfying $a_n, c_n > 0$, $b_n \geq 0$ and $a_n + b_n + c_n = 1$.

Let $P_n(x)$, $n \in N_0 = N \cup \{0\}$, be a sequence of polynomials on R such that each $P_n(x)$ is of degree n and the following recurrence relation is satisfied

$$P_0(x) = 1, \quad P_{-1}(x) = 0,$$
$$xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n \in N_0. \tag{1.1}$$

Favard's theorem says that the polynomials $P_n(x)$ are orthogonal on an infinite subset K of R w.r. to a positive measure π if and only if $a_{n-1}c_n > 0$ for $n \in N$. The measure π is called the Plancherel measure of $\{P_n(x)\}$.

Received April 25, 2001; in revised form October 21, 2002.
1991 *Mathematics Subject Classification.* MSC 2000: 60B05, 60J25, 60J35, 60J80.
Key words and phrases. Convolution, dual hypergroup, Lévy process, Jacobi polynomials.
The paper is partially supported by the National Basic Research Program in Natural Science, Vietnam.

The product $P_n(x)P_m(x)$ is of the form

$$P_n(x)P_m(x) = \sum_{k=0}^{2m} A(n,m,k)P_{n+m-k}(x). \tag{1.2}$$

The coefficients in (1.2) are called the linearization coefficients of the polynomials $P_n(x)$. If all linearization coefficients are nonnegative then we say that the polynomials $P_n(x)$ have the property ($\mathbb{P}$) (cf. Lasser [8, 9, 10, 11]). Such a property ($\mathbb{P}$) together with the above mentioned properties guarantees the hypergroup structure on N_0.

In what follows we fix (a_n), (b_n), (c_n) such that ($\mathbb{P}$) implies that the polynomials (P_n) are orthogonal on K w.r. to a Plancherel measure π.

Following Lasser [8] we define a convolution operation, say $\square$, on N_0 by

$$\delta_0 \square \delta_n = \delta_n \square \delta_0 = \delta_n \quad \text{for} \quad n \in N_0$$

and

$$\delta_n \square \delta_m = \sum_{k=0}^{2m} A(n,m,n+m-k)\delta_{n+m-k} \tag{1.3}$$

for $n, m \in N$.

It should be noted that $\delta_n \square \delta_m$ is a p.m. on N_0 with δ_0 as unit element. Thus $(N_0, \square)$ becomes a hypergroup (cf. Lasser [8, 9]).

Following Lasser [9] we define, for $z \in C$,

$$\alpha_z(n) = \mathbb{P}_n(n)$$

and let

$$D = \{z \in C : (\mathbb{P}_n) \text{ is bounded}\} \text{ and } D_s = D \cap \mathbb{R}.$$

It has been proved in [9] that both D and D_s are compact, $D_s \subseteq [1 - 2a_0, 1]$ and the map z $\Rightarrow \alpha_z$ is a homeomorphism. Therefore, $\mathcal{D}_s$ can be regarded as the dual object to the hypergroup $(N_0, \square)$. Moreover, for the Plancherel measure (orthogonal measure) π we have

$$\text{supp}\pi \subseteq \mathcal{D}_s.$$

2. A condition for the existence of the dual hypergroup of $(N_0, \square)$

Given $x, y \in D_s$ define a linear functional $\omega(x,y)$ on $\mathcal{H} = \text{spand}\,\{P_n : n \in N_0)\}$ by

$$(\omega(x,y)P_n = P_n(x)P_n(y) \tag{2.1}$$

for $n \in N_0$. Obviously,

$$\omega(x,y)P_0 = \omega(x,y)1 = 1 \tag{2.2}$$

and $\omega(x,y)$ is continuous in $C(D_s)$-norm if and only if there exists a constant $K_{x,y} > 0$ such that for any $f \in \mathcal{H}$,

$$|\omega(x,y)f| \leq K_{x,y}\|f\| \tag{2.3}$$

which together with the property ($\mathbb{P}$) and (2.2) implies the existence of a unique probability measure $\mu_{x,y}$ on D_s such that for each $n \in N_0$,

$$\int\limits_{D_s} P_n(u)\mu_{x,y}(du) = P_n(x)P_n(y).$$

Putting

$$\mu_{x,y} = \delta_x \circ \delta_y \tag{2.4}$$

and taking into account (2.1), (2.2), (2.3) we get a binary operation $\circ$ on $P(D_s)$ such that each $\delta_x \circ \delta_y$ is a p.m and $\circ$ is commutative with δ_1 as the unit element. Note that in such a case the constant $K_{x,y}$ in (2.3) can be 1. Thus, $(P(D_s), \circ)$ is a weak hypergroup and we have proved the following theorem.

Theorem 2.1. *Suppose that* ($\mathbb{P}$) *holds. Then there exists a convolution operation* $\circ$ *such that* $(P(D_s), \circ)$ *is a weak hypergroup if and only if for any* $\lambda_0, \lambda_1, ..., \lambda_n \in R$ *and* $x, y \in D_s$

$$\Big|\sum_{j=0}^{n} \lambda_j P_j(x)P_j(y)\Big| \leq \sup_{u \in D_s} \Big|\sum_{j=0}^{n} \lambda_j P_j(u)\Big|. \tag{2.5}$$

Remark 2.1. If there exists convolution $\circ$ with the property (2.4) then the associated generalized translation operators τ^x, $x \in D_s$, satisfy the following equation:

$$\tau^x P_n(y) = P_n(x)P_n(y). \tag{2.6}$$

3. Jacobi Polynomials

Let us consider the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, where $\alpha \geq \beta > -1$, $\alpha + \beta + 1 \geq 0$. For the defining sequences (a_n), (b_n), (c_n) we have

$$a_n = \frac{2(n+\alpha+\beta+1)(n+\alpha+1)(\alpha+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)2(\alpha+1)},$$

$$b_n = \frac{\alpha-\beta}{2(\alpha+1)}\Big[1 - \frac{(\alpha+\beta+2)(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}\Big],$$

$$c_n = \frac{2n(n+\beta)(\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)2(\alpha+1)}.$$

One can check that $a_n > 0$, $c_n > 0$, $b_n \geq 0$ and $a_n + b_n + c_n = 1$.

It was proved in [9] (see also [1]) that property ($\mathbb{P}$) holds. Moreover, in this case $\widehat{N}_0$ is a hypergroup and can be identified with $[-1, 1]$. The Plancherel measure π is given on $[-1, 1]$ by

$$d\pi(x) = (1-x)^{\alpha}(1+x)^{\beta}dx. \tag{3.1}$$

Thus $[-1,1] = \operatorname{supp}\pi \subseteq [1-2a_0, 1]$.

Let $\otimes = \circ_{\alpha,\beta}$ denote the stochastic convolution on $D_s = [-1,1]$ such that $\mathcal{P}(D_s, \circ_{\alpha,\beta})$ stands for the dual hypergroup of $\widehat{N}_0$. In particular, for any $x, y \in [-1,1]$

$$P_n^{(\alpha,\beta)}(x)\ P_n^{(\alpha,\beta)}(y) = \int_{-1}^{1} P_n^{(\alpha,\beta)}(u)\delta_x \circ_{\alpha,\beta} \delta_y(du). \tag{3.2}$$

Let τ^x, $x \in [-1,1]$, denote the generalized translation operator associated to $\circ_{\alpha,\beta}$. By a similar way as in Thu ([15], formula 3.1) we define

$$D^{\otimes} f(x) = \lim_{y\to 1-} \frac{\tau^x f(y) - f(x)}{1-y}\,, \tag{3.3}$$

where the convergence is taken in $C([-1,1])$-norm. The operator $D^{\otimes}$ is called a characteristic operator for $\otimes$.

By virtue of (3.2) and by the fact that

$$\lim_{y\to 1} \frac{1 - P_n^{(\alpha,\beta)}(y)}{1-y} = P_n^{(\alpha,\beta)\prime}(1) = \frac{n(n+1+\alpha+\beta)}{\alpha+\beta+2}\,, \tag{3.4}$$

it follows that $\{P_n^{(\alpha,\beta)}(y)\}$ are eigenvectors of $D^{\otimes}$ and the corresponding eigenvalues are $\dfrac{n(n+1+\alpha+\beta)}{\alpha+\beta+2}$.

Let ξ_t, $t \geq 0$, be an $\otimes$-Lévy process on $[-1,1]$ corresponding to an $\otimes$-semigroup $\{\mu_t\}$ of p.m.'s on $[-1,1]$. Then there exists a p.m. $H \in \mathcal{P}([-1,1])$ such that

$$(1-y)t^{-1}\mu_t(dy) \to H \qquad \text{weakly.}$$

Let $\{\mu_t\}$ be an $\otimes$-semigroup corresponding to an $\otimes$-Lévy process $\{\xi_t\}$ with the infinitesimal operator A. Then we have

Theorem 3.1. *The following inclusion holds:*

$$\mathcal{D}(D^{\otimes}) \subset \mathcal{D}(A),$$

where $\mathcal{D}(S)$ denotes the domain of operator S. Moreover, for $f \in \mathcal{D}(D^{\otimes})$, we have

$$Af(x) = \int_{-1}^{1} \frac{\tau^x f(y) - f(x)}{1-y} H(dy), \tag{3.5}$$

H being a p.m. in $\mathcal{P}(D_s)$ and the integrand assumes the value $D^{\otimes} f(x)$ at $y = 1-$. The measure H is uniquely determined.

Conversely, for every p.m. H on $[-1,1]$ the formula (3.5) defines an $\otimes$-Lévy process with the infinitesimal operator A given by (3.5). Proof is similar to that of Theorem 3.3 in (Thu [15]) following from the Lévy-Hinčin formula (cf. Lasser [10], Theorem 5).

Corollary 3.1. *If $H = \delta_x,\ x \in [-1,1)$, the process $\{\xi_t\}$ is of Poisson type and $H = \delta_1$ corresponds to the "Gaussian" case. In the last case, $\{\xi_t\}$ becomes a "Brownian" motion and the corresponding infinitesimal operator τ satisfies*

(a) $P_n^{(\alpha,\beta)}(x)$ *belongs to* $\mathcal{D}(\tau)$

(b) $\tau P_n^{(\alpha,\beta)}(x) = \dfrac{n(n+1+\alpha+\beta)}{\alpha+\beta+2} P_n^{(\alpha,\beta)}(x), \quad n = 0,1,2,...$

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