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DEGENERACY OF HOLOMORPHIC CURVES IN $\mathbb{P}^n$

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Dedicated to the memory of Le Van Thiem

Abstract. By using the Nevanlinna-Cartan theory we establish some conditions for degeneracy of holomorphic curves in the complex projective space $\mathbb{P}^n$.

1. Introduction

Let $X \subset \mathbb{P}^n$ be a projective subvariety of $\mathbb{P}^n$, by which we mean an irreducible algebraic subset. A holomorphic curve in the projective subvariety $X \subset \mathbb{P}^n$ is said to be degenerate if it is contained in some proper algebraic subset of X. In 1979, M. Green and Ph. Griffths [3] conjectured that every holomorphic curve in a complex projective hypersurface of general type is degenerate. M. Green [2] proved the degeneracy of holomorphic curves in the Fermat variety of large degree. In [8], A. M. Nadel showed the validity of Green-Griffth's conjecture for some classes of hypersurfaces and applied this result to construct some explicit examples of hyperbolic surfaces in P^3 of degree $3e \geq 21$. Recently, H. H. Khoai [4] proved the conjecture for other classes of hypersurfaces, and gave examples of hyperbolic surfaces of arbitrary degree ≥ 22.

It is well-known that every holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ omiting $n+2$ hyperplanes in general position, is linearly degenerate (Bloch-Cartan). That is $f(\mathbb{C})$ is contained in some proper linear subspace of $\mathbb{P}^n$. In [10] again by using Borel's lemma, M. Ru proved that every holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ omiting at least three distinct hyperplanes which are linearly dependent, is linearly degenerate.

In this paper, by using the Nevanlinna-Cartan theory we obtain some conditions for the degeneracy of holomorphic curves in $\mathbb{P}^n$. The condition "omiting hyperplanes" of Bloch-Cartan and M. Ru can be weakened by the condition "ramifying over hyperplanes with large degree".

2. Generalized Bloch-Cartan's theorem

Let f be a holomorphic curve in the complex projective space, i.e, a holomorphic map from complex plane $\mathbb{C}$ into the n-dimensional complex projective space $\mathbb{P}^n$. Suppose that X is represented by a collection of holomorphic functions on $\mathbb{C}$:

$$f = (f_0, ..., f_n),$$

where the functions f_i, $0 \leq i \leq n$, have no common zeros.

Definition 2.1. The curve f is said to be *linearly non-degenerate* if the image of f is not contained in any linear subspace of $\mathbb{P}^n$ of dimension less than n.

Now let $H_1, H_2, ..., H_q$ be hyperplanes in P^n in *general position*. This means that these hyperplanes are linearly independent if $q \leq n$, and any $(n+1)$ of these hyperplanes are linearly independent if $q \geq n+1$.

Definition 2.2. Let f be a holomorphic curve from $\mathbb{C}$ into $\mathbb{P}^n$ and let H be a hyperplane of P^n such that $H \not\supset f(\mathbb{C})$.

Assume that the hyperplane H is defined by the linear equation $L = 0$. Then we define the *pull-backed divisor of f* over H by

$$f^*H = \sum \mathrm{ord}_a(L \circ f)a,$$

where the sum is taken on all of zeros a of $L \circ f(z)$. Let $\deg_z f^*H$ denote the degree of the pull-backed divisor f^*H at $z \in \mathbb{C}$.

Definition 2.3. We say that f *ramifies* at least d $(d > 0)$ over H if $\deg_z f^*H \geq d$ for all $z \in f^{-1}H$. This means every zero of the entire function $L \circ f$ has multiplicity at least d. In the case $f^{-1}H = \emptyset$, we set $d = \infty$.

Let H_j, $j = 1, 2, ..., q$, be hyperplanes of $\mathbb{P}^n$ in general position. Then the following statement is valid.

Lemma 2.1. (H. Cartan [1]) *Assume that f is linearly non-degenerate and ramifies at least d_j over H_j, $1 \leq j \leq q$. Then*

$$\sum_{j=1}^{q} \Big(1 - \frac{n}{d_j}\Big) \leq n+1.$$

We will apply Lemma 2.1 to prove following theorem.

Theorem 2.1. (Generalized Bloch-Cartan's Theorem) *Let $H_0, ..., H_{n+1}$ be $n+2$ hyperplanes of $\mathbb{P}^n$ in general position. Assume that f ramifies at least d_j over H_j, $0 \leq j \leq n+1$. Suppose that*

$$\sum_{j=0}^{n+1} \frac{1}{d_j} < \frac{1}{n}, \quad (n \geq 2). \tag{1}$$

Then f is linearly degenerate.

Proof. Let $L_0(x), ..., L_{n+1}(x)$ denote the linear forms defining the hyperplanes.

Because any set of $n+2$ hyperplanes in $\mathbb{P}^n$ is linearly dependent over $\mathbb{C}$, there exist constants c_j not all zeros such that

$$\sum_{j=0}^{n+1} c_j L_j(x) = 0.$$

Since $H_0, ..., H_{n+1}$ are in general position in $\mathbb{P}^n$, we have $c_j \neq 0$, $0 \leq j \leq n+1$. Moreover, $(n+1)$ is the smallest number such that we have such a relation.

Hence

$$\sum_{j=0}^{n+1} c_j L_j(f) \equiv 0.$$

We now prove that $L_j(f) = L_j \circ f$, $0 \leq j \leq n$, are linearly dependent. Assume that $L_j(f) = L_j \circ f$, $0 \leq j \leq n$, are linearly independent. We define a holomorphic curve g in $\mathbb{P}^n$ by setting

$$g(z) = (L_0(f)(z), ..., L_n f(z)) \quad \forall z \in \mathbb{C}.$$

Then g is linearly non-degenerate. Consider the following hyperplanes in general position in $\mathbb{P}^n$:

$$H_0 = \{x_0 = 0\}, ..., H_n = \{x_n = 0\}, H_{n+1} = \{c_0 x_0 + ... + c_n x_n = 0\}.$$

By the hypothesis, g ramifies at least d_j over H_j, $0 \leq j \leq n$. It follows from Lemma 2.1 that

$$\sum_{j=0}^{n+1} \left(1 - \frac{n}{d_j}\right) \leq n+1.$$

Hence

$$\sum_{j=0}^{n+1} \frac{1}{d_j} \geq \frac{1}{n}.$$

We have arrived at a contradiction, because $\sum_{j=0}^{n+1} \frac{1}{d_j} < \frac{1}{n}$. So there is a non-trivial linear relation

$$c'_0 L_0 \circ f + ... + c'_n L_n \circ f \equiv 0, \quad c'_j \in \mathbb{C}.$$

Then the image of f is contained in the linear subspace (hyperplane) defined by the equation

$$\sum_{j=0}^{n} c'_j L_j(x) = 0.$$

By the minimality of $n+1$, this subspace is proper. The proof is complete. □

Corollary 2.1. (Bloch-Cartan [6]) *Let $f : \mathbb{C} \to \mathbb{P}^n$ be a non-constant holomorphic curve with $n \geq 2$. Let $H_0, ..., H_{n+1}$ be $n+2$ hyperplanes in general position. If the image of f lies in the complement of $H_0 \cap ... \cap H_{n+1}$, then it lies in some hyperplane.*

Proof. It suffices to apply Theorem 2.1 with $d_j = \infty$, $0 \leq j \leq n+1$. □

Example. It is clear that

$$f : \mathbb{C} \longrightarrow \mathbb{P}^2,$$
$$z \longmapsto (z^5, -z^5, 1),$$

is a holomorphic curve in the complex projective plane $\mathbb{P}^2$. Take 4 hyperplanes of $\mathbb{P}^2$ in general position:

$$H_0 = \{x_0 = 0\}, \quad H_1 = \{x_1 = 0\}, \quad H_2 = \{x_2 = 0\}, \quad H_3 = \{x_0 + x_1 + x_2 = 0\}$$

Note that f does not omit H_0 and H_1. Since $\frac{2}{5} < \frac{1}{2}$, f is linearly degenerate (Theorem 2.1). The image of f is contained in the hyperplane defined by the equation $x_0 + x_1 = 0$.

3. Degeneracy of holomorphic curves

Definition 3.1. A projective variety $X \subset \mathbb{P}^n$ is said to be *Brody hyperbolic* if every holomorphic curve $f : \mathbb{C} \longmapsto X$ is constant. Similarly, if Y is a subset of X, we say that Y is *Brody hyperbolic* (in X) if every holomorphic curve $f : \mathbb{C} \to X$, whose image is contained in Y, is constant.

Recent studies suggest that the hyperbolicity of a complex space X is related to the finiteness of the number of rational or integral points of X (see [10]).

It is well-known that the complement of $2n+1$ hyperplanes in general position in $\mathbb{P}^n$ is Brody hyperbolic (Bloch, Dufresnoy, Green, Fujimoto, see [6]). The question is that given a set $\mathcal{H}$ of hyperplanes in $\mathbb{P}^n$ (not necessarily in general position), what is necessary and sufficient condition for $\mathcal{H}$ such that $\mathbb{P}^n - |\mathcal{H}|$ is Brody hyperbolic and how do we verify it ? In [10], M. Ru answered this question by providing an algorithm (in term of linear algebra) to determine whether or not $\mathbb{P}^n - |\mathcal{H}|$ is Brody hyperbolic. Here $|\mathcal{H}|$ denotes the finite union of hyperplanes in $\mathcal{H}$.

Definition 3.2 ([10]). Let $\mathcal{H}$ be a set of hyperplanes in $\mathbb{P}^n$. Let V be a linear subspace of $\mathbb{P}^n$. V is called $\mathcal{H}$- *admissible* if V is not contained in any hyperplane in $\mathcal{H}$. $\mathcal{H}$ is said to be *nondegenerate* (over $\mathbb{C}$) if for every $\mathcal{H}$- admissible subspace V of $\mathbb{P}^n$ of projective dimension greater than or equal to one, $\mathcal{H} \cap V$ contains at least three distinct hyperplanes of V which are linearly dependent over $\mathbb{C}$.

In [10], M. Ru proved that the complement of $\mathcal{H}$ in $\mathbb{P}^n$ is Brody hyperbolic if and only if $\mathcal{H}$ is nondegenerate over $\mathbb{C}$. This means that every holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n - |\mathcal{H}|$ is constant if and only if $\mathcal{H}$ nondegenerate (over $\mathbb{C}$).

In this section we study the degeneracy of holomorphic curves ramifying over hyperplanes in $\mathcal{H}$.

Defenition 3.3. Let $\mathcal{H} = \{H_1, H_2, ..., H_q\}$, $q \geq 3$, be a set of q hyperplanes in $\mathbb{P}^n$. We say that a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ *ramifies with large degree* over $\mathcal{H}$ if the image of f is not contained in the intersection of any three hyperplanes in $\mathcal{H}$ and for every $j = 1, ..., q$, f ramifies at least d_j over $H_j \in \mathcal{H}$ such that

$$\sum_{j=1}^{q} \frac{1}{d_j} < \frac{1}{q-2}\,. \tag{2}$$

Theorem 3.1. *Let $\mathcal{H} = \{H_1, ..., H_q\}$ be a set of q hyperplanes of $\mathbb{P}^n$ with $q \geq 3$. Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve in $\mathbb{P}^n$. Assume that f ramifies with large degree over $\mathcal{H}$. Then f linearly degenerate if $\mathcal{H}$ contains at least three distinct hyperplanes which are linear dependent over $\mathbb{C}$.*

Proof. Let $L_1(x), ..., L_q(x)$ $(q \geq 3)$ be the linear forms defining the hyperplanes in $\mathcal{H}$. By the linear dependence assumption, there exist non-zero constants a_i such that

$$\sum_{i=1}^{q} a_i L_i(x) \equiv 0.$$

Without loss of generality, by shrinking the set of hyperplanes, we can assume that q is the smallest integer such that we have such a relation (i.e. $a_i \neq 0$ for all i). Since the hyperplanes are distinct, we have $q \geq 3$. Now

$$\sum_{i=1}^{q} a_i L_i \circ f \equiv 0.$$

We are going to prove that the functions $L_1 \circ f, ..., L_{q-1} \circ f$ are linearly dependent. Assume that $L_j \circ f$, $1 \leq j \leq q-1$, are linearly independent. Because the image of f is not contained in the intersection of any three distinct hyperplanes in $\mathcal{H}$, we can define a holomorphic curve g in $\mathbb{P}^{q-2}$ by

$$g : z \in \mathbb{C} \longmapsto (L_1 \circ f(z), ..., L_{q-1} \circ f(z)).$$

Consider the following hyperplanes in general position in $\mathbb{P}^{q-2}$:

$$H_1 = \{z_1 = 0\}, ..., H_{q-1} = \{z_{q-1} = 0\}, H_q = \{a_1 z_1 + ... + a_{q-1} z_{q-1} = 0\}.$$

By the hypothesis, g ramifies at least d_j over H_j, $1 \leq j \leq q$. It follows from Lemma 2.1 that

$$\sum_{j=1}^{q} \Big(1 - \frac{q-2}{d_j}\Big) \leq q - 1.$$

Hence

$$\sum_{j=1}^{q} \frac{1}{d_j} \geq \frac{1}{q-2}\,.$$

This contracdicts our assumption. Thus there is a non-trivial linear relation.

$$a'_1 L_1 \circ f + ... + a'_{q-1} L_{q-1} \circ f \equiv 0.$$

So the image of f is contained in the linear subspace (hyperplane) defined by the equation

$$\sum_{j=1}^{q-1} c_j L_j(x) = 0,$$

and this is a proper subspace of $\mathbb{P}^n$ by the condition that q is minimal. □

Corollary 3.1. (M. Ru's Theorem, see [10]). *Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve. If $f(\mathbb{C})$ omits at least three dictinct hyperplanes in $\mathbb{P}^n$ which are linearly dependent over $\mathbb{C}$, then f must be linearly degenerate.*

Proof. Apply Theorem 3.1 with $q = 3$, $d_1 = d_2 = d_3 = \infty$. □

Theorem 3.2. *Let $\mathcal{H}$ be a set of q hyperplanes in $\mathbb{P}^n$, $q \geq 3$. Then $\mathcal{H}$ is non-degenerate over $\mathbb{C}$ if and only if every holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ ramifying with large degree over $\mathcal{H}$, is constant.*

Proof. Let $\mathcal{H}$ be nondegenerate over $\mathbb{C}$. Then $\mathcal{H}$ contains at least three distinct hyperplanes which are linearly dependent. By Theorem 3.1, every holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ ramifying with large degree over $\mathcal{H}$, is linearly degenerate. This means that the image of f is contained in some proper linear subspace W of $\mathbb{P}^n$. We have $\dim W < n$. Since f ramifies at least d_j over all H_j in $\mathcal{H}$, W is $\mathcal{H}$-admissible. By the assumption that $\mathcal{H}$ is nondegenerate over $\mathbb{C}$, $\mathcal{H} \cap W$ still contains at least three distinct hyperplanes of W which are linearly dependent.

By $\mathrm{Im} f \subset W$ we have $H_j \cap \mathrm{Im} f = (H_j \cap W) \cap \mathrm{Im} f$ for all $H_j \in \mathcal{H}$. It follows that $f^*H = f^*(H_j \cap W)$ for all $H_j \in \mathcal{H}$. Hence

$$\deg_z f^*(H_j \cap W) = \deg_z f^* H_j \geq d_j$$

for all $z \in f^{-1}(H_j \cap W)$. Therefore f still ramifies at least d_j over $H_j \cap W$ in $\mathcal{H} \cap W$ for all $j = 1, ..., q$. We know that inequality (2) still holds in this case. So we can apply Theorem 3.1 again. By induction, we conclude that f is constant.

Conversely, if $\mathcal{H}$ is not degenerate over $\mathbb{C}$, then we will construct a nonconstant holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n - |\mathcal{H}|$. Since $\mathcal{H}$ is not degenerate, there exists an $\mathcal{H}$-admissible subspace V of $\mathbb{P}^n$ of projective dimension greater than or equal to one such that $\mathcal{H} \cap V$ does not contain at least three distinct hyperplanes which are linearly dependent over $\mathbb{C}$. Without loss of generality, we can assume that $W = \mathbb{P}^n$. Then $q \leq n+1$, and $H_1, ..., H_q$ are linearly independent. We may assume that $H_1, ..., H_q$ are the first q coordinate planes, then the holomorphic curve f represented by $f = (1, e^z, ..., e^z)$ is non-constant and satifies our conditions. □

Corollary 3.2. (M. Ru's Theorem; see [10]). $\mathbb{P}^n - |\mathcal{H}|$ *Brody hyperbolic if and only if $|\mathcal{H}|$ is nondegenerate over $\mathbb{C}$.*

Proof. Note that every holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n - |\mathcal{H}|$, is a holomorphic curve in $\mathbb{P}^n$ ramifying with large degree over $\mathcal{H}$. □

4. The Fermat variety

By using Theorem 2.1 we can prove Green's theorem (in [6]). The Fermat variety X in $\mathbb{P}^n$, of degree d, is defined by the equation

$$x_0^d + ... + x_n^d = 0.$$

Theorem 4.1. (Green [6]). *Let $f = (f_0, ..., f_n) : \mathbb{C} \to \mathbb{P}^n$ with $n \geq 2$ be a holomorphic curve in the Fermat variety X, so*

$$f_0^d + ... + f_n^d \equiv 0.$$

If $d \geq n^2$ then the functions $f_0^d, ..., f_{n-1}^d$ are linearly dependent.

Proof. We define a holomorphic curve g in $\mathbb{P}^{n-1}$ by the relation

$$z \in \mathbb{C} \longmapsto (f_0^d(z), ..., f_{n-1}^d(z)) \in \mathbb{P}^{n-1}.$$

Consider the following $(n+1)$ hyperplanes in general position in $\mathbb{P}^{n-1}$:

$$H_0 = \{x_0 = 0\}, ..., H_{n-1} = \{x_{n-1} = 0\}, H_n = \{x_0 + ... + x_{n-1} = 0\}.$$

We know that g ramifies at least $d_j \geq d$ over H_j, $0 \leq j \leq n$, and the following condition holds

$$\sum_{j=0}^{n} \frac{1}{d_j} \leq \sum_{j=0}^{n} \frac{1}{d} = \frac{n+1}{d} \leq \frac{n+1}{n^2} < \frac{n+1}{n^2-1} = \frac{1}{n-1} \cdot$$

By Theorem 2.1, g is linearly degenerate. The proof is complete. □

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