A NEW ALEXANDER-EQUIVALENT ZARISKI PAIR

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Dedicated to the memory of Le Van Thiem

1. Statement of the result

Consider a moduli \( M(\sigma; n) \) of plane curves with a given degree \( n \) and having prescribed set of finite singularities \( \sigma \). Let \( C, C' \in M \). The pair of curves \( (C, C') \) is called a Zariski pair if the pairs of spaces \( (\mathbb{P}^2, C) \) and \( (\mathbb{P}^2, C') \) are not homeomorphic. A Zariski pair \( (C, C') \) is called Alexander-equivalent if their generic Alexander polynomials coincide. The first example of Alexander-equivalent Zariski pair \( (C, C') \) for irreducible plane curves are given in [5]. They are plane curves of degree 12 with 27 cusps. Here \( C \) is a generic \((3, 3)\)-covering of a three cuspidal quartic and \( C' \) is constructed using a six cuspidal non-conical sextic.

The purpose of this note is to construct an Alexander-equivalent Zariski pair \( (D, D') \) of irreducible curves of degree 8 with 12 cusps. We give a brief recipe for the construction. Consider the moduli space \( M(cA_2; n) \) plane curves of degree \( n \) with \( c \) cusps of type \( y^2 - x^3 = 0 \). As we only consider cuspidal curves in this note, we simply denote \( M(c; n) \) in stead of \( M(cA_2; n) \). It is well-known that \( M(3; 4) \) is irreducible. In fact, its dual is the moduli of plane curves of degree 3 with one node by the Plücker’s formula (see [N]). The fundamental group of the complement \( \mathbb{P}^2 - C, C \in M(3; 4) \), is a finite non-abelian group of order 12 ([8, 3]).

The first curve \( D \) is given by the generic \((2,2)\)-cyclic covering \( C_{2,2}(Z) \) of a quartic \( Z \) in \( M(3; 4) \). Thus the fundamental group \( \pi_1(\mathbb{P}^2 - D) \) is a finite group of order 24 and the Alexander polynomial \( \Delta_D(t) \) is equal to that of \( C \) by [4] and therefore it is trivial. Actually we know that the generic Alexander polynomials of any cuspidal curves of degree \( 2m \), \( m = 1, 2, \ldots \) are trivial by [1]. Thus the cuspidal curves of degree 8 is also interesting in this sense.

To construct the second curve \( D' \), we start from a two cuspidal quartic \( Q \) i.e., \( Q \in M(2; 4) \). If \( Q \) is generic, it has 8 flexes and a bi-tangent line by the Plücker’s flex formula ([7, 2, 6]). Namely the dual curve is a 8 cuspidal sextic with one node. Let \( L_\infty \) be the line at infinity. We choose a generic line at infinity \( L_\infty \) and two flexes \( P_1, P_2 \) with tangent line \( L_1 \) and \( L_2 \) respectively such that \( L_1 \cap L_2 \cap L_\infty = \emptyset \). Then in the affine space \( \mathbb{C}^2 := \mathbb{P}^2 - L_\infty \) we take the change of linear coordinates so that \( L_1 \) and \( L_2 \) are given by the coordinate axis \( x = 0 \).
and \( y = 0 \). Let \( f(x, y) = 0 \) be the defining polynomial of \( Q \) and let \( P_1, P_2 \) be the cusps and let \( R_1 = (\alpha^2, 0) \) and \( R_2 = (0, \beta^2) \) be the flex points. First take the flex double covering at \( R_1 \) and let \( Q' := \mathcal{F}(2)(R_1) \) ([5]). Namely \( Q' \) is the pull-back of \( Q \) by the covering mapping \((x, y) \mapsto (x, y^2)\) and \( Q' \) is defined by \( f(x, y^2) = 0 \). \( Q' \) has 5 cusps: two cusps \( P_{i,\pm} \) from each \( P_i \), \( i = 1, 2 \) and one cusp comes from \( R_1 \) and two flexes \( R_{2,\pm} \) on \( x = 0 \) coming from \( R_2 \). They have the same tangent line \( x = 0 \). Then we take the flex double covering along \( x = 0 \) and let \( D' \) be the pull-back of \( Q' \). We see that \( D' \) is defined by \( f(x^2, y^2) = 0 \) and it has 12 cusps: 10 cusps come from the five cusps of \( Q' \) and two cusps come from flexes \( R_{2,\pm} \). More precisely, we have 4 cusps coming from each cusp of \( Q \) and four cusps coming from flex points \( R_1, R_2 \), which are given by \((\pm \alpha, 0), (0, \pm \beta)\). We will show that \( \pi_1(\mathbb{P}^2 - D') \cong \mathbb{Z}_8 \) by a direct computation using the Zariski’s pencil method.

2. Fundamental groups

2.1. Construction. For the practical computation of flex coverings, we need to know the locus of flex points explicitly. To construct such a two cuspidal quartic, we start from a curve \( C \) of type \((1, 2; 4)\) with one cusp and a flex with the tangent line \( y = 0 \). We denote the moduli of such curve by \( \mathcal{M}_1 \). By the action of automorphisms, we may assume that the cusp is at \((3, 1)\) and the flex is at \((2, 0)\). Then the generic curve is described by one parameter family

\[
g(x, y) = y^2 + (52 - 48s + 9s^2)y + (34s - 36 - 6s^2)xy + (-6s + 6 + s^2)x^2y - 80 + 48s + (144 - 88s)x + (60s - 96)x^2 + (28 - 18s)x^3 + (2s - 3)x^4.
\]

Now take a generic curve \( C \) in \( \mathcal{M}_1 \) which is defined by \( g(x, y) = 0 \). We take the symmetric double covering \( \varphi : \mathbb{C}^2 \to \mathbb{C}^2 \), defined by \( \varphi(u, v) = (u + v, uv) \) ([3]) and let \( S_2(C) \) be the quartic defined by the pull-back of \( C \). The branching locus of the symmetric covering is given by \( \Delta = \{ (x, y); x^2 - 4y = 0 \} \). Thus we must assume that any cusps or the marked flex point of \( C \) are not located on \( \Delta \). \( S_2(C) \) is defined by the symmetric polynomial \( g'(u, v) = 0 \) where \( g'(u, v) := g(u + v, uv) \) and \( S_2(C) \) has two cusps at \((\beta_1, \beta_2)\) and \((\beta_2, \beta_1)\) where \( \beta_1, \beta_2 \) are the root of \( t^2 - 3t + 1 = 0 \). The flex at \((2, 0)\) splits into two flexes \( R_1 = (2, 0) \) and \( R_2 = (0, 2) \) in \( S_2(C) \). Their tangent lines are given by \( y = 0 \) and \( x = 0 \) respectively. See [3] for the detail about symmetric coverings.

Remark. We remark here that the pull-back of a flex of \( C \) is not necessarily a flex of \( S_2(C) \) in general, as the pull-back of the tangent line is not a line in general. However this is the case if the flex is on \( x\)-axis with the tangent line \( y = 0 \) as the pull-back of \( y = 0 \) is the the union of two lines \( u = 0 \) and \( v = 0 \).

We denote by \( \mathcal{M}_2 \) the set of symmetric quartic with two cusps and two marked flexes whose tangent lines are coordinate axis. Note that \( S_2(C) \subseteq \mathcal{M}_2 \) for any \( C \in \mathcal{M}_1 \) and conversely any \( Q \in \mathcal{M}_2 \) is presented as a symmetric double covering \( S_2(C) \) of some \( C \in \mathcal{M}_1 \).

We construct a correspondence \( \varphi : \mathcal{M}_2 \to \mathcal{M}(12; 8) \). For any quartic \( Q \in \mathcal{M}_2 \) defined by \( f(x, y) = 0 \), we take twice flex covering and we define \( Q \mapsto \tilde{Q} \), where \( \tilde{Q} \)
of degree 8 with 12 cusps which is defined by \( f(x^2, y^2) = 0 \). We denote the image of \( \mathcal{M}_2 \) in \( \mathcal{M}(12; 8) \) by \( \mathcal{M}_3 \) and the image of \( \mathcal{M}(3; 4) \) by the generic (2,2)-covering by \( \mathcal{M}_{Zar} \). So \( \mathcal{M}_{Zar} = \{ \mathcal{C}_{2,2}(Q); Q \in \mathcal{M}(3; 4) \} \).

**Theorem 2.1.** (1) For any \( D' \in \mathcal{M}_3 \), we have \( \pi_1(\mathbb{P}^2 - D') \cong \mathbb{Z}_8 \) and the generic Alexander polynomial \( \Delta_{D'}(t) \) is trivial.

(2) For any \( D \in \mathcal{M}_{Zar} \), \( \pi_1(\mathbb{P}^2 - D) \) is a finite non-abelian group of order 24 and the generic Alexander polynomial is trivial.

In particular, the pair of irreducible curves \((D, D')\) is a Alexander-equivalent Zariski pair for \( D' \in \mathcal{M}_3 \) and \( D \in \mathcal{M}_{Zar} \) and therefore the moduli space \( \mathcal{M}(12; 8) \) is not irreducible.

**Remark.** Any generic curve \( C \in \mathcal{M}(2; 4) \) has 8 flexes and a bi-tangent line by the flex formula (see [7, 2],[6]). Thus the dual curves have degree 6 and the singularities are 8 cusps and a node. It is easy to show that \( \mathcal{M}(2; 4) \) is irreducible variety. Let \( \mathcal{M}' \) be the moduli of two cuspidal quartics with two marked flexes. There is a surjective forgetting morphism \( \psi : \mathcal{M}' \to \mathcal{M}(2; 4) \). There exists a canonical (but not unique) rational mapping from \( \mathcal{M}' \) to \( \mathcal{M}(12; 8) \) as follows.

For any \( C' \in \mathcal{M}' \), we have a linear change of coordinates so that \( C \) is defined by \( f(x, y) = 0 \) and two marked flex tangents are given by \( y = 0 \) and \( x = 0 \). Then we can take the mapping \( C \mapsto \varphi(C) := \tilde{C} \) where \( \tilde{C} \) is defined by \( f(x^2, y^2) = 0 \) as above. The mapping \( \varphi \) is unique if we fix the line at infinity and is well-defined on \( C \) if two tangent lines at marked flex points intersect outside of the line at infinity.

By a direct computation, it seems that \( \mathcal{M}' \) has two irreducible components and one components is equal to the PSL\( (3; \mathbb{C}) \)-orbit of \( \mathcal{M}_2 \). We do not know whether the image of these components are in a same component of the moduli \( \mathcal{M}(12; 8) \) or not.

2.2. **Computation of the fundamental group.** The second assertion of Theorem 2.1 follows from Theorem 5.5 of [4]. To prove the assertion (1), we consider the following symmetric polynomial

\[
 f(x, y) := (xy - \frac{3}{2}(x + y - 2)^2)^2 + xy - 2(x + y - 2)^3 + \frac{3}{4}(x + y - 2)^4
\]

which is the pull-back of

\[
 g(x, y) = (y - \frac{3}{2}(x - 2)^2)^2 + y - 2(x - 2)^3 + \frac{3}{4}(x - 2)^4
\]

by the symmetric covering. Let \( C_1 := C^0(f) \) and \( C_2 := \{(x, y); f(x, y^2) = 0\} \) and \( C_3 := \{(x, y); f(x, y^2) = 0\} \). The quartic \( C_1 \) has two cusps at \( P_1 := (\beta_2, \beta_1) \) and \( P_2 := (\beta_1, \beta_2) \) where \( \beta_1 = D(3 - \sqrt{5})/2, \beta_2 = (3 + \sqrt{5})/2 \) and two flex points at \( R_1 := (2, 0) \) and \( R_2 := (0, 2) \) where the tangent lines are given by \( y = 0 \) and \( x = 0 \). The discriminant polynomial of \( f \) with respect to \( y \) is given by

\[
 \Delta_y(f)(x) = x^2(3x - 8)(111x^3 - 441x^2 + 311x + 216)(x^2 - 3x + 1)^3
\]

For the computation of the fundamental group, we consider the vertical pencil lines \( L_\eta = \{ x = \eta \}, \eta \in \mathbb{C} \). The three roots of \( 111x^3 - 441x^2 + 311x + 216 = 0 \),
which we denote by $\alpha_1, \alpha_2, \alpha_3$, and $x = 8/3$ corresponds to the singular pencil lines which are simply tangent to $C_1$. They are real numbers which are given by $\alpha_1 = -0.419..., \alpha_2 = 1.771..., \alpha_3 = 2.620...$. The roots $x^2 - 3x + 1 = 0$ corresponds to cusps and they are given by $\beta_1$ and $\beta_2$. We note that $\beta_2 = 2.618...$ is slightly smaller than $\alpha_3 = 2.620...$. See Figure 2. The graph of $f$ is given in Figure 1 and the local enlarged graph is given in Figure 2.

We are going to show that $\pi_1(\mathbb{P}^2 - C_3) \cong \mathbb{Z}_8$ using the pencil $x = \eta, \eta \in \mathbb{C}$ and the information for $C_1$. This implies also the commutativity: $\pi_1(\mathbb{P}^2 - C_2) \cong \mathbb{Z}_8$. The singular pencils of $C_i$, $i = 1, 2, 3$ corresponds to $\Sigma_i$ which are given by Lemma 2.4 of [5] as

$$Si_1 = \{0, \alpha_1, \alpha_2, \alpha_3, 8/3, \beta_1, \beta_2\}, \quad \Sigma_2 = \Sigma_1 \cup \{2\} \quad \Sigma_3 = \{\pm \sqrt{\eta}; \eta \in \Sigma_2\}$$

We use the same notation as in [4] and [5]. Thus the bullets in the following Figures are the intersections of $C_3$ (of $C_1$ in Figure 4) and the pencil lines. A path ending to a bullet denotes a small loop going around that intersection counterclockwise (which is called a lasso in [4]). Monodromy relations are read from the behavior of four points $L_\eta \cap C_1$ over the real line, which is sketched in Figure 4. For $\eta \in \Sigma_i$, we denote $\eta^\pm = \eta \pm \varepsilon$ where $\varepsilon > 0$ is sufficiently small.
We take generators $\rho_1, \ldots, \rho_4$, $\xi_1, \ldots, \xi_4$ of $\pi_1(L_0^+ - L_0^+ \cap C_3)$ as in Figure 3. The base point is chosen to be $[0, 1, 0]$ which is the base point of the pencil $L_\eta$, $\eta \in \mathbb{C}$ and also equal to the point at infinity of $L_\eta \cong \mathbb{P}^1$. Note that $\rho_1, \ldots, \rho_4$ are symmetric to $\xi_1, \ldots, \xi_4$ with respect to the origin. Thus any relation in $\rho_1, \ldots, \rho_4$ is also true for $\xi_1, \ldots, \xi_4$. First we have the relation:

\[(2.2) \quad \xi_4 \ldots \xi_1 \rho_4 \ldots \rho_1 = e.\]
The monodromy relation at $x = 0$ is the cusp relation and it is given by

$$\rho_2 = \rho_4, \quad \{\rho_2, \rho_3\} = e, \quad \xi_2 = \xi_4, \quad \{\xi_2, \xi_3\} = e,$$

where $\{a, b\} = abab^{-1}a^{-1}b^{-1}$ as in [4]. The relation from the singular line $x = \alpha_1$ is equivalent to:

$$\rho_1 = \rho_3^{-1}\rho_4\rho_3, \quad \xi_1 = \xi_3^{-1}\xi_4\xi_3.$$  

At this point, we have reduced our generators to $\{\rho_2, \rho_3, \xi_2, \xi_3\}$. Now the main point is the following observation. Let $y_1(x), y_2(x)$ be the roots of $f(x, y) = 0$ for $\beta_1 < x < \alpha_2$ such that $\Im(y_i(x)) > 0, i = 1, 2$. The other two roots are given by their complex conjugates. We may assume that $\Re(y_1(\beta_1^+)) < \Re(y_2(\beta_1^+))$. Then

**Assertion 1.** The inequality $\Re(y_1(x)) < \Re(y_2(x))$ is preserved on the interval $(\beta_1, \alpha_2)$.

Assuming this, the monodromy relation at $x = \alpha_2$ is given as

$$\rho_3 = \xi_3'' = (\xi_3\xi_2\xi_1)^{-1}\xi_4(\xi_3\xi_2\xi_1), \quad \xi_3 = \rho_4'' = (\rho_3\rho_2\rho_1)^{-1}\rho_4(\rho_3\rho_2\rho_1)$$

where $\rho_4', \rho_4'', \xi_4', \xi_4''$ are defined as in Figure 5 and we have

$$\rho_4' = \rho_3^{-1}\rho_4\rho_3, \quad \xi_4' = \xi_3^{-1}\xi_4\xi_3, \quad \xi_4'' = \xi_2, \quad \rho_4'' = \rho_2$$

by (2.3) and (2.4). These relations reduce to

$$\rho_3 = \xi_2, \quad \xi_3 = \rho_2.$$  

The monodromy relations at $x = 2$ and $\beta_2$ do not give any new relations. At $x = \beta_2^+$, we consider the generators as in Figure 6 where $\hat{\rho}_4$ is defined as in Figure 6. By an easy computation we have $\hat{\rho}_4 = \rho_3^{-1}\rho_2\rho_3$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Deformation of $C_1 \cap L_\eta$}
\end{figure}
The monodromy relation at \( x = \alpha_3 \) is given by \( \hat{\rho}_4 = \rho_1 \) which reduces to \( \rho_2 \rho_3 = \rho_3 \rho_2 \). Using the cusp relation \( \rho_2 \rho_3 \rho_2 = \rho_3 \rho_2 \rho_3 \) we get the relation \( \rho_2 = \rho_3 \). Thus the generators \( \rho_2, \rho_3, \xi_2, \xi_3 \) reduces to the single element \( \rho_2 \), which implies that 
\[ \pi_1(\mathbb{P}^2 - C_3) \cong \mathbb{Z}_8. \]

2.3. Appendix: Proof of Assertion 1. We give a brief proof of Assertion 1. First, the four roots of \( f(x,y) = 0 \) in \( y \), with a real \( x \) being fixed, are closed by complex conjugation. So we look at those roots with positive imaginary part, say \( y_1(x) \) and \( y_2(x) \). Assume that there exists a \( x_0 \in (\beta_1, \alpha_2) \) such that \( \Re(y_1(x_0)) = \Re(y_2(x_0)) \) and put \( u_0 = \Re(y_1(x_0)) \) and \( v_1, v_2 \) be the imaginary parts. Consider \( f(x, u + iv) \) and put \( f_e(x, u, v) \) and \( f_o(x, u, v) \) be the real and the imaginary parts. By an easy computation, \( f_e \) and \( f_o \) are polynomials of \( v \) of degree 4 and 3 respectively and

\[
f_e(x, u, v) = 3v^4 + 3(xu - \frac{3}{2}(x + u - 2)^2)v^2 - (xv - 3(x + u - 2)v)^2
+ 6(x + u - 2)v^2 - \frac{9}{2}(x + u - 2)^2v^2 + (xu - \frac{3}{2}(x + u - 2)^2)^2
+ xu - 2(x + u - 2)^3 + \frac{3}{4}(x + u - 2)^4
\]

\[
f_o(x, u, v) = v^3(-9x + 26 - 12u) + v(157x + 12u^3 - 78u^2 - 120
+ 27xu^2 - 132xu + 9x^3 + 168u - 66x^2)\]
By the assumption, $f_e(x_0, u_0, v) = 0$ and $f_o(x_0, u_0, v) = 0$ has four common solutions $\pm v_1, \pm v_2$. As $\deg_v f_o(x_0, u_0, v) = 3$, we must to have $f_o(x_0, u_0, v) \equiv 0$. For this, it is necessary that the coefficients $c_3 := -9x + 26 - 12u$ and $c_1 := 157x + 12u^3 - 78u^2 - 120 + 27xu^2 - 132xu + 9x^3 + 168u - 66x^2$ should vanish at $(x, u) = (x_0, u_0)$. Thus $u_0 = -3/4x_0 + 13/6$ and $x_0$ is the solution of $c_1 = -1/9 - 3/8x^3 - 3/2x + 19/12x^2 = 0$. Thus the only possibility for $x_0$ in the interval $(\beta_1, \alpha_2)$ is $x_0 = 1.590...$. However $f_e(x_0, -3/4x_0 + 13/6, v) = 0$ does not have four real solutions in this case. By contradiction, this completes the proof of Assertion 1.

References


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