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A NEW ALEXANDER-EQUIVALENT ZARISKI PAIR

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Dedicated to the memory of Le Van Thiem

1. Statement of the result

Consider a moduli $\mathcal{M}(\sigma; n)$ of plane curves with a given degree n and having prescribed set of finite singularities σ. Let $C, C' \in \mathcal{M}$. The pair of curves (C, C') is called *a Zariski pair* if the pairs of spaces $(\mathbf{P}^2, C)$ and $(\mathbf{P}^2, C')$ are not homeomorphic. A Zariski pair (C, C') is called *Alexander-equivalent* if their generic Alexander polynomials coincide. The first example of Alexander-equivalent Zariski pair (C, C') for irreducible plane curves are given in [5]. They are plane curves of degree 12 with 27 cusps. Here C is a generic $(3,3)$-covering of a three cuspidal quartic and C' is constructed using a six cuspidal non-conical sextic.

The purpose of this note is to construct an Alexander-equivalent Zariski pair (D, D') of irreducible curves of degree 8 with 12 cusps. We give a brief recipe for the construction. Consider the moduli space $\mathcal{M}(cA_2; n)$ plane curves of degree n with c cusps of type $y^2 - x^3 = 0$. As we only consider cuspidal curves in this note, we simply denote $\mathcal{M}(c; n)$ in stead of $\mathcal{M}(cA_2; n)$. It is well-known that $\mathcal{M}(3; 4)$ is irreducible. In fact, its dual is the moduli of plane curves of degree 3 with one node by the Plücker's formula (see [N]). The fundamental group of the complement $\mathbf{P}^2 - C,\ C \in \mathcal{M}(3; 4)$, is a finite non-abelian group of order 12 ([8, 3]).

The first curve D is given by the generic (2,2)-cyclic covering $\mathcal{C}_{2,2}(Z)$ of a quartic Z in $\mathcal{M}(3; 4)$. Thus the fundamental group $\pi_1(\mathbf{P}^2 - D)$ is a finite group of order 24 and the Alexander polynomial $\Delta_D(t)$ is equal to that of C by [4] and therefore it is trivial. Actually we know that the generic Alexander polynomials of any cuspidal curves of degree $2^m,\ m = 1, 2, \dots$ are trivial by [1]. Thus the cuspidal curves of degree 8 is also interesting in this sense.

To construct the second curve D', we start from a two cuspidal quartic Q i.e., $Q \in \mathcal{M}(2; 4)$. If Q is generic, it has 8 flexes and a bi-tangent line by the Plücker's flex formula ([7, 2, 6]). Namely the dual curve is a 8 cuspidal sextic with one node. Let L_∞ be the line at infinity. We choose a generic line at infinity L_∞ and two flexes P_1, P_2 with tangent line L_1 and L_2 respectively such that $L_1 \cap L_2 \cap L_\infty = \emptyset$. Then in the affine space $\mathbf{C}^2 := \mathbf{P}^2 - L_\infty$ we take the change of linear coordinates so that L_1 and L_2 are given by the coordinate axis $x = 0$

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and $y=0$. Let $f(x,y)=0$ be the defining polynomial of Q and let P_1,P_2 be the cusps and let $R_1=(\alpha^2,0)$ and $R_2=(0,\beta^2)$ be the flex points. First take the flex double covering at R_1 and let $Q':=\mathcal{F}^{(2)}(R_1)$ ([5]). Namely Q' is the pull-back of Q by the covering mapping $(x,y)\mapsto(x,y^2)$ and Q' is defined by $f(x,y^2)=0$. Q' has 5 cusps: two cusps $P_{i,\pm}$ from each $P_i,\ i=1,2$ and one cusp comes from R_1 and two flexes $R_{2,\pm}$ on $x=0$ coming from R_2. They have the same tangent line $x=0$. Then we take the flex double covering along $x=0$ and let D' be the pull-back of Q'. We see that D' is defined by $f(x^2,y^2)=0$ and it has 12 cusps: 10 cusps come from the five cusps of Q' and two cusps come from flexes $R_{2,\pm}$. More precisely, we have 4 cusps coming from each cusp of Q and four cusps coming from flex points R_1,R_2, which are given by $(\pm\alpha,0),(0,\pm\beta)$. We will show that $\pi_1(\mathbf{P}^2-D')\cong\mathbf{Z}_8$ by a direct computation using the Zariski's pencil method.

2. Fundamental groups

2.1. **Construction.** For the practical computation of flex coverings, we need to know the locus of flex points explicitly. To construct a such two cuspidal qurtic, we start from a curve C of type (1,2;4) with one cusp and a flex with the tangent line $y=0$. We denote the moduli of such curve by $\mathcal{M}_1$. By the action of automorphisms, we may assume that the cusp is at $(3,1)$ and the flex is at $(2,0)$. Then the generic curve is described by one parameter family

$$\begin{aligned}g(x,y)=y^2&+(52-48s+9s^2)y+(34s-36-6s^2)xy+(-6s+6+s^2)x^2y-80\\&+48s+(144-88s)x+(60s-96)x^2+(28-18s)x^3+(2s-3)x^4.\end{aligned}$$

Now take a generic curve C in $\mathcal{M}_1$ which is defined by $g(x,y)=0$. We take the symmetric double covering $\varphi:\mathbf{C}^2\to\mathbf{C}^2$, defined by $\varphi(u,v)=(u+v,uv)$ ([3]) and let $\mathcal{S}_2(C)$ be the quartic defined by the pull-back of C. The branching locus of the symmetric covering is given by $\Delta=\{(x,y);x^2-4y=0\}$. Thus we must assume that any cusps or the marked flex point of C are not located on Δ. $\mathcal{S}_2(C)$ is defined by the symmetric polynomial $g'(u,v)=0$ where $g'(u,v):=g(u+v,uv)$ and $\mathcal{S}_2(C)$ has two cusps at (β_1,β_2) and (β_2,β_1) where β_1,β_2 are the root of $t^2-3t+1=0$. The flex at $(2,0)$ splits into two flexes $R_1=(2,0)$ and $R_2=(0,2)$ in $\mathcal{S}_2(C)$. Their tangent lines are given by $y=0$ and $x=0$ respectively. See [3] for the detail about symmetric coverings.

Remark. We remark here that the pull-back of a flex of C is not necessarily a flex of $\mathcal{S}_2(C)$ in general, as the pull-back of the tangent line is not a line in general. However this is the case if the flex is on x-axis with the tangent line $y=0$ as the pull-back of $y=0$ is the the union of two lines $u=0$ and $v=0$.

We denote by $\mathcal{M}_2$ the set of symmetric quartic with two cusps and two marked flexes whose tangent lines are coordinate axis. Note that $\mathcal{S}_2(C)\in\mathcal{M}_2$ for any $C\in\mathcal{M}_1$ and conversely any $Q\in\mathcal{M}_2$ is presented as a symmetric double covering $\mathcal{S}_2(C)$ of some $C\in\mathcal{M}_1$.

We construct a correspondence $\varphi:\mathcal{M}_2\to\mathcal{M}(12;8)$. For any quartic $Q\in\mathcal{M}_2$ defined by $f(x,y)=0$, we take twice flex covering and we define $Q\mapsto\widehat{Q}$, where $\widehat{Q}$

of degree 8 with 12 cusps which is defined by $f(x^2, y^2) = 0$. We denote the image of $\mathcal{M}_2$ in $\mathcal{M}(12;8)$ by $\mathcal{M}_3$ and the image of $\mathcal{M}(3;4)$ by the generic (2,2)-covering by $\mathcal{M}_{Zar}$. So $\mathcal{M}_{Zar} = \{\mathcal{C}_{2,2}(Q); Q \in \mathcal{M}(3;4)\}$.

Theorem 2.1. (1) *For any $D' \in \mathcal{M}_3$, we have $\pi_1(\mathbf{P}^2 - D') \cong \mathbf{Z}_8$ and the generic Alexander polynomial $\Delta_{D'}(t)$ is trivial.*

(2) *For any $D \in \mathcal{M}_{Zar}$, $\pi_1(\mathbf{P}^2 - D)$ is a finite non-abelian group of order 24 and the generic Alexander polynomial is trivial.*

In particular, the pair of irreducible curves (D, D') is a Alexander-equivalent Zariski pair for $D' \in \mathcal{M}_3$ and $D \in \mathcal{M}_{Zar}$ and therefore the moduli space $\mathcal{M}(12;8)$ is not irreducible.

Remark. Any generic curve $C \in \mathcal{M}(2;4)$ has 8 flexes and a bi-tangent line by the flex formula (see [7, 2],[6]). Thus the dual curves have degree 6 and the singularities are 8 cusps and a node. It is easy to show that $\mathcal{M}(2;4)$ is irreducible variety. Let $\mathcal{M}'$ be the moduli of two cuspidal quartics with two marked flexes. There is a surjective forgetting morphism $\psi : \mathcal{M}' \to \mathcal{M}(2;4)$. There exists a canonical (but not unique) rational mapping from $\mathcal{M}'$ to $\mathcal{M}(12;8)$ as follows. For any $C' \in \mathcal{M}'$, we have a linear change of coordinates so that C is defined by $f(x,y) = 0$ and two marked flex tangents are given by $y = 0$ and $x = 0$. Then we can take the mapping $C \mapsto \varphi(C) := \widehat{C}$ where $\widehat{C}$ is defined by $f(x^2, y^2) = 0$ as above. The mapping φ is unique if we fix the line at infinity and is well-defined on C if two tangent lines at marked flex points intersect outside of the line at infinity. By a direct computation, it seems that $\mathcal{M}'$ has two irreducible components and one components is equal to the PSL$(3;\mathbf{C})$-orbit of $\mathcal{M}_2$. We do not know whether the image of these components are in a same component of the moduli $\mathcal{M}(12;8)$ or not.

2.2. **Computation of the fundamental group.** The second assertion of Theorem 2.1 follows from Theorem 5.5 of [4]. To prove the assertion (1), we consider the following symmetric polynomial

$$f(x,y) := (xy - \frac{3}{2}(x+y-2)^2)^2 + xy - 2(x+y-2)^3 + \frac{3}{4}(x+y-2)^4$$

which is the pull-back of

$$g(x,y) = (y - \frac{3}{2}(x-2)^2)^2 + y - 2(x-2)^3 + \frac{3}{4}(x-2)^4$$

by the symmetric covering. Let $C_1 := C^a(f)$ and $C_2 := \{(x,y); f(x,y^2) = 0\}$ and $C_3 := \{(x,y); f(x^2,y^2) = 0\}$. The quartic C_1 has two cusps at $P_1 := (\beta_2, \beta_1)$ and $P_2 := (\beta_1, \beta_2)$ where $\beta_1 = D(3-\sqrt{5})/2$, $\beta_2 = (3+\sqrt{5})/2$ and two flex points at $R_1 := (2,0)$ and $R_2 := (0,2)$ where the tangent lines are given by $y = 0$ and $x = 0$. The discriminant polynomial of f with respect to y is given by

$$\Delta_y(f)(x) = x^2(3x-8)(111x^3 - 441x^2 + 311x + 216)(x^2 - 3x + 1)^3$$

For the computation of the fundamental group, we consider the vertical pencil lines $L_\eta = \{x = \eta\}$, $\eta \in \mathbf{C}$. The three roots of $111x^3 - 441x^2 + 311x + 216 = 0$, which we denote by $\alpha_1, \alpha_2, \alpha_3$, and $x = 8/3$ corresponds to the singular pencil lines which are simply tangent to C_1. They are real numbers which are given by $\alpha_1 = -0.419...$, $\alpha_2 = 1.771...$, $\alpha_3 = 2.620....$ The roots $x^2 - 3x + 1 = 0$ corresponds to cusps and they are given by β_1 and β_2. We note that $\beta_2 = 2.618...$ is slightly smaller than $\alpha_3 = 2.620....$ See Figure 2. The graph of f is given in Figure 1 and the local enlarged graph is given in Figure 2.

FIGURE 1. Graph of C_1

We are going to show that $\pi_1(\mathbf{P}^2 - C_3) \cong \mathbf{Z}_8$ using the pencil $x = \eta, \eta \in \mathbf{C}$ and the information for C_1. This implies also the commutativity: $\pi_1(\mathbf{P}^2 - C_2) \cong \mathbf{Z}_8$. The singular pencils of C_i, $i = 1, 2, 3$ corresponds to Σ_i which are given by Lemma 2.4 of [5] as

$$Si_1 = \{0, \alpha_1, \alpha_2, \alpha_3, 8/3, \beta_1, \beta_2\}, \quad \Sigma_2 = \Sigma_1 \cup \{2\} \quad \Sigma_3 = \{\pm\sqrt{\eta}; \eta \in \Sigma_2\}$$

We use the same notation as in [4] and [5]. Thus the bullets in the following Figures are the intersections of C_3 (of C_1 in Figure 4) and the pencil lines. A path ending to a bullet denotes a small loop going around that intersection counter-clockwise (which is called a lasso in [4]). Monodromy relations are read from the behavior of four points $L_\eta \cap C_1$ over the real line, which is sketched in Figure 4. For $\eta \in \Sigma_i$, we denote $\eta^\pm = \eta \pm \varepsilon$ where $\varepsilon > 0$ is sufficiently small.

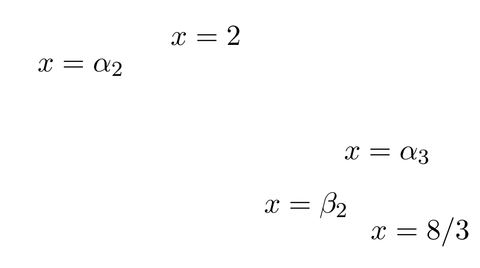

FIGURE 2. Local graph of C_1

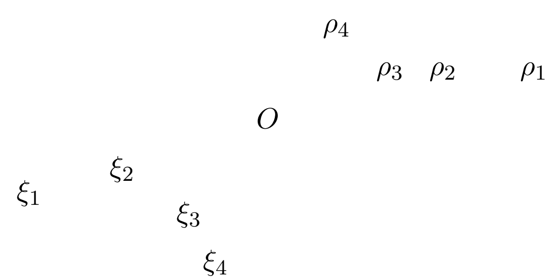

FIGURE 3. Generators ($x = 0^+$)

We take generators $\rho_1, \dots, \rho_4$, $\xi_1, \dots, \xi_4$ of $\pi_1(L_{0^+} - L_{0^+} \cap C_3)$ as in Figure 3. The base point is chosen to be $[0, 1, 0]$ which is the base point of the pencil L_η, $\eta \in \mathbf{C}$ and also equal to the point at infinity of $L_\eta \cong \mathbf{P}^1$. Note that $\rho_1, \dots, \rho_4$ are symmetric to $\xi_1, \dots, \xi_4$ with respect to the origin. Thus any relation in $\rho_1, \dots, \rho_4$ is also true for $\xi_1, \dots, \xi_4$. First we have the relation:

$$\xi_4 \dots \xi_1 \rho_4 \dots \rho_1 = e. \tag{2.2}$$

$x = \alpha_1^+$ $x = 0^+$ $x = \beta_1^+$

$x = \alpha_2^+$ $x = \beta_2^-$ $x = \alpha_3^-$

FIGURE 4. Deformation of $C_1 \cap L_\eta$

The monodromy relation at $x = 0$ is the cusp relation and it is given by

$$\rho_2 = \rho_4, \quad \{\rho_2, \rho_3\} = e, \quad \xi_2 = \xi_4, \quad \{\xi_2, \xi_3\} = e, \tag{2.3}$$

where $\{a, b\} = abab^{-1}a^{-1}b^{-1}$ as in [4]. The relation from the singular line $x = \alpha_1$ is equivalent to:

$$\rho_1 = \rho_3^{-1}\rho_4\rho_3, \quad \xi_1 = \xi_3^{-1}\xi_4\xi_3. \tag{2.4}$$

At this point, we have reduced our generators to $\{\rho_2, \rho_3, \xi_2, \xi_3\}$. Now the main point is the following observation. Let $y_1(x), y_2(x)$ be the roots of $f(x, y) = 0$ for $\beta_1 < x < \alpha_2$ such that $\Im(y_i(x)) > 0, i = 1, 2$. The other two roots are given by their complex conjugates. We may assume that $\Re(y_1(\beta_1^+)) < \Re(y_2(\beta_1^+))$. Then

Assertion 1. The inequality $\Re(y_1(x)) < \Re(y_2(x))$ is preserved on the interval (β_1, α_2).

Assuming this, the monodromy relation at $x = \alpha_2$ is given as

$$\rho_3 = \xi_4'' = (\xi_3\xi_2\xi_1)^{-1}\xi_4(\xi_3\xi_2\xi_1), \quad \xi_3 = \rho_4'' = (\rho_3\rho_2\rho_1)^{-1}\rho_4(\rho_3\rho_2\rho_1) \tag{2.5}$$

where $\rho_4', \rho_4'', \xi_4', \xi_4''$ are defined as in Figure 5 and we have

$$\rho_4' = \rho_3^{-1}\rho_4\rho_3, \quad \xi_4' = \xi_3^{-1}\xi_4\xi_3, \quad \xi_4'' = \xi_2, \quad \rho_4'' = \rho_2 \tag{2.6}$$

by (2.3) and (2.4).These relations reduce to

$$\rho_3 = \xi_2, \quad \xi_3 = \rho_2. \tag{2.7}$$

The monodromy relations at $x = 2$ and β_2 do not give any new relations. At $x = \beta_2^+$, we consider the generators as in Figure 6 where $\hat\rho_4$ is defined as in Figure 6. By an easy computation we have $\hat\rho_4 = \rho_3^{-1}\rho_2\rho_3$.

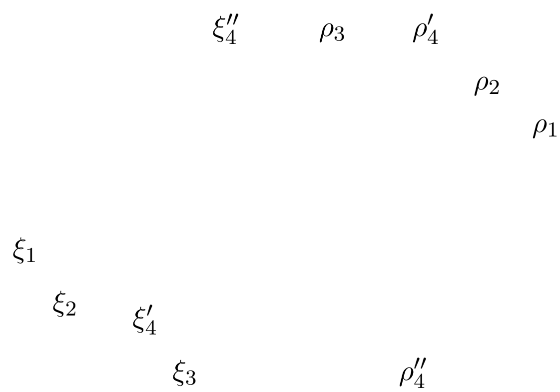

FIGURE 5. Generators ($x = \beta_1^+$)

The monodromy relation at $x = \alpha_3$ is given by $\hat{\rho}_4 = \rho_1$ which reduces to $\rho_2\rho_3 = \rho_3\rho_2$. Using the cusp relation $\rho_2\rho_3\rho_2 = \rho_3\rho_2\rho_3$ we get the relation $\rho_2 = \rho_3$. Thus the generators $\rho_2, \rho_3, \xi_2, \xi_3$ reduces to the single element ρ_2, which implies that $\pi_1(\mathbf{P}^2 - C_3) \cong \mathbf{Z}_8$.

2.3. **Appendix: Proof of Assertion 1.** We give a brief proof of Assertion 1. First, the four roots of $f(x, y) = 0$ in y, with a real x being fixed, are closed by complex conjugation. So we look at those roots with positive imaginary part, say $y_1(x)$ and $y_2(x)$. Assume that there exists a $x_0 \in (\beta_1, \alpha_2)$ such that $\Re(y_1(x_0)) = \Re(y_2(x_0))$ and put $u_0 = \Re(y_1(x_0))$ and v_1, v_2 be the imaginary parts. Consider $f(x, u + iv)$ and put $f_e(x, u, v)$ and $f_o(x, u, v)$ be the real and the imaginary parts. By an easy computation, f_e and f_o are polynomials of v of degree 4 and 3 respectively and

$$\begin{aligned} f_e(x,u,v) = {} & 3v^4 + 3(xu - \frac{3}{2}(x+u-2)^2)v^2 - (xv - 3(x+u-2)v)^2 \\ & + 6(x+u-2)v^2 - \frac{9}{2}(x+u-2)^2v^2 + (xu - \frac{3}{2}(x+u-2)^2)^2 \\ & + xu - 2(x+u-2)^3 + \frac{3}{4}(x+u-2)^4 \end{aligned}$$

$$\begin{aligned} f_o(x,u,v) = {} & v^3(-9x + 26 - 12u) + v(157x + 12u^3 - 78u^2 - 120 \\ & + 27xu^2 - 132xu + 9x^3 + 168u - 66x^2) \end{aligned}$$

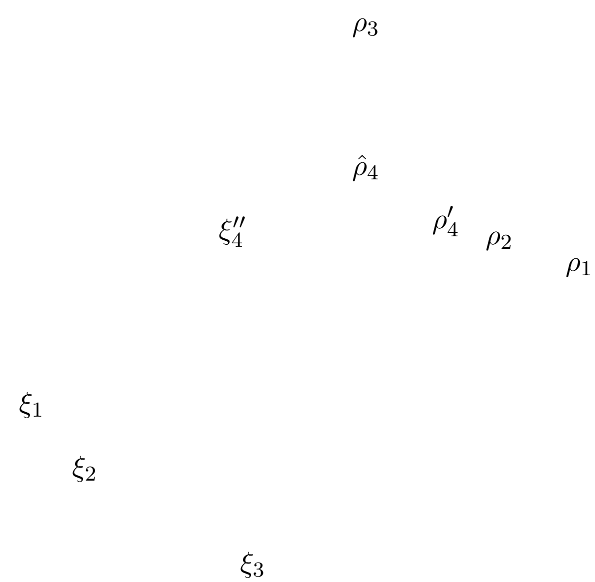

FIGURE 6. Generators ($x = \beta_2^+$)

By the assumption, $f_e(x_0, u_0, v) = 0$ and $f_o(x_0, u_0, v) = 0$ has four common solutions $\pm v_1, \pm v_2$. As $\deg_v f_o(x_0, u_0, v) = 3$, we must to have $f_o(x_0, u_0, v) \equiv 0$. For this, it is necessary that the coefficients $c_3 := -9x + 26 - 12u$ and $c_1 := 157x + 12u^3 - 78u^2 - 120 + 27xu^2 - 132xu + 9x^3 + 168u - 66x^2$ should vanish at $(x, u) = (x_0, u_0)$. Thus $u_0 = -3/4x_0 + 13/6$ and x_0 is the solution of $c_1 = -1/9 - 3/8x^3 - 3/2x + 19/12x^2 = 0$. Thus the only possibility for x_0 in the interval (β_1, α_2) is $x_0 = 1.590...$ However $f_e(x_0, -3/4x_0 + 13/6, v) = 0$ does not have four real solutions in this case. By contradiction, this completes the proof of Assertion 1.

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