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UNIQUE RANGE SETS FOR HOLOMORPHIC CURVES

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Dedicated to the memory of Le Van Thiem

Abstract. The purpose of this paper is to give a uniqueness result [Theorem 2.2] for holomorphic curves from $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$.

1. Preliminaries

In 1926, Nevanlinna proved that two non-constant meromorphic functions of a complex variable which attain five distinct values at the same points must be identical. In the present note, by using Nochka theorem [4], we prove a theorem on the unique range set for the case of holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$.

Let f be a meromorphic function in the complex plane $\mathbb{C}$ and $a \in \mathbb{C}$ be a complex number. Nevanlinna has constructed the following functions.

Let $n(f, a, r)$ denote the number of points $z \in \mathbb{C}$ for which $f(z) = a$ and $|z| \leq r$, counting with multiplicity. We set

$$N_f(a, r) = \int_0^r \frac{n(f, a, t) - n(f, a, 0)}{t} dt + n(f, a, 0) \log r,$$

$$m_f(a, r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta,$$

where $\log^+ x = \max(0, \log x)$, and set

$$T_f(a, r) = m_f(a, r) + N_f(a, r).$$

Nevanlinna's First Main Theorem asserts that for every meromorphic function $f(z)$ there exists a function $T_f(r)$ such that for all $a \in \mathbb{C}$.

$$T_f(a, r) = T_f(r) + O(1),$$

where $O(1)$ is bounded when $r \longrightarrow \infty$.

Definition 1.1. Let $f : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve from $\mathbb{C}$ into the $n-$dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$. *The Cartan characteristic*

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function of f is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \max_{1 \le j \le n+1} \log \left| f_j(re^{i\theta}) \right| d\theta.$$

Let H be a hyperplane of $\mathbb{P}^n(\mathbb{C})$ defined by the equation $F = 0$. For every positive integer k, we define *the counting function* of $F \circ f$ truncated at k by

$$N_{k,f}(H, r) = \sum_{0 \ne a \in D_r} \log^+ \frac{r}{|a|},$$

where every zero a of function $F \circ f$ is counted with multiplicity if its multiplicity is less than or equal to k, and k times otherwise.

For any positive integers k, ℓ, by $N_{k,f}^{\le \ell}(t_{(m)})$ (resp. $N_{k,f}^{>\ell}(t_{(m)})$) we denote the sum taken over all the zeros a with multiplicity less than or equal to ℓ (resp. at least $\ell + 1$). Then

$$N_{k,f}(H, r) = N_{k,f}^{\le \ell}(H, r) + N_{k,f}^{>\ell}(H, r).$$

For every $k \ge 1$, we have

$$N_{1,f}(H, r) \le N_{k,f}(H, r) \le k N_{1,f}(H, r), N_{k,f}(H, r) \le N_f(H, r),$$

$$\frac{1}{\ell + 1} N_{k,f}^{\le \ell}(H, r) + N_{k,f}^{>\ell}(H, r) \le \frac{1}{\ell + 1} N_f(H, r).$$

Set

$$\overline{E}_f(H) = \{z \in \mathbb{C} : F \circ f(z) = 0 \text{ ignoring multiplicities}\}.$$

For every positive integer k, define a set

$$\overline{E}_f(H, k) = \big\{z \in \mathbb{C} : F \circ f(z) = 0 \text{ ignoring multiplicities with } ord_f z \le k\big\}.$$

Hyperplanes $H_1, \dots, H_q$ in $\mathbb{P}^n(\mathbb{C})$, $q \ge n + 1$, are said to be in *general position* if any $n + 1$ of them are linearly independent.

Theorem 1.1. (See [4]). *Let $f = (f_1, \dots, f_{n+1}) : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a linearly m-nondegenerate holomorphic curve and $H_1, \dots H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that $f(\mathbb{C}) \not\subset H_j$, $j = 1, \dots, q$. Then*

$$(q - 2n + m - 1) T_f(r) \le \sum_{i=1}^{q} N_{m,f}(H_i, r) + S(r),$$

where $S(r) = 0(\log(r.T_f(r)))$.

2. The unique range set for holomorphic curves

From Theorem 1.1 we can deduce the following.

Theorem 2.1. *Let $f = (f_1, \dots, f_{n+1}) : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a linearly m-nondegenerate holomorphic curve, $k_1, \dots, k_q$ be positive integers and $H_1, \dots H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that $f(\mathbb{C}) \not\subset H_j$, $j = 1, \dots, q$. Then*

$$\Big(\sum_{i=1}^{q} \frac{k_i}{k_i+1} - 2n + m - 1\Big) T_f(r) \leq \sum_{i=1}^{q} \frac{k_i}{k_i+1} N_{m,f}^{\leq k_i}(H_i, r) + S(r),$$

where $S(r) = 0(\log(r.T_f(r)))$.

Proof. For $H_i \in \{H_1, \dots, H_q\}$ and $k_i \in \{k_1, \dots, k_q\}$, we have

$$\begin{aligned}
N_{m,f}(H_i, r) &= N_{m,f}^{\leq k_i}(H_i, r) + N_{m,f}^{> k_i}(H_i, r) \\
&\leq \frac{k_i}{k_i+1} N_{m,f}^{\leq k_i}(H_i, r) + \frac{1}{k_i+1} N_{m,f}^{\leq k_i}(H_i, r) + N_{m,f}^{> k_i}(H_i, r) \\
&\leq \frac{k_i}{k_i+1} N_{m,f}^{\leq k_i}(H_i, r) + \frac{1}{k_i+1} N_f(H_i, r) \\
&\leq \frac{k_i}{k_i+1} N_{m,f}^{\leq k_i}(H_i, r) + \frac{1}{k_i+1} T_f(r) + 0(1).
\end{aligned}$$

It follows that

$$\sum_{i=1}^{q} N_{m,f}(H_i, r) \leq \sum_{i=1}^{q} \frac{k_i}{k_i+1} N_{m,f}^{\leq k_i}(H_i, r) + \sum_{i=1}^{q} \frac{1}{k_i+1} T_f(r) + 0(1).$$

On the other hand, by Theorem 1.1, we have

$$(q - 2n + m - 1) T_f(r) \leq \sum_{i=1}^{q} N_{m,f}(H_i, r) + S(r).$$

The conclusion follows from the last two inequalities. □

Theorem 2.2. *Let $f = (f_1, \dots, f_{n+1})$, $g = (g_1, \dots, g_{n+1}) : \mathbb{C} \longrightarrow \mathbb{P}^m(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ be two linearly m-nondegenerate holomorphic curves and $H_1, \dots, H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that $f(\mathbb{C}) \not\subset H_j$, $j = 1, \dots, q$. Let $k_1, \dots, k_q$ be positive integers such that $k_1 \geq \dots \geq k_q$ and $\sum_{i=2mn+1}^{q} \frac{k_i}{k_i+1} > 2n - m + 1$. Assume that*

$$\overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C}), k_i) = \overline{E}_g(H_i \cap \mathbb{P}^m(\mathbb{C}), k_i), \quad i = 1, \dots, q,$$

and $f(z) = g(z)$ for any $z \in \bigcup_{i=1}^{q} \overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C}), k_i)$. Then $f \equiv g$.

Proof. Assume to the contrary $f_i g_j \not\equiv f_j g_i$. From the hypothesis it follows that

$$1 \geq \frac{k_1}{k_1+1} \geq \dots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2}.$$

By Theorem 2.1, we have

$$\Big\langle\sum_{i=1}^{q}\frac{k_i}{k_i+1}-2n+m-1\Big\rangle T_f(r)\le\sum_{i=1}^{q}\frac{k_i}{k_i+1}N^{\le k_i}_{m,f}(H_i,r)+S_f(r)$$
$$\le\frac{k_{2mn}}{k_{2mn}+1}\sum_{i=1}^{q}N^{\le k_i}_{m,f}(H_i,r)$$
$$+\sum_{i=1}^{2mn}\Big(\frac{k_i}{k_i+1}-\frac{k_{2mn}}{k_{2mn}+1}\Big)N^{\le k_i}_{m,f}(H_i,r)+S_f(r)$$
$$\le\frac{k_{2mn}}{k_{2mn}+1}\sum_{i=1}^{q}N^{\le k_i}_{m,f}(H_i,r)$$
$$+\sum_{i=1}^{2mn}\Big(\frac{k_i}{k_i+1}-\frac{k_{2mn}}{k_{2mn}+1}\Big)T_f(r)+S_f(r)+0(1),$$

where $S_f(r)=0(\log(r.T_f(r)))$. This gives

$$\Big\langle\sum_{i=2mn+1}^{q}\frac{k_i}{k_i+1}+\frac{2mnk_{2mn}}{k_{2mn}+1}-2n+m-1\Big\rangle T_f(r)$$
$$\le\frac{k_{2mn}}{k_{2mn}+1}\sum_{i=1}^{q}N^{\le k_i}_{m,f}(H_i,r)+S_f(r)+0(1)$$
$$\le\frac{mk_{2mn}}{k_{2mn}+1}\sum_{i=1}^{q}N^{\le k_i}_{1,f}(H_i,r)+S_f(r)+0(1)$$
$$\le\frac{mnk_{2mn}}{k_{2mn}+1}N_{\frac{f_i}{f_j}-\frac{g_i}{g_j}}(r)+S_f(r)+0(1).$$

On the other hand, by [3, Lemma 3.1, Chapter VII] we have

$$N_{\frac{f_i}{f_j}-\frac{g_i}{g_j}}(r)\le N_{\frac{f_i}{f_j}}(r)+N_{\frac{g_i}{g_j}}(r)\le T_f(r)+T_g(r)+0(1).$$

Therefore

$$\Big\langle\sum_{i=2mn+1}^{q}\frac{k_i}{k_i+1}+\frac{2mnk_{2mn}}{k_{2mn}+1}-2n+m-1\Big\rangle T_f(r)$$
$$\le\frac{mnk_{2mn}}{k_{2mn}+1}\big(T_f(r)+T_g(r)\big)+S_f(r)+0(1).$$

Similarly,

$$\Big\langle\sum_{i=2mn+1}^{q}\frac{k_i}{k_i+1}+\frac{2mnk_{2mn}}{k_{2mn}+1}-2n+m-1\Big\rangle T_g(r)$$
$$\le\frac{mnk_{2mn}}{k_{2mn}+1}\big(T_f(r)+T_g(r)\big)+S_g(r)+0(1).$$

From the above two inequalities we obtain

$$\Big\langle \sum_{i=2mn+1}^{q} \frac{k_i}{k_i+1} - 2n + m - 1\Big\rangle \big(T_f(r) + T_g(r)\big) \leq S_f(r) + S_g(r) + 0(1).$$

Hence

$$\Big\langle \sum_{i=2mn+1}^{q} \frac{k_i}{k_i+1} - 2n + m - 1\Big\rangle \leq \frac{S_f(r) + S_g(r) + 0(1)}{T_f(r) + T_g(r)}\cdot \tag{2.1}$$

Since f, g are two linearly nondegenerate holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^m(\mathbb{C})$ and $H_i \cap \mathbb{P}^m(\mathbb{C})$, $i = 1, \dots, q$, are hyperplanes of $\mathbb{P}^m(\mathbb{C})$ in general position, by [1, Theorem 5.2.1] we have

$$\lim_{r\to+\infty} \inf \frac{S_f(r)}{T_f(r)} \leq 0.$$

From (2.1) and the hypothesis, we get a contradiction. Hence $f_i g_j \equiv f_j g_i$. Thus, $f \equiv g$. □

Corollary 2.1. *Let $f = (f_1, \dots, f_{n+1})$, $g = (g_1, \dots, g_{n+1}) : \mathbb{C} \longrightarrow \mathbb{P}^m(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ be two linearly m-nondegenerate holomorphic curves and $H_1, \dots H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that $f(\mathbb{C}) \not\subset H_j$, $j = 1, \dots, q$, with $q > 2mn + 2n - m + 1$. Assume that*

$$\overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C})) = \overline{E}_g(H_i \cap \mathbb{P}^m(\mathbb{C})), \quad i = 1, \dots, q,$$

and $f(z) = g(z)$ for any $z \in \underset{i=1}{\rightarrow} \overset{q}{\rightarrow} \bigcup \overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C}))$. Then $f \equiv g$.

Proof. In Theorem 2.2, take $k_1 = k_2 = \dots = k_q = k$ and let $k \longrightarrow \infty$. □

Corollary 2.2. *Let $f = (f_1, \dots, f_{n+1})$, $g = (g_1, \dots, g_{n+1}) : \mathbb{C} \longrightarrow \mathbb{P}^m(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ be two linearly m-nondegenerate holomorphic curves and $H_1, \dots, H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that $f(\mathbb{C}) \not\subset H_j$, $j = 1, \dots, q$. Let $k_1, \dots, k_q$ be positive integers such that $k_1 \geq \dots \geq k_q$ and $\sum_{i=2m+1}^{q} \frac{k_i}{k_i+1} > 2n - m + 1$. Assume that*

$$\overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C}), k_i) \cap \overline{E}_f(H_j \cap \mathbb{P}^m(\mathbb{C}), k_i) = \emptyset, \quad \forall i \neq j,$$
$$\overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C}), k_i) = \overline{E}_g(H_i \cap \mathbb{P}^m(\mathbb{C}), k_i), \quad i = 1, \dots, q,$$

and $f(z) = g(z)$ for any $z \in \bigcup_{i=1}^{q} \overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C}), k_i)$. Then $f \equiv g$.

Corollary 2.3. *Let $f = (f_1, \dots, f_{n+1})$, $g = (g_1, \dots, g_{n+1}) : \mathbb{C} \longrightarrow \mathbb{P}^m(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ be two linearly m-nondegenerate holomorphic curves and $H_1, \dots, H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that $f(\mathbb{C}) \not\subset H_j$, $j = 1, \dots, q$, with $q > 2n + m + 1$. Assume that*

$$\overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C})) \cap \overline{E}_f(H_j \cap \mathbb{P}^m(\mathbb{C})) = \emptyset, \quad \forall i \neq j,$$
$$\overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C})) = \overline{E}_g(H_i \cap \mathbb{P}^m(\mathbb{C})), \quad i = 1, \dots, q$$

and $f(z) = g(z)$ *for any* $z \in \bigcup_{i=1}^{q} \overline{E}_f(H_i \cap \mathbb{P}^m(\mathbb{C}))$. *Then* $f \equiv g$.

Proof. In Corollary 2.2, take $k_1 = k_2 = \cdots = k_q = k$ and $k \longrightarrow \infty$. □

Note that for $m = n$ from Corollary 2.3, we obtain the uniqueness theorem for holomorphic curves of Stoll [6].

Corollary 2.4. ([6]) *Let* $f = (f_1, \ldots, f_{n+1}), g = (g_1, \ldots, g_{n+1}) : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ *be two linearly non-degenerate holomorphic curves and* $H_1, \ldots, H_{3n+2}$ *be hyperplanes of* $\mathbb{P}^n(\mathbb{C})$ *in general position. Assume that*

$$f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset, \quad \forall i \neq j,$$
$$f^{-1}(H_i) = g^{-1}(H_i), \quad i = 1, \ldots, 3n+2,$$

and $f(z) = g(z)$ *for any* $z \in \bigcup_{i=1}^{3n+2} f^{-1}(H_i)$. *Then* $f \equiv g$.

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