ACTA MATHEMATICA VIETNAMICA
Volume 27, Number 3, 2002, pp. 297-300

EXCEPTIONAL SETS FOR HOLOMORPHIC MAPPINGS

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Dedicated to the memory of Le Van Thiem

Since Professor Le Van Thiem made many original contributions to the theory of value distribution in one complex variable, it would be appropriate to use the time that we have to give a survey of some recent results for value distribution of holomorphic mappings.

Where the results that we present will often be valid in a larger setting, in order to simplify the presentation, we shall only treat entire mappings. Thus, $F : \mathbb{C}^n \longrightarrow \mathbb{C}^m$, $F = (f_1, ..., f_m)$ and f_j is holomorphic for every j. In addition, we shall assume that F is non-degenerate, that is that $\max\limits_{z\in\mathbb{C}^n}(\operatorname{rank} J_F(z)) = m$, where J_F is Jacobian matrix of F. If $m = 1$, that is if F is a holomorphic function, then the results of value distribution theory are similar to those for functions of one complex variable. However, when $m > 1$, they diverge significantly.

Let $B(a, r)$ be the Euclidean ball of center a and radius r in $\mathbb{C}^n$. If X is an analytic variety of pure dimension p in $\mathbb{C}^n$, $\sigma_X(r)$ is the Euclidean area of $X \cap B(0, r)$ and if $0 \notin X$, $\nu_x(r)$ is projective area of X, that is the average of $X \cap l \cap B(0, r)$ over all complex lines l in the projective space; there exists a constant c_n such that $\nu_X(r) = c_n\sigma_X(r)r^{-2p}$ (cf. [5]).

We set $M_F(r) = \sup\limits_{z\in B(0,r)} \|F(z)\|$, where $\|.\|$ is the Euclidean norm. A subset E of $\mathbb{C}^n$ is said to be pluripolar if E is contained in the set where some plurisubharmonic function takes on the value $-\infty$.

1. Upper bounds for $F^{-1}(a)$

If $m = 1$, then it follows from the theorem of Jensen-Gauss, as when $n = 1$, that for every $a \in \mathbb{C}$ and every $\varepsilon > 0$, there exists a constant $C(a, \varepsilon)$ such that if $X_a = F^{-1}(a)$,

$$\nu_{X_a}(r) \le C(a, \varepsilon) \log(M_F(r(\varepsilon + 1)) + 1).$$

However, for $m > 1$, there is no general function of $M_F(r)$ which bounds $\nu_{X_a}(r)$ for every mapping F (cf. [2]).

On the other hand, one has the following results (cf. [3]).

Theorem 1. *Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be a non-degenerate holomorphic mapping. Then for every $\beta > 1$ the set of $a \in \mathbb{C}^m$ such that*

$$\lim_{r\to\infty} \sup \frac{\nu_{X_a}(r)}{(\log r)^\beta \{\log[M_F(r(1+\varepsilon))] + 1\}^m} = +\infty, \quad \textit{for some } \varepsilon > 0$$

is pluripolar in $\mathbb{C}^m$.

The idea of the proof quite simple. It is well known that for every $A \subset \mathbb{C}^m$ non-pluripolar, there exists a plurisubharmornic function V such that $V(z) \leq C + \log^+ \|z\|$ and the Monge-Ampere mass of V is concentrated on A. Thus, one studies $V \circ F$ and applies an iteration of a generalized Jensen formula to obtain an upper bound.

2. Estimates for the restriction of analytic varieties to subspaces

The following problem has no equivalent for functions of one complex variable since in that case zero sets are of dimension zero.

Suppose that $X \subset \mathbb{C}^n$ is an analytic variety of complex dimension $n-1$. Then there exists an entire function F such that $X = F^{-1}(0)$ and for every $\varepsilon > 0$

$$\log M_F(r) \leq C_\varepsilon \nu_X(r(1+\varepsilon)) \quad \text{(cf. [7], [10], [6])}$$

If $G_p(\mathbb{C}^n)$ is a Grassmannian manifold of complex subspaces of $\mathbb{C}^n$ of dimension p ($\dim_{\mathbb{C}} G_p(\mathbb{C}^n) = p(n-p)$) then it follows from Jensen's formula that for every $l \in G_p(\mathbb{C}^n)$

$$\nu_{X\cap l}(r) \leq C_\varepsilon \nu_X(r(1+\varepsilon)).$$

However, if $\dim_{\mathbb{C}} X < (n-1)$, no such general upper bound is possible (cf. [2]).

In a similar vain, since $\log|F|$ is plurisubhamonic and

$$\int\limits_{l\cap B(0,r)} \log|F| d\lambda = C_p \int\limits_0^r \nu_{X\cap l}(t) \frac{dt}{t}$$

from Jensen's Theorem (where $d\lambda$ is the Lebesgue measure on $l \cap B(0,r)$), it follows from general results for exceptional sets for plurisubharmonic functions that

$$\Big\{ l \in G_p(\mathbb{C}^n) : \forall t > 0,\ \lim_{r\to+\infty} \sup \frac{\nu_{X\cap l}(r)}{\nu_X(tr)} = 0 \Big\}$$

is pluripolar in $G_p(\mathbb{C}^n)$ (cf. [6]). In fact, better estimates for the exceptional sets are possible (cf. [1]). For analytic sets of arbitrary dimension, we have the following result (cf. [3]).

Theorem 2. *Let $X \in \mathbb{C}^n$ be an analytic set of pure dimension k. Then for $p \geq n-k$*

$$\Big\{ l \in G_p(\mathbb{C}^n) : \forall t > 0 \ \lim_{r\to+\infty} \sup \frac{\nu_{X\cap l}(r)}{\nu_X(Ar)} = \infty \Big\}$$

is of measure zero in $G_p(\mathbb{C}^n)$ *and*

$$\Big\{l \in G_p(\mathbb{C}^n) : \forall t > 0 \ \lim_{r\to+\infty} \sup \frac{\nu_{X\cap l}(r)}{\nu_X(tr)} = 0\Big\}$$

is of measure zero in $G_p(\mathbb{C}^n)$.

If X is of finite order, that is if there exists $\rho > 0$ such that $\sigma_X(r) \le Cr^\rho$, then estimates can be obtained for the exceptional sets in terms of small pluripolar sets in $G_p(\mathbb{C}^n)$ (cf. [4]).

3. Nevanlinna type estimates for $F^{-1}(E)$ for discrete sets E

The theorem of Picard tells us that the image of a meromorphic mapping in $\mathbb{C}$ cannot omit a set of more than two points in the Riemann sphere, and Nevanlinna's Second Main Theorem, the crown jewel of value distribution theory, gives an estimate for the asymptotic growth of $f^{-1}(E)$, for E a set and f a meromorphic map, in terms of the asymptotic growth of the characteristic fuction of f.

On the other hand, if $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is an entire non-degenerate holomorphic mapping, the image can omit any bounded set, even if $J_F \equiv 1$ and F is one-to-one.

However, there are certain discrete sets E such that $F^{-1}(E)$ is not empty.

Let $E \subset \mathbb{C}^n$. We say that E is unavoidable for a family $\mathcal{F}$ of non-degenerate holomorphic mappings $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ if for every r large enough

$$F(C(B(0,r) \cap E)) \cap B(0,r) \neq \emptyset.$$

We let

$$\mathcal{F} = \{F : \mathbb{C}^n \longrightarrow \mathbb{C}^n : J_F \not\equiv 0\} \quad \text{and} \quad \tilde{\mathcal{F}} = \{F : \mathbb{C}^n \longrightarrow \mathbb{C}^n : J_F \equiv \text{const}(\neq 0)\}.$$

Theorem 3. (cf. [5]) *Let* $\tilde{E}$ *be a set such that for some* $\varepsilon > 0$*, no point of* $B(0,m+1) - \overline{B}(0,m)$ *is at a distance greater than* $m^{-(2n-1)-\varepsilon}$ *from* $\tilde{E}$*. Then* $\tilde{E}$ *is unavoidable for* $\tilde{\mathcal{F}}$*.*

Let E *be a set such that for* $\beta > 1$*, no point of* $B(0,m+1) - \overline{B}(0,m)$ *is at a distance greater than* $m^{-(\log\log m)^\beta}$ *from* E*. Then* E *is unavoidable for* $\mathcal{F}$*.*

One can obtain Nevanlinna type estimates for sets E as in the above theorem. For $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ a holomorphic mapping, we define the characteristic function $T_F(v)$ of F, by stting:

$$T_f(r) = \int_{B(0,r)} (r^2 - \|z\|^2)\|F(z)\|^{2m} d\lambda(z), \quad (m > 2n).$$

Then we have the following result (cf. [8]).

Theorem 4. *If* $m > 2n$ *and* $\tilde{E}$ *is as in the above theorem, then*

$$\lim_{r\to\infty} \sup \frac{\operatorname{card} F^{-1}(\tilde{E}) \cap B(0,r)}{T_F(r)} > 0,$$

for every $F \in \tilde{\mathcal{F}}$.

Similar, though less precise, results hold for mappings in $\mathcal{F}$ (cf. [9]).

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