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A MODEL FOR HOMOTOPY TYPE OF THE COMPLEMENT

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Dedicated to the memory of Le Van Thiem

Abstract. We construct a cellular complex of the same homotopy type of the complement for a complex line arrangement in $\mathbb{C}^2$. The construction based on the labyrinth of a line arrangement, defined by us, and on the braid monodromy presentation for its fundamental group.

1. Introduction

Let $\mathcal{A}$ be an ℓ-arrangement, that is a finite set of hyperplanes in $\mathbb{C}^\ell$. The hyperplanes under consideration are not necessarily linear. Each hyperplane is defined by a linear form. If all hyperplanes of $\mathcal{A}$ are defined by real defining forms, we have a real arrangement. The complement of $\mathcal{A}$ is the open 2ℓ-manifold $M = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. Many efforts have been done to find out a simple model for the homotopy type of this complement M. M. Salvetti [9] has constructed a regular CW-complex which is a strong deformation of the complement for the case of real arrangements. P. Orlik [7] proved the existence of such a complex for an arbitrary arrangement of subspaces, the most general case. However, cell structure of the Orlik's complex can not be described explicitly. Another result was due to A. Libgober [5], where he proved that the canonical 2-complex associated to the braid monodromy presentation for the fundamental group of the complement of a plane curve is homotopy to its complement. Again, the braid monodromy presentation is somewhat unwieldy. For the case of complexification of arrangement of real lines, from Salvetti's complex, M. Falk [4] has deduced another complex which is homotopy equivalent to the complement. It is nothing but a modification of the canonical 2-complex modeled on the Randell's presentation [8], using Tietze's transformations.

In this paper we will deal with the case of arrangements of complex lines in $\mathbb{C}^2$. More precisely, we shall construct a cellular complex of the same homotopy type as the complement of $\mathcal{A}$ in $\mathbb{C}^2$. Our construction will base on the method to describe braid monodromy from the labyrinth of a line arrangement.

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2. Braid monodromy presentation

In this section we recall the definition of braid monodromy of a plane curve defined by B. Moishezon [6]. In the context of line arrangements we introduce the notion of labyrinth. Then we shall show how we can read off explicitly all braids of an arrangement of lines from its labyrinth.

Let $\mathcal{C} = \{f(x,y) = 0\} \in \mathbb{C}^2$ be a plane curve. Suppose that the projection $pr_1 : \mathbb{C}^2 \to \mathbb{C}^1$ onto the x-axis is generic with respect to the curve $\mathcal{C}$. Denote by $S(\mathcal{C})$ the set $\{\alpha \in \mathcal{C}; \partial f(\alpha)/\partial y = 0\}$ and $D(\mathcal{C})$ the image of $S(\mathcal{C})$ under the projection pr_1. For a point $\tilde{x}$ of the x-plane $\mathbb{C}^1$ let $\mathbb{C}_{\tilde{x}}$ denote the fiber of the projection pr_1 over the point $\tilde{x}$, $\mathbb{C}_{\tilde{x}} = \{(x,y) \in \mathbb{C}^2\,;\, x = \tilde{x}\}$. Given a path $\gamma : I \to \mathbb{C}^1 \setminus D(\mathcal{C})$ on the x-coordinate $\mathbb{C}^1$. We see easily that outside of $S(\mathcal{C})$ the restriction $pr_1|_{\mathcal{C}}$ of pr_1 is a trivial bundle. We have then a homeomorphisms

$$(pr_1^{-1}(\gamma(0)), pr_1^{-1}(\gamma(0)) \cap \mathcal{C}) \longrightarrow (pr_1^{-1}(\gamma(t)), pr_1^{-1}(\gamma(t)) \cap \mathcal{C}),$$

$t \in [0,1]$, induced by a given trivialization of $(pr_1, pr_1|_{\mathcal{C}})$. We call this homeomorphism the braid homeomorphism defined over the path γ, or simply the braid defined over γ. Let fix a base point x_0 of the x-axis, $x_0 \in \mathbb{C}^1 \setminus D(\mathcal{C})$. When γ is a loop beginning and ending at x_0, we obtain a homeomorphism

$$(\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap \mathcal{C}) \;\longrightarrow\; (\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap \mathcal{C}).$$

This defines a homomorphism

$$\theta \;:\; \pi_1(\mathbb{C}^1 \setminus D(\mathcal{C}); x_0) \longrightarrow B[\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap \mathcal{C}],$$

which is called the braid monodromy of the curve $\mathcal{C}$ (see [6]). Here by $B[P,K]$ we mean the group of isotopy classes of compact support homeomorphisms of a 2-plane P which preserves a fixed finite subset $K \subset P$.

The determination of the braid monodromy is usually carried out in two steps. First, for a point $x_k \in D(\mathcal{C})$ we denote by $D^{\varepsilon}_{x_k}$ a small disk of radius ε, centered at x_k. Let fix a point x_k^{ε} on the boundary $\partial D^{\varepsilon}_{x_k}$ of this disk and $\mathbb{C}_{x_k^{\varepsilon}}$ the fiber over this point x_k^{ε}. By moving this fiber $\mathbb{C}_{x_k^{\epsilon}}$ counterclockwise along the boundary of the disk $D^{\varepsilon}_{x_k}$ we obtain a homeomorphism of $\mathbb{C}_{x_k^{\varepsilon}}$ into itself, preserving $\mathbb{C}_{x_k^{\epsilon}} \cap \mathcal{C}$. It gives rise an element of the braid group $B[\mathbb{C}_{x_k^{\varepsilon}}, \mathbb{C}_{x_k^{\varepsilon}} \cap \mathcal{C}]$ and will be called the local braid monodromy of $\mathcal{C}$ at x_k.

Next, suppose that $D(\mathcal{C}) = \{x_1, ..., x_N\}$. Let $\Gamma_1, ..., \Gamma_N$ be a system of simple paths in $\mathbb{C}^1 \setminus D(\mathcal{C})$ satisfying

1) $\Gamma_i \cap \Gamma_j = x_0,\ 1 \le i < j \le N$;

2) Each Γ_i connects x_0 with x_i^{ε} and $\Gamma_i \cap D(\mathcal{C}) = \emptyset$.

We call those Γ_i's, $i = 1, ..., N$, a good system of simple paths. Let $\gamma_i \in \pi_1(\mathbb{C}^1 \setminus D(\mathcal{C}))$ be the element represented by $\Gamma_i \cdot \partial D^{\varepsilon}_{x_i} \cdot \Gamma_i^{-1}$. The set of all those γ_i's is called a good ordered system of generators of $\pi_1(\mathbb{C}^1 \setminus D(\mathcal{C}))$. To determine the braid monodromy θ means to find all $\theta(\gamma_i)$, $1 \le i \le N$. Let $\theta(\Gamma_i)$ be the braid homeomorphism defined over the path Γ_i. Then it is clear that $\theta(\gamma_i)$ can be completely determined by the local braid monodromy at x_i and the braid $\theta(\Gamma_i)$.

From now on we consider only the case when $\mathcal{C}$ is an arrangement $\mathcal{A}$ of complex lines in $\mathbb{C}^2$.

Suppose that each line $H_i \in \mathcal{A}$ is defined by an equation $y = \alpha_i(x)$, where α_i is a linear function $\alpha_i : \mathbb{C} \to \mathbb{C}$. Let $R_i(x) = Re(\alpha_i(x))$. For any $1 \leq i < j \leq n$, the subset $L_{i,j}$ of the x-axis $\mathbb{C}^1$, defined by

$$L_{i,j} \ = \ \{x \in \mathbb{C}^1; R_i(x) = R_j(x)\},$$

is a *(real)* line in $\mathbb{C}^1$.

Definition 2.1. We call the set

$$\mathcal{L}(\mathcal{A}) \ = \ \{L_{i,j} \ ; \ 1 \leq i < j \leq n\}$$

the *labyrinth* of the arrangement $\mathcal{A}$.

Remark 2.1. For each line $L \in \mathcal{L}(\mathcal{A})$, there might be $i_1, ..., i_k$ with $1 \leq i_1 < ... < i_k \leq n$ such that

$$L \ = \ \{x \in \mathbb{C}^1 \ ; R_{i_s}(x) = R_{i_t}(x), \ 1 \leq s < t \leq k\}.$$

The number k will be called the multiplicity of L. It is easy to see that after a suitable change of coordinates we can always assume that the multiplicity of any line L in $\mathcal{L}(\mathcal{A})$ equals to 2.

In the arrangement context, the points of $S(\mathcal{C})$ are nothing but multiple points of the arrangement $\mathcal{A}$. By definition, a multiple point P of the arrangement $\mathcal{A}$ is the nonempty intersection of two or more hyperplanes of $\mathcal{A}$. The assumption on the genericity of the projection pr_1 implies that the multiple points of the arrangement $\mathcal{A}$ are distinct by their x-coordinates. In other words, the images of multiple points of $\mathcal{A}$ on the x-plane $\mathbb{C}^1$ are pairwise distinct.

Let $x_k \in \mathbb{C}^1$ be the image of a multiple point $P_k = (x_k, y_k)$ of $\mathcal{A}$ under the projection pr_1. Suppose that $P_k = \bigcap_{j=1}^{r} H_{i_j}$. Then it is clear that x_k belongs to the lines L_{i_s,i_t}, $1 \leq s < t \leq r$ of the labyrinth $\mathcal{L}(\mathcal{A})$. However, there might be another line $L \in \mathcal{L}(\mathcal{A})$, which does not belong to $\{L_{i_s,i_t}; 1 \leq s < t \leq r\}$, going through this point x_k.

Definition 2.2. (i) The labyrinth $\mathcal{L}(\mathcal{A})$ is said to be good with respect to the multiple point $P_k = \bigcap_{j=1}^{r} H_{i_j}$ if there is not any line of $\mathcal{L}(\mathcal{A})$ except L_{i_s,i_t}; $1 \leq s < t \leq r$, going through x_k.

(ii) The labyrinth $\mathcal{L}(\mathcal{A})$ of an arrangement $\mathcal{A}$ is said to be *proper* if any line of $\mathcal{L}(\mathcal{A})$ has multiplicity 2 and it is good with respect to all multiple points of $\mathcal{A}$.

Remark 2.2. After a suitable change of coordinates we can assume that the labyrinth $\mathcal{L}(\mathcal{A})$ is good with respect to all multiple points of $\mathcal{A}$. So, from now on we always assume that the labyrinth $\mathcal{L}(\mathcal{A})$ of an arrangement $\mathcal{A}$ is proper.

Now we show how we can read off the braids of an arrangement $\mathcal{A}$ from its labyrinth. The intersection of $\mathbb{C}_{\tilde{x}} = \{(x, y) \in \mathbb{C}^2; x = \tilde{x}\}$, the fiber of pr_1 over the point $\tilde{x} \in \mathbb{C}^1 \backslash D(\mathcal{A})$, with lines of $\mathcal{A}$, $\mathbb{C}_{\tilde{x}} \cap \big(\bigcup_{i=1}^{n} H_i\big)$, consists of n distinct points. When we move the point $\tilde{x}$ along a path in $\mathbb{C}^1 \setminus D(\mathcal{A})$, the fiber $\mathbb{C}_{\tilde{x}}$ will move correspondingly. These n points form a braid on n strings. We will call the string corresponding to the hyperplane H_i the i^{th} string. In general, these points have distinct real parts. A braiding will occur when the path intersects a line of the labyrinth $\mathcal{L}(\mathcal{A})$. Precisely, following two steps suggested by Moishezon as described in Section 2, suppose that the path Γ_k intersects the line $L_{i,j}$ of the labyrinth $\mathcal{L}(\mathcal{A})$. Then we will obtain a braiding of the i^{th} string and j^{th} string. The received braid is determined up to sign. The sign of this braid depends on the fact that which of these strings moves over the other one. This can also be determined by the labyrinth $\mathcal{L}(\mathcal{A})$. Recording successively all these braids when the fiber moves along the path γ_k, we will get the braids $\theta(\gamma_k)$.

3. A CW-complex model for $\mathbb{C}^2 \setminus \mathcal{A}$

We begin by recalling results of [3]. Suppose that $\mathcal{P} = \{P_1, ..., P_N\}$ denotes the set of all multiple points of $\mathcal{A}$. For each multiple point P_k, let I_k be its local index (see [3]). Let choose a good system of simple paths Γ_k, $k = 1, ..., N$ as in Section 2. It gives us a good ordered system of generators $\{\gamma_1, ..., \gamma_N\}$ of $\pi_1(\mathbb{C}^1 \setminus D(\mathcal{A}))$.

The following is the main result of [3].

Theorem 3.1. *The braid monodromy of $\mathcal{A}$ is determined by*

$$\theta(\gamma_k) = \beta_k \cdot A_{I_k} \cdot \beta_k^{-1}, \quad 1 \leq k \leq N,$$

where A_{I_k} is the full twist on I_k, β_k is a braid which can be read off from the labyrinth $\mathcal{L}(\mathcal{A})$.

The full twists A_{I_k}'s and the braids β_k's are determined from the labyrinth $\mathcal{L}(\mathcal{A})$ as mentioned after the Remark 2.2.

As noted in [6], the braid monodromy of $\mathcal{A}$ is closely related to the fundamental group of its complement $\pi_1\big(\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H\big)$. The intersection of the fiber $\mathbb{C}_{x_0}$ of pr_1 over x_0 with hyperplanes of $\mathcal{A}$ consists of n points. Then $\mathbb{C}_{x_0} \setminus (\mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H))$ is a punctured complex line with n removed points. Let $g_1, ..., g_n$ denote the generators of the free group $\pi_1(\mathbb{C}_{x_0} \setminus (\mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H)))$. We identify these generators with their images in $\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ via the homomorphism induced from the embedding $\mathbb{C}_{x_0} \setminus (\mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H)) \subset \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. Let consider the braid group $B[\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H)]$ as a group of automorphisms of $\pi_1(\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H)$. For each multiple point P_k, $1 \leq k \leq N$, we denote by $\mathcal{I}_k$ the set of indices of all lines of $\mathcal{A}$ going through P_k. Then we have the following corollary (cf. [5]).

Corollary 3.1. *The fundamental group of the complement to the arrangement* $\mathcal{A}$, $\pi_1(\mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H)$, *is generated by elements* $g_1, ..., g_n$, *with the defining relations*

$$g_i = \beta_k \cdot A_{I_k} \cdot \beta_k^{-1} \cdot g_i, \quad i \in \mathcal{I}_k,\ k = 1, ..., N.$$

Remind that in [8], Randell gave a presentation for the fundamental group of the complement of any real arrangement. This presentation can be simplified (see [4]) to get a smaller presentation having one generator for each line and $m-1$ commutation relations for each multiple point of multiplicity m. Also, in [4], M. Falk has proved that the canonical 2-complex, modeled on this presentation is homotopy equivalent to the complement M.

Now we proceed on the construction of our model. Consider the set $\mathcal{P} = \{P_1, ..., P_N\}$ of all multiple points of $\mathcal{A}$. These multiple points $P_k = (x_k, y_k)$ are the only singularities of $\mathcal{A}$. Let x_0 be a fixed base point in $\mathbb{C}^1 \setminus D(\mathcal{A})$. Remind that $\mathcal{I}_k = \{i_1, ..., i_r\}$ is the set of all indices of those lines of $\mathcal{A}$ passing through the multiple point P_k. Let denote by $\mathcal{A}_k$ the arrangement of hyperplanes H_{i_s}; $s = 1, ..., r$. Observe that, locally at the point P_k, after a suitable isotopy, we can consider that we have a real arrangement. First, we associate with x_0 a cellular complex $C_0 = \underbrace{\mathbb{S}^1 \vee ..... \vee \mathbb{S}^1}_{n}$. Let denote each copy of $\mathbb{S}^1$ by e_i; $i = 1, ..., n$ respectively. For each multiple point P_k, $1 \le k \le N$, we associate a complex C_k as follows. Denote by R_k the simplified Randell's complex corresponding to the arrangement $\mathcal{A}_k$ as defined in [4]. Then we set $C_k = R_k \vee (\underbrace{\mathbb{S}^1 \vee ... \vee \mathbb{S}^1}_{n-r})$. We denote the 1-cells of R_k by $e_{i_s}^{(k)}$; $s = 1, ..., r$ respectively, and each copy of $\mathbb{S}^1$ by $e_i^{(k)}$, $i \in \{1, ..., n\} \setminus \mathcal{I}_k$. Next we have to attach these complexes C_i's to each other. In fact, each C_k will be attach to C_0 and the attachment will be done along the chosen simple paths Γ_k connecting x_0 to a point x_k^ε near the point x_k. We have two cases.

Case I: If the path Γ_k does not cut any line of the labyrinth $\mathcal{L}(\mathcal{A})$, we will attach a 1-cell l_k connecting the 0-cell of C_0 to the 0-cell of C_k. Then for each $i = 1, ..., n$ we attach a 2-cell having the boundary $e_i \cdot l_k \cdot (e_i^{(k)})^{-1} l_k^{-1}$.

Case II: If the path Γ_k cuts some lines of the labyrinth $\mathcal{L}(\mathcal{A})$, the attachment will be done in several steps.

Step 1: Suppose that beginning from x_0, Γ_k cuts the line $L_{m,n}$ first. We take a new copy of complex C_0, denote it by $C_0^{(1)}$ and denote its 1-cells by $g_i^{(1)}$, $i = 1, ..., n$, respectively. We attach $C_0^{(1)}$ to C_0 first by attaching a 1-cell connecting two 0-cells of these complexes. Then, for each $i \in \{1, ..., n\} \setminus \{m, n\}$ we attach a 2-cell as in the case I. Finally, we attach a new 2-cell having the boundary $e_m \cdot e_n \cdot (g_m^{(1)})^{-1} \cdot (g_n^{(1)})^{-1}$.

Step 2: Suppose that after $L_{m,n}$, the path Γ_k will cut next another line $L_{s,t}$ of $\mathcal{L}(\mathcal{A})$. Then we will repeat the step 1, using the the complex $C_0^{(1)}$ instead of the complex C_0, the 1-cells $g_i^{(1)}$, $i = 1, ..., n$, instead of 1-cells e_i; $i = 1, ..., n$, and the indices s, t instead of indices m, n. Continuing this way, after a finite number of steps we will come to a complex $C_0^{(h)}$ and from here the path Γ_k will go to the point x_k^ε, without cutting any other line of the labyrinth. We denote the 1-cells of this complex by $e_i^{(h)}$, $i = 1, ..., n$.

Step 3: Now, the complex $C_0^{(h)}$ is attached to the complex C_k in the same way as it has been done in the Case I.

By this attachment procedure, we obtain a CW-complex, denoted by $C(\mathcal{A})$.

Theorem 3.2. *The CW-complex $C(\mathcal{A})$ is homotopy equivalent to the complement of the arrangement $\mathcal{A}$ in $\mathbb{C}^2$.*

Proof. The proof is quite elementary and can be deduced from the construction of this CW-complex. If the arrangement $\mathcal{A}$ is central, as noted above, after a suitable isotopy we can consider it to be a real arrangement. Then, the complex $C(\mathcal{A})$ is nothing but the simplified Randell's complex. And the theorem follows from [4].

Suppose that the arrangement $\mathcal{A}$ has more than one multiple points. From the complex $C(\mathcal{A})$ we first collapse the new 2-cells occurring in the attachment. By this way, we have identified the 1-cells $e_i^{(k)}$, $i = 1, ..., n$, $k = 1, ..., N$ to e_i respectively, modulo a conjugation. Note that according to the construction of the complex $C(\mathcal{A})$ these conjugations are the same conjugations appear in the determining of braid monodromy as in [3]. Observe that the union of all new 1-cells appear in the construction of $C(\mathcal{A})$ is a copy of the union of all Γ_k, $k = 1, ..., N$. Because Γ_k, $k = 1, ..., N$, is a good system of simple paths, this union is contractible. So, we can collapse all new 1-cells to the only 0-cell of C_0. Clearly, the resulting complex is the canonical 2-complex associated to the braid monodromy presentation of the fundamental group of the complement given in [3], [2]. According to Libgober [5], it is homotopy equivalent to the complement of the arrangement $\mathcal{A}$.

The Theorem was proved. □

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