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A NEW CLASS OF UNIQUE RANGE SETS FOR MEROMORPHIC FUNCTIONS ON $\mathbb{C}$

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Dedicated to the memory of Le Van Thiem

ABSTRACT. In this paper, we give a new class of unique range sets for meromorphic functions.

1. Introduction and the result

In 1926, Nevanlinna [N] proved the well-known Five-point Theorem: "Let f and g be two meromorphic functions on $\mathbb{C}$. If $f^{-1}(a_i) = g^{-1}(a_i)$ for five distinct points a_i, $i = 1, \dots, 5$, then $f \equiv g$. Since then such a similar uniqueness property of meromorphic functions has been studied extensively.

Let f be a non-constant meromorphic function on the complex plane $\mathbb{C}$ and S be a subset of $\mathbb{C}$. Define

$$E_f(S) = \bigcup_{a \in S} E_f(a),$$

where $E_f(a) = \{(m, z) \in \mathbb{N} \times \mathbb{C} | f(z) - a = 0, \text{ with multiplicity } m\}$.

A set S is called a *unique range set for meromorphic functions* ($URSM$) if, for any pair of non-constant meromorphic functions f and g, the condition $E_f(S) = E_g(S)$ implies $f \equiv g$. A set S is called a *unique range set for entire functions* ($URSE$) if for any pair of non-constant entire functions f and g, the condition $E_f(S) = E_g(S)$ implies $f \equiv g$. A natural question arises: *Which conditions warrant S to be a unique range set*?.

There are several contributions to this quetion. For example, in 1982 Gross and Yang [GY] showed that the set $S = \{z \in \mathbb{C} \mid z + e^z = 0\}$ is a $URSE$ (note that S contains an infinite number of elements). Afterwards, $URSE$ and also $URSM$ with finitely many elements have been found by Yi ([Y1], [Y2]), Li and Yang ([LY1], [LY2]), Mues and Reinders [MR], Hu and Yang [HY], and others. In fact, in these papers, they showed that the set $\{\omega, \omega^q + a\omega^{q-r} + b = 0\}$ gives small unique range sets for meromorphic functions or entire functions under suitable conditions on constants a, b and positive interges q, r. Recently, Frank

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and Reinders [FR] gave a unique set for meromorphic functions with 11 elements, which are given as the set of all zeros of the polynomoial

$$P(\omega) = \frac{(q-1)(q-2)}{2}\omega^q - q(q-2)\omega^{q-1} + \frac{q(q-1)}{2}\omega^{q-2} - c$$

for $q = 11$ and a constant $c \neq 0, 1$. Forthermore, the method used in the above cited papers involves estimations of Nevanlinna characteristic functions. In [FR] the authors remarked that by the method of estimations of Nevanlinna characteristic functions one cannot obtain a lower bound less than 11, while there is a conjecture saying that $\lambda_M = 6$ ([LY]) (recall that $\lambda_M = \inf\{\#(S) \mid S \text{ is a URSM}\}$, where $\#(S)$ is the number of elements of the set S.)

In this paper, we give a new class of URSMs (with 13 elements), using a recent result of Y. T. Siu and S. K. Yeung [SY]. The main idea is to relate the problem of finding URSM to the problem of proving some curves to be degenerate.

Theorem 1.1. *Suppose that n and $2m$ are two positive integers such that n and $2m$ have no common factors and $n > 8 + 4m$. Let*

$$S = \big\{z \in \mathbb{C} \mid\ z^n + az^{n-m} + bz^{n-2m} + c = 0\big\},$$

where a, b, $c \in \mathbb{C}^*$ such that $a^2 - 4b \neq 0$ and the algebraic equation

$$z^n + az^{n-m} + bz^{n-2m} + c = 0$$

has no mutiple roots. Then S is a unique range set for meromorphic functions.

2. Some lemmas

The following lemmas will be needed in the proof of our theorem.

Lemma 2.1. [SY] *Let $g_j(x_0, \dots, x_n)$ be a homogeneous polynomial of degree δ_j for $0 \leq j \leq n$. Suppose there exists a holomorphic map $f : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ so that its image lies in the curve described by*

$$\sum_{j=0}^{n} x_j^{p-\delta_j} g_j(x_0, \dots, x_n) = 0,$$

$$\textit{and} \quad p > (n+1)(n-1) + \sum_{j=0}^{n} \delta_j.$$

Then the polynomials

$$x_1^{p-\delta_1} g_1(x_0, \dots, x_n), \dots, x_n^{p-\delta_n} g_n(x_0, \dots, x_n)$$

are linearly dependent on the image of f.

Lemma 2.2. (The Second Main Theorem, [L]). *Let f be a meromorphic function and $a_1, \dots, a_q$ be distinct complex constants. Then*

$$(q-2)T(r) \leq \sum_{j=1}^{q} N_1(a_j, r) + S(r, f).$$

Furthermore,

$$\sum_{a\in\mathbb{C}} \delta_f(a) \leq 2,$$

where

$$\delta_f(a) = 1 - \lim_{r\to\infty} \sup \frac{N_1\big(r, \dfrac{1}{f-a}\big)}{T(r,f)} \cdot$$

Lemma 2.3. [M] *If X is a irreducible curve of degree d and genus g in the complex projective plane, then*

$$\frac{1}{2}(d-1)(d-2) = g + \sum \delta_z, \quad 2\delta_z = \mu_z + r_z - 1,$$

where μ is multiplicity of X at the singular point z, and r branches of X pass through the singular point z.

3. Proof of Theorem

Let $a_1, a_2, \dots, a_n$ be the distinct roots of the polynomial $z^n + az^{n-m} + bz^{n-2m} + c = 0$ and (z_1, z_2, z_3, z_4) be homogeneous projective coordinates in $\mathbb{P}^3(\mathbb{C})$.

Consider the surface X in $\mathbb{P}^3(\mathbb{C})$ define by

$$X : (z_1 - a_1z_2)(z_1 - a_2z_2)\dots(z_1 - a_nz_2) - (z_3 - a_1z_4)(z_3 - a_2z_4)\dots(z_3 - a_nz_4) = 0.$$

Let f, g be non-constant meromorphic functions such that $E_f(S) = E_g(S)$. Represent $f = \dfrac{f_1}{f_2}$ and $g = \dfrac{l_1}{l_2}$, where (f_1, f_2) and (l_1, l_2) are some pairs of entire functions without common factors. Then there exists an entire function h such that

$$(f_1 - a_1f_2)\dots(f_1 - a_nf_2) = e^h(\ell_1 - a_1\ell_2)\dots(\ell_1 - a_n\ell_2).$$

Put $g_1 = e^{\frac{l}{n}}l_1$, $g_2 = e^{\frac{l}{n}}l_2$, and define $\Phi = (f_1, f_2, g_1, g_2)$. Then Φ is a holomorphic curve in X. Hence

(1)

$$f_1^{n-2m}\left(f_1^{2m} + af_1^m f_2^m + bf_2^{2m}\right) + cf_2^n - g_1^{n-2m}\left(g_1^{2m} + ag_1^m g_2^m + bg_2^{2m}\right) - cg_2^n = 0.$$

Since $n > 8 + 4m$, the hypothesis of Lemma 2.1 is satisfied (with $k = 3$, $\delta_0 = \delta_2 = 2m$, $\delta_1 = \delta_3 = 0$). Without loss of generality we can suppose that there are numbers α_1, α_2, α_3, not all are zero, such that

$$\alpha_1 f_1^{n-2m}(f_1^{2m} + af_1^m f_2^m + bf_2^{2m}) + \alpha_2 f_2^n - \alpha_3 g_2^n = 0$$

We consider the possible cases:

Case 1: $\alpha_1\alpha_2\alpha_3 \neq 0$.

Using again Lemma 2.1 (with $k = 2$, $\delta_0 = 2m$, $\delta_1 = \delta_2 = 0$), we obtain

$$\alpha_1' f_1^{n-2m}(f_1^{2m} + af_1^m f_2^m + bf_2^{2m}) + \alpha_2' f_2^n = 0,$$

where not all α_i' are zeros. This implies that f is constant.

Case 2: $\alpha_3 = 0$. It is clear that f is constant.

Case 3: $\alpha_2 = 0$. Clearly, $\alpha_1\alpha_3 \neq 0$. Then

$$f_1^n + af_1^{n-m}f_2^m + bf_1^{n-2m}f_2^{2m} = \gamma g_2^n,$$

where $\gamma = \dfrac{\alpha_3}{\alpha_1}$.

The image of the holomorphic mapp

$$(f_1, f_2, g_2) : \mathbb{C} \longrightarrow \mathbb{P}^2$$

is contained in the curve L defined by the equation

$$x^n + ax^{n-m}y^m + bx^{n-2m}y^{2m} - \gamma z^n = 0.$$

Denoting by $F(x, y, z)$ the polynomial in the equation of the curve L, we have

$$(2) \qquad \begin{cases} \dfrac{\partial F}{\partial x} = x^{n-2m-1}[nx^{2m} + (n-m)ax^m y^m + (n-2m)by^{2m}], \\ \dfrac{\partial F}{\partial y} = my^{m-1}x^{n-2m}(ax^m + 2by^m), \\ \dfrac{\partial F}{\partial z} = \gamma n z^{n-1}. \end{cases}$$

If $t = (x_0, y_0, z_0) \in \mathbb{P}^2$ is a singular point of the curve, then

$$\frac{\partial F}{\partial x}\Big|_t = \frac{\partial F}{\partial y}\Big|_t = \frac{\partial F}{\partial z}\Big|_t = 0$$

From (2) and the hypothesis it follows that L has a unique singular point at $(0, 1, 0)$.

We prove that the curve L is irreducible. If not, since $(0, 1, 0)$ is the only singular point of L, the irreducible components of L must go through this point. This implies that the point $(0, 1, 0)$ must be an irreducible singularity of L, which contradicts to the assumption $(n, 2m) = 1$. In this case, by Lemma 2.3, the genus of L is

$$g = \frac{(n-1)(n-2)}{2} - \frac{(n-2m-1)(n-1)}{2} = \frac{(n-1)(2m-1)}{2}\cdot$$

The conditions on n, m imply that the genus of L is at least 6. Then L is hyperbolic and (f_1, f_2, g_2) is constant mapping.

Case 4: $\alpha_1 = 0$. It is clear that $\alpha_2\alpha_3 \neq 0$. Further more,

$$\alpha f_2^n = g_2^n,$$

where $\alpha = \dfrac{\alpha_2}{\alpha_3}$. From (1) we obtain

$$\begin{aligned} &f_1^{n-2m}(f_1^{2m} + af_1^m f_2^m + bf_2^{2m}) \\ &+ c(1-\alpha)f_2^n - g_1^{n-2m}(g_1^{2m} + a\varepsilon^m g_1^m f_2^m + b\varepsilon^{2m} f_2^{2m}) = 0, \end{aligned}$$

where $\varepsilon^n = \alpha$.

We claim that $\alpha = 1$. Indeed, if $\alpha \neq 1$, then using Lemma 2.1 (with $\delta_0 = \delta_2 = 2m$, $\delta_1 = 0$, $k = 2$, $n > 3 + 4m$) we obtain that $f_1^{n-2m}(f_1^{2m} + af_1^m f_2^m + bf_2^{2m})$

and f_2^n are linearly dependent, and then $\dfrac{f_1}{f_2}$ = constant. This is a contracdition. Hence $f_2^n = g_2^n$.

Putting $h = \dfrac{f}{g}$, we conclude from (1) that

$$(h^n - 1)g^{2m} + a(h^{n-m} - 1)g^m + b(h^{n-2m} - 1) = 0. \tag{3}$$

We prove that h is constant. In fact, if it is not the case, we write (3) in the form

$$\Big[(h^n - 1)g^m + \frac{a}{2}(h^{n-m} - 1)\Big]^2 = \Psi(h), \tag{4}$$

where Ψ is defined by

$$\Psi(z) = -b(z^n - 1)(z^{n-2m} - 1) + \frac{a^2}{4}(z^{n-m} - 1)^2.$$

Since

$$\Psi'(z) = z^{n-2m-1}\left[(n-m)\frac{a^2 - 4b}{2}z^n + bnz^{2m} - \frac{a^2(n-m)}{2}z^m + b(n-2m)\right],$$

and $\Psi(0) \neq 0$, the polynomial Ψ has at least $(2n - 2m) - n = n - 2m$ distinct zeros. From (4), we obtain that the roots of $\Psi(h) = 0$ have multiplicity at least 2. It follows from Lemma 2.2 that $\dfrac{n-2m}{2} \leq 2$, which contradicts the condition $n > 8 + 4m$. Hence h is constant.

Furthermore, since g is not constant, equation (3) give $h^n - 1 = 0$ and $h^{n-1} - 1 = 0$. It follows $h = 1$ and hence $f \equiv g$. So S is a unique range set for meromorphic functions.

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