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POLYNOMIAL MAPS OF THE COMPLEX PLANE WITH THE BRANCHED VALUE SETS ISOMORPHIC TO THE COMPLEX LINE

NGUYEN VAN CHAU

Abstract. We present a completed list of the polynomial dominating maps of $\mathbb{C}^2$ with the branched value curves isomorphic to the complex line $\mathbb{C}$, up to polynomial automorphisms.

1. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial dominating map, $\mathrm{Close}(f(\mathbb{C}^n)) = \mathbb{C}^n$, and denote by $\deg_f$ the geometric degree of f-the number of solutions of the equation $f = a$ for generic points $a \in \mathbb{C}^n$. The *branched value set* E_f of f is the smallest subset of $\mathbb{C}^n$ such that the map

$$(*) \qquad f : \mathbb{C}^n \setminus f^{-1}(E_f) \longrightarrow \mathbb{C}^n \setminus E_f$$

gives an unbranched $\deg_f$ −sheeted covering. It is well-known (see [M]) that the branched value set E_f is either empty set or an algebraic hypersurface, and

$$E_f = \{a \in \mathbb{C}^n : \# f^{-1}(a) \neq \deg_f\}.$$

If $E_f = \emptyset$, then f is injective, and hence, f is an automorphism of $\mathbb{C}^n$ by the well-known fact that injective polynomial maps of $\mathbb{C}^n$ are automorphisms (see [R]). The famous Jacobian conjecture ([BCW]) asserts that f must have a singularity if $E_f \neq \emptyset$. A question naturally raises as what can be said about polynomial dominating maps of $\mathbb{C}^n$ if their branched value sets are isomorphic to a given algebraic hypersurface E.

In this article we consider polynomial dominating maps of $\mathbb{C}^2$ with the branched value sets isomorphic to the complex line $\mathbb{C}$. We are interested in finding a list of such maps, up to polynomial automorphisms. We say that two maps $f, g : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ are *equivalent* if there are polynomial automorphisms α and β of $\mathbb{C}^2$ such that $\alpha \circ f \circ \beta = g$.

Theorem 1. *A polynomial dominating map f of $\mathbb{C}^2$ with finite fibres and the branched value set isomorphic to $\mathbb{C}$ is equivalent to the map $(x, y) \mapsto (x^{\deg_f}, y)$.*

In view of this theorem, the equivalence classes of polynomial dominating maps of $\mathbb{C}^2$ with finite fibres and the branched value sets isomorphic to $\mathbb{C}$ are completely determined by the geometric degree of maps.

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Theorem 1 is an immediate consequence of the following

Theorem 2. *A polynomial dominating map of $\mathbb{C}^2$ with the branched value set isomorphic to $\mathbb{C}$ is equivalent to one of the following maps*

i) $(x, y) \mapsto (x^{\deg_f}, y)$;

ii) $(x, y) \mapsto (x^{\deg_f}, x^m y)$, $m \geq 1$;

iii) $(x, y) \mapsto \Big(x^{\deg_f}, x^m\big(x^n y + \sum_{i=0}^{n-1} a_i x^i\big)\Big)$, $m \geq 1$, $n \geq 1$, $a_0 \neq 0$ *and* $a_i = 0$ *for* $i + m = 0(mod \deg_f)$.

In fact, in the above list, only maps of type (i) have finite fibres. The fiber at $(0, 0)$ of a map of the types (ii) and (iii) is the line $x = 0$. Further, as shown in Section 4, the topology of maps of type (ii) and type (iii) are quite different. The proof of Theorem 2 presented in Section 3 is an application of the famous theorem of Abhyankar, Moh and Suzuki ([AM], [S]) on the embedding of the complex line into the complex plane. Section 4 is devoted to some remarks and open questions.

2. Let us recall some elementary facts on the topology of polynomials in two-variables. Let $h(x, y) \in \mathbb{C}[x, y]$. By the *exceptional value set* E_h of h we mean a minimal set $E_h \subset \mathbb{C}$ such that the map

$$h : \mathbb{C}^2 \setminus h^{-1}(E_h) \longrightarrow \mathbb{C} \setminus E_h$$

gives a smooth locally trivial fibration. The set E_h is at most a finite set (see [V]). The fiber of this fibration, denoted by Γ_h, is called *the generic fiber* of h. A polynomial h is *primitive* if its generic fiber is connected. By the Stein factorization a polynomial $h(x, y)$ can be represented in the form $h(x, y) = \phi(r(x, y))$ for a primitive polynomial $r(x, y)$ and an one-variable polynomial $\phi(t)$ (see [F]).

The following lemma is an immediate consequence of Abhyankar-Moh-Suzuki theorem on the embedding of the complex line into the complex plane, which asserts that regular embbedings of $\mathbb{C}$ in $\mathbb{C}^2$ are equivalent to the natural embbeding, or equivalently, that if $p(x, y)$ is irreducible and if the curve $p = 0$ is a smooth contractible algebraic curve, then $p \circ \alpha(x, y) = x$ for some polynomial automorphism α of $\mathbb{C}^2$.

Lemma 1. *Let $h \in \mathbb{C}[x, y]$. Suppose that the generic fiber Γ_h has d connected components and each of them is diffeomorphic to $\mathbb{C}$. Then, there exists a polynomial automorphism α in $\mathbb{C}^2$ such that*

$$h \circ \alpha(x, y) = x^d + a_1 x^{d-1} + \cdots + a_d.$$

We will use this lemma in the situation when all fibres of h, except for at most one, are diffeomorphic to a dictinct union of d lines $\mathbb{C}$. Then, the lemma shows that $h(\alpha(x, y)) = x^d + a_d$ for an automorphism α.

Proof of Lemma 1. By the Stein factorization, we can represent $h(x, y) = \phi(r(x, y))$ for a primitive polynomial $r \in \mathbb{C}[x, y]$ and $\phi \in \mathbb{C}[t]$. Further, one can choose r

and ϕ so that

$$\phi(t) = t^{\deg \phi} + \text{lower terms}.$$

Observe that for each $c \in \mathbb{C}$ the fiber $h^{-1}(c)$ consists of the curves $r(x, y) = c_i$, $i = 1, \dots \deg \phi$, where c_i are zero points of $\phi(t) - c = 0$. Since the generic fiber Γ_h has d connected components and each of them is diffeomorphic to $\mathbb{C}$, the generic fiber Γ_r of r is diffeomorphic to $\mathbb{C}$, $\deg \phi = d$, and

$$\phi(t) = t^d + a_1 t^{d-1} + \cdots + a_d.$$

Let $\gamma \in \mathbb{C}$ be a fixed generic value of r. Then the polynomial $r(x, y) - \gamma$ is irreducible and the curve $r(x, y) - \gamma = 0$ is diffeomorphic to $\mathbb{C}$. So, in view of the Abhyankar-Moh-Suzuki theorem, there exists a polynomial automorphism α of $\mathbb{C}^2$ such that $r(\alpha(x, y)) - \gamma = x$. Then we get

$$h(\alpha(x, y)) = \phi(r(\alpha(x, y)) = \phi(x + \gamma) = x^d + a_1 x^{d-1} + \cdots + a_d.$$

□

3. *Proof of Theorem 2.* Let $f : \mathbb{C}^2_{(x,y)} \longrightarrow \mathbb{C}^2_{(u,v)}$ be a polynomial dominating map with the branched value set E_f isomorphic to $\mathbb{C}$, where (x, y) and (u, v) stand for coordinates in $\mathbb{C}^2$. In view of the Abhyankar-Moh-Suzuki theorem we can choose a polynomial automorphism α of $\mathbb{C}^2$ so that the image $\alpha(E_f)$ is the line $u = 0$.

Let $\bar{f} := \alpha \circ f$, $\bar{f} = (\bar{f}_1, \bar{f}_2)$. Then

$$E_{\bar{f}} = \{u = 0\}, \quad \bar{f}^{-1}(E_{\bar{f}}) = \bar{f}_1^{-1}(0),$$

and the map $\bar{f} : \mathbb{C}^2 \setminus \bar{f}_1^{-1}(0) \longrightarrow \mathbb{C}^2 \setminus \{u = 0\}$ gives a unbranched $\deg_f$ −sheeted covering. This covering induces unbranched $\deg_f$ −sheeted coverings

$$\bar{f} : \bar{f}_1^{-1}(c) \longrightarrow \{u = c\} \simeq \mathbb{C}, \;\; c \neq 0.$$

Since $\mathbb{C}$ is simply connected, for every $0 \neq c \in \mathbb{C}$ the fiber $\bar{f}_1^{-1}(c)$ consists of exactly $\deg_f$ connected components and each of these components is diffeomorphic to $\mathbb{C}$. So, applying Lemma 1 we see that there exists an automorphism β of $\mathbb{C}^2$ such that

$$\bar{f}_1(\beta(x, y)) = x^{\deg_f} + c.$$

Let $\tilde{f} := \bar{f} \circ \beta - (c, 0)$. Then $\tilde{f}(x, y) = (x^{\deg_f}, \tilde{f}_2(x, y))$. Note that $E_{\tilde{f}} = \{u = 0\}$ and $\tilde{f}^{-1}(\{u = 0\}) = \{x = 0\}$. So, by definition, for each $(a, b) \in \mathbb{C}^2$, $a \neq 0$, the equation $\tilde{f}(x, y) = (a, b)$ has exactly $\deg_f$ dictinct solutions. This implies that for each $(a, b) \in \mathbb{C}^2$, $a \neq 0$, the equation $\tilde{f}_2(\epsilon, y) = b$ has a unique solution for each $\deg_f$ radical ε of a. Such a polynomial $\tilde{f}_2(x, y)$ must be of the form

$$\tilde{f}_2(x, y) = ax^k y + x^l g(x), \quad 0 \neq a \in \mathbb{C}, \quad k \geq 0 \;\; l \geq 0, \quad g \in \mathbb{C}[x].$$

For $k = 0$, we have $\tilde{f} \circ \gamma(x, y) = (x^{\deg_f}, y)$ for the automorphism $\gamma(x, y) := (x, a^{-1}y - ax^l g(x))$.

Consider the case $k > 0$. Put $m = \min\{k; l\}$ and $n = k - m$.

For $n = 0$ we can represent $\tilde{f}_2(x, y)$ in the form

$$\tilde{f}_2(x, y) = ax^m(y + h(x)) + c(x^{\deg_f}),$$

where $h(x), c(x) \in \mathbb{C}[x]$ and

$$h(x) = \sum_{i+m \neq 0 (mod \deg_f)} a_i x^i.$$

Define

$$\gamma_1(u, v) := (u, a^{-1}(v - c(u))), \quad \gamma_2(x, y) := (x, y - h(x)).$$

Then we get

$$\gamma_1 \circ \tilde{f} \circ \gamma_2(x, y) = (x^{\deg_f}, x^m y).$$

For the case $n > 0$ we can represent

$$\tilde{f}_2(x, y) = ax^m(x^n y + h(x) + x^n b(x)) + c(x^{\deg_f}),$$

where $h(x), b(x),\ c(x) \in \mathbb{C}[x]$ and

$$h(x) = \sum_{i=0,\dots n-1,\ i+m \neq 0 (mod \deg_f)} a_i x^i.$$

Let

$$\gamma_1(u, v) := (u, a^{-1}(v - c(u))), \quad \gamma_2(x, y) := (x, y - b(x)).$$

Then

$$\gamma_1 \circ \tilde{f} \circ \gamma_2(x, y) = (x^{\deg_f}, x^m(x^n y + h(x))).$$

Thus, the map f under consideration is always equivalent to one of maps of the types (i)-(iii). □

4. Let us to conclude the paper by some remarks and open questions.

From the topological point of view, maps of the types (i), (ii) and (iii) behave quite differently. The maps of type (i) have finite fibres, while the fiber at $(0, 0)$ of a map of type (ii) or (iii) is the line $\{x = 0\}$. Furthermore, for an irreducible germ curve $\gamma \subset \mathbb{C}^2$ located at $(0, 0)$ and intersecting with the line $\{u = 0\}$ at $\{(0, 0)\}$, the inverse image of γ by a map of type (ii) is connected and consists of the line $x = 0$ and a branch located at $(0, 0)$. But, the inverse image of γ by a map of type (iii) consists of the line $x = 0$ and a branch at infinity.

As shown in Theorem 2 and its proof, the polynomial dominating maps of $\mathbb{C}^2$ with the branched value curve isomorphic to $\mathbb{C}$ form a small class among polynomial dominating maps of $\mathbb{C}^2$. The structure of covering (*) associated to such a map is very simple. Up to automorphisms of $\mathbb{C}^2$, this covering looks like an unbranched covering from $\mathbb{C}^2 \setminus \{x = 0\} \longrightarrow \mathbb{C}^2 \setminus \{u = 0\}$. The singular set of such a map is isomorphic to $\mathbb{C}$. In particular, *a polynomial map of $\mathbb{C}^2$ with the branched value curve isomorphic to $\mathbb{C}$ must has a singularity.* This is true even when the branched value curve is only homeomorphic to $\mathbb{C}$ (see [C1]). It is worth to present here the following observation, which is a deduced from the results of [C2].

Theorem 3. (see [Thm. 4.4, C2]) *Suppose* $f = (P, Q)$ *is a non-zero constant Jacobian polynomial map of* $\mathbb{C}^2$*. If* $E_f \neq \emptyset$*, then every irreducible component* ℓ *of* E_f *is a singular curve parametrized by a polynomial map* $(p_\ell, q_\ell) : \mathbb{C} \longrightarrow \mathbb{C}^2$ *with*

$$\frac{\deg p_\ell}{\deg q_\ell} = \frac{\deg P}{\deg Q}.$$

Note that the situation is quite different in the case of holomorphic maps. Orevkov in [O] had constructed a nonsingular holomorphic map from a Stein manifold homeomorphic to $\mathbb{R}^4$ onto an open ball in the complex plane $\mathbb{C}^2$ which gives a three-sheeted branching covering with the branched value set diffeomorphic to $\mathbb{R}^2$.

It is worth to find analogous statements of Theorem 1 for high-dimensional cases. Let $F = (F_1, F_2, \dots F_n) : \mathbb{C}^n_{(x_1,\dots x_n)} \longrightarrow C^n_{(u_1,\dots u_n)}$ be a polynomial dominating map with finite fibres.

Problem. Suppose $E_F = \{u_1 = 0\}$.

i) Does there exists an automorphism α of $\mathbb{C}^n$ such that

$$F_1 \circ \alpha(x_1, \dots x_n) = x_1^{\deg_F} \ ?$$

ii) Is F equivalent to the map

$$(x_1, \dots x_n) \mapsto (x_1^{\deg_F}, x_2, \dots x_n) \ ?$$

Such a map F gives a locally trivial fibration $F_1 : \mathbb{C}^n \setminus \ F^{-1}(0) \longrightarrow \mathbb{C} \setminus \{0\}$ with fiber diffeomorphic to a distinct union of $\deg_f$ space $\mathbb{C}^{n-1}$. Further, every connected component of the fibers $F_1^{-1}(c)$, $c \neq 0$, is isomorphic to $\mathbb{C}^{n-1}$. The situation here seems to be simpler than those in the problem of embeddings $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$- a generalization of the Abhyankar-Moh-Suzuki theorem, which asks whether a regular embbeding of $\mathbb{C}^{n-1}$ in $\mathbb{C}^n$ is equivalent to the natural embbeding, is still open for $n > 2$.

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Hanoi Institute of Mathematics,
P.O. Box 631, Boho 10000, Hanoi, Vietnam

E-mail address: nvchau@thevinh.ncst.ac.vn