ACTA MATHEMATICA VIETNAMICA
Volume 27, Number 2, 2002, pp. 185-195

APPROXIMATE RECOVERY OF MULTIVARIATE PERIODIC FUNCTIONS USING WAVELET DECOMPOSITIONS

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Abstract. We study the optimal recovery of multivariate periodic functions of the Besov class of common smoothness $SB^{\omega}_{p,\theta}$ from their values at n points in terms of the quantity $R_n(SB^{\omega}_{p,\theta}, L_q)$, which is a characterization of optimality of methods of recovery. The smoothness of $SB^{\omega}_{p,\theta}$ is defined via modulus of smoothness dominated by a function ω of modulus of smoothness type. With some restrictions on ω and p, q, we give the asymptotic order of this quantity when $n \to \infty$. An asymptotically optimal method of recovery is constructed by using the wavelet family formed from the integer translates of the dyadic scales of multivariate de la Vallée Poussin kernels.

1. Introduction

We investigate optimal methods of recovery of multivariate periodic functions from their values at n points. Multivariate periodic functions are considered as functions defined on d-torus $\mathbf{T}^d := [-\pi, \pi]^d$. Let X be a normed space of functions on $\mathbf{T}^d$, $\{x^1, ..., x^n\} \subset \mathbf{T}^d$ a selection of n points and $P_n(y_1, ..., y_n)$ a mapping from $\mathbf{R}^n$ into a linear manifold in X of dimension at most n. We can naturally consider the method of approximate recovery of a function $f \in X$ from its values $f(x^1), ..., f(x^n)$ by the element $g = P_n(f(x^1), ..., f(x^n))$. The recovery error is measured by $\|f - g\|$. Let W be a subset in X. We are interested in the optimal recovery of functions $f \in W$ over all such methods of approximate recovery. The error of this optimal recovery is given by

$$R_n(W, X) := \inf \sup_{f \in W} \|f - P_n(f(x^1), ..., f(x^n))\|,$$

where inf is taken over all selections of n points $\{x^1, ..., x^n\} \subset \mathbf{T}^d$ and mappings P_n from $\mathbf{R}^n$ into a linear manifold in X of dimension at most n.

In the present paper, we study optimal methods of recovery of functions from the Besov class of common smoothness $SB^{\omega}_{p,\theta}$ in terms of the quantity $R_n(SB^{\omega}_{p,\theta}, L_q)$ for $1 \le p, q \le \infty$. Its smoothness is defined via modulus of smoothness dominated by a function ω of modulus of smoothness type. With some restrictions on ω and p, q, we give the asymptotic order of this quantity

Received April 20, 2001.

when $n \to \infty$. An asymptotically optimal method of recovery is constructed using periodic wavelet decompositions of functions into the integer translates of the dyadic scales of multivariate de la Vallée Poussin kernels. Problems of recovery of periodic functions which are related to the present paper, were considered in [3–5], [7], [8].

Let us define smoothness Besov spaces of functions on $\mathbf{T}^d$. For a positive integer l, the symmetric difference operator Δ_h^l, $h \in \mathbf{T}^d$, is defined inductively by

$$\Delta_h^l := \Delta_h^1 \Delta_h^{l-1},$$

starting from

$$\Delta_h^1 f := f(\cdot + h/2) - f(\cdot - h/2).$$

Let

$$\omega_l(f,t)_p := \sup_{|h|<t} \|\Delta_h^l f\|_p, \quad t \geq 0,$$

is the lth p-integral modulus of smoothness of f where $\|\cdot\|_p$ denotes the usual p-integral norm of the space $L_p := L_p(\mathbf{T}^d)$.

We introduce the class MS_l of functions ω of modulus of smoothness type as follows. It consists of all non-negative functions ω on $[0, \infty)$ such that:

(i) $\omega(0) = 0$,
(ii) $\omega(t) \leq \omega(t')$ if $t \leq t'$,
(iii) $\omega(kt) \leq k^l \omega(t)$, for $k = 1, 2, ...$,
(iv) ω satisfies Condition Z_l, that is, there exist a positive number $a < l$ and positive constant C_l such that

$$\omega(t)t^{-a} \geq C_l h^{-a} \omega(h), \ 0 \leq t \leq h,$$

(v) ω satisfies Condition BS, that is, there exist a positive number b and positive constant C such that

$$\omega(t)t^{-b} \leq C h^{-b} \omega(h), \ 0 \leq t \leq h \leq 1.$$

Let $1 \leq p \leq \infty$, $0 < \theta \leq \infty$ and $\omega \in MS_l$. The Besov space $B_{p,\theta}^\omega$ consists of all functions $f \in L_p$ for which the Besov semi-quasi-norm

$$|f|_{B_{p,\theta}^\omega} := \begin{cases} \left(\int\limits_0^\infty \{\omega_l(f,t)_p / \omega(t)\}^\theta dt/t \right)^{1/\theta}, & \theta < \infty, \\ \sup\limits_{t>0} \omega_l(f,t)_p / \omega(t), & \theta = \infty \end{cases} \tag{1}$$

is finite. We define the Besov quasi-norm by

$$\|f\|_{B_{p,\theta}^\omega} := \|f\|_p + |f|_{B_{p,\theta}^\omega}. \tag{2}$$

The definition of $B_{p,\theta}^\omega$ does not depend on l, i. e., for a given ω, (1)–(2) determine equivalent norms for all l such that $\omega \in MS_l$. The function $\omega(t) = t^r$, $r > 0$ belongs to the class MS_l for any $l > r$. The space $B_{p,\theta}^r := B_{p,\theta}^\omega$ with $\omega(t) = t^r$, $r > 0$, is the classical Besov space.

We say that ω satisfies Condition R(ε) ($\varepsilon \geq 0$) if $\omega(t)t^{-\varepsilon}$ satisfies Condition BS. If ω satisfies Condition R(d/p), then $B^{\omega}_{p,\theta}$ is compactly embedded into $C(\mathbf{T}^d)$.

The Besov class

$$SB^{\omega}_{p,\theta} := \{f \in B^{\omega}_{p,\theta} : \|f\|_{B^{\omega}_{p,\theta}} \leq 1\}$$

is defined as the unit ball of the space $B^{\omega}_{p,\theta}$.

We denote $a_+ := \max\{a, 0\}$ and use the notation $F \asymp F'$ if $F \ll F'$ and $F' \ll F$, and $F \ll F'$ if $F \leq CF'$ with C being an absolute constant.

The main result of the present paper is the following

Theorem 1. *Let $1 \leq p,\ q \leq \infty,\ \ 0 < \theta \leq \infty$. Assume that either $p \geq q$ or $p < q \leq 2$ and ω satisfies Condition* R(d/p)*. Then we have*

$$R_n(SB^{\omega}_{p,\theta}, L_q) \ \asymp\ \omega(d/n)n^{(d/p-d/q)_+}.$$

Theorem 1 was proved in [4] for the univariate classical Besov-Hölder class $SB^{r}_{p,\infty}$.

We now construct an asymptotically optimal method of recovery using periodic wavelet decompositions of functions into the integer translates of the dyadic scales of multivariate de la Vallée Poussin kernels. Let

$$V_{m,r}(t) := \ \frac{1}{r}\sum_{k=m}^{m+r-1} D_k(t) = \frac{\sin(rt/2)\sin((2m+r)t/2)}{r\sin^2(t/2)}$$

be the de la Vallée Poussin kernel of order m, where

$$D_m(t) := \sum_{|k|\leq m} e^{ikt}$$

is the univariate Dirichlet kernel of order m. The multivariate de la Vallée Poussin kernel $V_{m,r}$ is defined by

$$V_{m,r}(x) := V_{m,r}(x_1)V_{m,r}(x_2)\cdots V_{m,r}(x_d).$$

We use the abbreviation $V_m(x) := V_{m,m}(x)$.

Next we let

$$v_k := (3 \times 2^k)^{-d}V_{2^k},\ k = 0, 1, 2, ...$$

be the periodic dyadic scaling functions, and the periodic wavelets

$$v_{k,s} := v_k(\cdot - sh_k),\ s \in Q_k,$$

be defined as the integer translates of v_k, where

$$h_k := \frac{2\pi}{3 \times 2^k} \quad \text{and} \quad Q_k := \{s \in \mathbf{Z}^d :\ 0 \leq s_j < 3 \times 2^k,\ j = 0, ..., d\}.$$

Consider the recovery of a functions f on $\mathbf{T}^d$ from its values at the points

$$\{sh_k :\ s \in Q_k\}$$

by the method S_k defined as follows

$$S_k(f) := \sum_{s \in Q_k} f(sh_k) v_{k,s}.$$

Notice that the number of the points $\{sh_k : \ s \in Q_k\}$ is $3^d \times 2^{dk}$. For the sake of completeness we put $S_{-1}(f) := f(0)$.

Let

$$R_k(f) := f - S_k(f)$$

be the error of recovering f by $S_k(f)$. For functions from $SB^\omega_{p,\theta}$, we have the following

Theorem 2. *Let* $1 \le p,\ q \le \infty,\ \ 0 < \theta \le \infty$ *and* ω *satisfy Condition* $\mathrm{R}(d/p)$. *For a given natural number* n, *let* k *be the largest non-negative integer such that* $3^d \times 2^{dk} \le n$. *Then we have*

$$\sup_{f \in SB^\omega_{p,\theta}} \|R_k(f)\|_q \ \asymp\ \omega(d/n) n^{(d/p - d/q)_+}.$$

In this way, under the assumptions of Theorem 1 the method of recovery S_k defined in Theorem 2, is asymptotically optimal for $R_n(SB^\omega_{p,\theta}, L_q)$. The lower bound of Theorem 2 was proved in [8] (see Theorem 6.2 of Chapter 1) for the univariate classical Besov-Hölder class $SB^r_{p,\infty}$.

2. Estimation of the recovery error via best trigonometric approximation

For functions $f \in L_q$, we define the convolution operator I_k by letting

$$I_k(f) := f * V_{2^k},$$

for $k \ge 0$, and

$$I_{-1}(f) := \pi^{-d} \int_{\mathbf{T}^d} f(x) dx.$$

Let $\mathcal{T}_m$ denote the space of trigonometric polynomials of order at most m in each variable. Obviously, $S_k(f)$ and $I_k(f)$ belong to $\mathcal{T}_{2^{k+1}-1}$. From the definition it is easy to see that

$$I_k(f) = \ f, \quad f \in \mathcal{T}_{2^k}. \tag{3}$$

Let us give some properties of the method of recovery S_k. The recovery method S_k is precise for trigonometric polynomials $\mathcal{T}_{2^{k-1}}$, i.e.,

$$S_k(f) = f, \quad f \in \mathcal{T}_{2^k}. \tag{4}$$

This property is derived from (3) and the following assertion (see [5]). If the integers m, n, s satisfy the inequality $m + n < s$, then the following equality holds for any $f \in \mathcal{T}_m$ and $g \in \mathcal{T}_n$

$$f * g = s^{-1} \sum_k f(2\pi k/s) g(\cdot - 2\pi k/s),$$

where the sum is taken over all $k \in \mathbf{Z}^d$ such that $0 \le k_j < s,\ j = 1, 2, ..., d$. The method of recovery S_k interpolates f at the points $sh_k,\ s \in Q_k$, i.e.,

$$S_k(f, sh_k) = f(sh_k),\ s \in Q_k.$$

For the following property see [5]. For any $f \in \mathcal{T}_{2^s},\ s \ge k$,

$$\|S_k(f)\|_q \le C_q 2^{d(s-k)/q} \|f\|_q. \tag{5}$$

From properties of de la Vallée Poussin kernels it follows that for any continuous function f on $\mathbf{T}^d$

$$\lim_{k\to\infty} \|R_k(f)\|_\infty = 0,$$

and for any function $f \in L_q(\mathbf{T}^d)$, $1 \le q \le \infty$,

$$\lim_{k\to\infty} \|f - I_k(f)\|_q = 0.$$

Hence, using (4) we deduce that any continuous function f on $\mathbf{T}^d$ can be decomposed into the wavelets $v_{k,s}$ by the series

$$f(x) = \sum_{k=0}^{\infty} \delta S_k(f) = \sum_{k=0}^{\infty} \sum_{s\in Q_k} \lambda_{k,s} v_{k,s}, \tag{6}$$

where both the series converge uniformly on $\mathbf{T}^d$. Here

$$\delta S_0(f) := S_{-1}(f); \quad \delta S_k(f) := S_{k-1}(f) - S_{k-2}(f),\ k = 1, 2, 3, ...,$$

$$\lambda_{k,s} = \lambda_{k,s}(\{f(s'h_k)\}_{s'\in Q_k}),$$

and $\lambda_{k,s}(\cdot)$ are linear functions defined on $\mathbf{R}^{3^d \times 2^{dk}}$.

Similarly, due to (3), any function $f \in L_q(\mathbf{T}^d)$, $1 \le q \le \infty$, can be decomposed into the wavelets $v_{k,s}$ by the series

$$f(x) = \sum_{k=0}^{\infty} \delta I_k(f) = \sum_{k=0}^{\infty} \sum_{s\in Q_k} \mu_{k,s} v_{k,s}, \tag{7}$$

where both the series converge in the norm of $L_q(\mathbf{T}^d)$. Here $\delta I_k(f)$ is defined in the same way as $\delta S_k(f)$,

$$\mu_{k,s} = \mu_{k,s}(\{\delta I_k(f, s'h_k\}_{s'\in Q_k}),$$

and $\mu_{k,s}(\cdot)$ are linear functions defined on $\mathbf{R}^{3^d \times 2^{dk}}$.

Let

$$E_m(f)_q := \inf_{g\in\mathcal{T}_m} \|f - g\|_q \ (1 \le q \le \infty)$$

be the best approximation to f by trigonometric polynomials of order at most m. Then we have for any continuous function f on $\mathbf{T}^d$,

$$\|R_k(f)\|_\infty \le CE_{2^{k-1}}(f)_\infty.$$

and for any function $f \in L_q(\mathbf{T}^d)$, $1 \le q \le \infty$,

$$\|f - I_k(f)\|_q \le 3E_{2^{k-1}}(f)_q. \tag{8}$$

For $1 \le q < \infty$, the recovery error $\|R_k(f)\|_q$ is estimated as follows.

Theorem 3. *Let $1 \le q < \infty$. Then we have for any continuous function f on $\mathbf{T}^d$*

$$\|R_k(f)\|_q \ \le \ C_q \sum_{s=-1}^{\infty} 2^{ds/q} E_{2^{k+s}}(f)_q. \tag{9}$$

Proof. Let f be a continuous function on $\mathbf{T}^d$. By (7) we have

$$f = I_{k-1}(f) + \sum_{s=k}^{\infty} \delta I_s(f)$$

On the other hand, from (3) we obtain $S_k(I_{k-1}(f)) = I_{k-1}(f)$. Therefore, using (8) we get

$$\begin{aligned}
\|R_k(f)\|_q \ &\le \ \|f - I_{k-1}(f)\|_q + \sum_{s=k}^{\infty} \|S_k(\delta I_s(f))\|_q \\
&\le \ 3E_{2^{k-1}}(f)_q + C_q \sum_{s=k}^{\infty} 2^{d(s-k)/q} \|\delta I_s(f)\|_q \\
&\le \ 3E_{2^{k-1}}(f)_q + 3C_q \sum_{s=k}^{\infty} 2^{d(s-k)/q} (E_{2^s}(f)_q + E_{2^{s-1}}(f)_q).
\end{aligned}$$

Hence we obtain (9). ◻

Theorem 3 was proved in [4] for the univariate case. A similar inequality earlier than that in [4], was proved in [7] for the univariate case and $1 \le q \le 2$.

3. Optimal recovery for Besov classes

Let $1 \le p \le \infty$, $0 < \theta \le \infty$. Then the following quasi-norms equivalence holds

$$\|f\|_{B^\omega_{p,\theta}} \ \asymp \ \left(\sum_{k=0}^{\infty} \{\|\delta I_k(f)\|_p / \omega(2^{-k})\}^\theta \right)^{1/\theta} \quad (\theta < \infty). \tag{10}$$

Moreover, a function f belongs to $B^\omega_{p,\theta}$ iff f can be represented as a series

$$f = \sum_{s=0}^{\infty} f_s, \quad f_s \in \mathcal{T}_{2^s}, \tag{11}$$

converging in L_p-norm, and the quasi-norm

$$\left(\sum_{k=0}^{\infty}\{\|f_k\|_p/\omega(2^{-k})\}^{\theta}\right)^{1/\theta} \;<\; \infty \quad (\theta<\infty) \tag{12}$$

is finite. In addition, the quasi-norm (12) is equivalent to $\|f\|_{B^{\omega}_{p,\theta}}$. (The sum in (10) and (12) is changed to the supremum for $\theta = \infty$). The quasi-norms equivalence (10) as well as (11) and (12) can be proved by a standard method of establishing equivalence of different Besov quasi-norms, using the inequality (8), Stechkin's theorem of trigonometric approximation (see, e.g., Theorem 2.3 in p. 205 of [1]), Bernstein's inequality (see, e.g., Theorem 2.5 in p. 102 of [1]) and the following generalization of discrete Hardy inequality.

Let ϕ be a positive non-decreasing function on $(0,\infty)$ and $0<\theta\le\infty$. We define the quasi-norm $\|a\|_{\theta,\phi}$ for a sequence $a=\{a_k\}_{k\in\mathbf{Z}}$ as follows

$$\|a\|_{\theta,\phi} := \left(\sum_{k\in\mathbf{Z}}\{|a_k|/\phi(2^{-k})\}^{\theta}\right)^{1/\theta}, \;(\theta<\infty)$$

(the sum is changed to sup when $\theta=\infty$).

Let $0<\theta\le\infty$ and let ϕ satisfy Conditions $\mathrm{R}(\varepsilon)$ and Z_l $(\varepsilon<l)$. If the sequences $a=\{a_k\}_{k\in\mathbf{Z}}$ and $b=\{b_k\}_{k\in\mathbf{Z}}$ with $a_k, b_k\ge 0$, satisfy the condition

$$b_k \;\le\; M\sum_{s\in\mathbf{Z}} a_s \min\{2^{\varepsilon(s-k)}, 2^{l(s-k)}\},$$

then

$$\|b\|_{\theta,\phi} \;\le\; C_{\theta,\phi} M \|a\|_{\theta,\phi}. \tag{13}$$

The proof of this inequality is similar to that for the case $\varepsilon=0$ in [2].

Theorem 4. *Let $1\le p\le\infty$, $0<\theta\le\infty$ and ω satisfy Condition* R*(d/p). Then there hold the following quasi-norms equivalences*

$$\|f\|_{B^{\omega}_{p,\theta}} \;\asymp\; \left(\sum_{k=0}^{\infty}\{\|\delta S_k(f)\|_p/\omega(2^{-k})\}^{\theta}\right)^{1/\theta} \quad (\theta<\infty), \tag{14}$$

and

$$\|f\|_{B^{\omega}_{p,\theta}} \;\asymp\; \left(\sum_{k=0}^{\infty}\{\|R_k(f)\|_p/\omega(2^{-k})\}^{\theta}\right)^{1/\theta} \quad (\theta<\infty),$$

with the sum changing to the supremum when $\theta=\infty$.

Proof. We will prove the quasi-norms equivalence (14). The other can be proved similarly. Due to (10), we need to prove that the quasi-norm in the right side of (14) is equivalent to that in the right side of (10). By the assumptions of the theorem, each function $f\in B^{\omega}_{p,\theta}$ is represented as series (6) and (7), and the quasi-norm equivalence (10) holds for the series (7). By (4), $\delta S_k(\delta I_s(f)) = 0$ for all k, s such that $s \le k$. Hence

$$\delta S_k(f) = \sum_s \delta S_k(\delta I_s(f)) = \sum_{s>k} \delta S_k(\delta I_s(f)).$$

The inequality (5) gives

$$\|\delta S_k(f)\|_p \ \le\ M_p \sum_{s>k} 2^{(s-k)d/p} \|\delta I_s(f)\|_p.$$

Applying the generalized Hardy inequality (13) for $\phi = \omega$ satisfying Conditions $\mathrm{R}(d/p)$ and Z_l $(d/p < l)$, and $b_k = \|\delta S_k(f)\|_p$, $a_k = \|\delta I_k(f)\|_p$, we deduce that the right side of (14) is not greater than the right side of (10) multiplied by some positive constant. The inverse inequality can be proved in the same way. $\square$

Proof of Theorem 2. Let us first prove the upper bound. By the inequality $\|\cdot\|_q \ll \|\cdot\|_p$ for $q \le p$ and the embedding of $B^\omega_{p,\theta}$ into $B^\omega_{p,\infty}$, it is enough to establish the upper bound for the case $p \le q$. From Theorem 3 and Stechkin's theorem of trigonometric approximation we have for any continuous function on $\mathbf{T}^d$

$$\|R_k(f)\|_q \ \le\ C_q \sum_{s=-1}^{\infty} 2^{ds/q} \omega_l(f, 2^{-k-s})_q. \tag{15}$$

A theorem of DeVore, Riemenschneider and Sharpley (see Theorem 3.4 on p. 181 of [1]) states that

$$\omega_l(f, 2^{-s})_q \ \le\ C_q \int_0^{2^{-s}} h^{1/q-1/p-1} \omega_l(f,h)_p dh. \tag{16}$$

Let $f \in SB^\omega_{p,\infty}$. This means that $\omega_l(f,h)_p \le \omega(h)$. Because Conditions $\mathrm{R}(d/p)$ implies Conditions $\mathrm{R}(d/p - d/q)$ and because $\omega \in MS_l$, there are a positive constant C and an $\delta > 0$ such that for $0 \le h \le 2^{-s}$ we have

$$h^{-d/p+d/q-\delta}\omega(h) \ \le\ C 2^{(d/p-d/q+\delta)s}\omega(2^{-s}).$$

Hence the integral in the right side of (16) can be esimated as follows

$$\begin{aligned} \int_0^{2^{-s}} h^{1/q-1/p-1}\omega_l(f,h)_p dh \ &\ll\ 2^{(d/p-d/q+\delta)s}\omega(2^{-s}) \int_0^{2^{-s}} h^{\delta-1} dh \\ &\ll\ 2^{(d/p-d/q)s}\omega(2^{-s}). \end{aligned}$$

Combining the last inequality with (15) and (16) gives

$$\|R_k(f)\|_q \ \ll\ 2^{(d/p-d/q)k} \sum_{s=-1}^{\infty} 2^{ds/p}\omega(2^{-k-s}). \tag{17}$$

Similarly as for the right side of (16), with integral changing to sum, using Conditions R(d/p) we can estimate the sum in (17) as $\ll \omega(2^{-k})$. Consequently,

$$\begin{aligned}\|R_k(f)\|_q &\ll \omega(2^{-k})2^{(d/p-d/q)k} \\ &\asymp \omega(d/n)n^{d/p-d/q}.\end{aligned}$$

The upper bound is proved.

We now prove the lower bound. It suffices to construct a function $g \in SB^{\omega}_{p,\theta}$ so that

$$\|R_k(g)\|_q \gg \omega(2^{-k})2^{(d/p-d/q)_+k}. \tag{18}$$

Let us first consider the case $p \geq q$. We define g by

$$g(x) := \lambda\omega(2^{-k})e^{i2^k x_1},$$

where λ is a positive number. By the equivalent quasi-norm (12) of $B^{\omega}_{p,\theta}$, we can define a value of λ so that $g \in SB^{\omega}_{p,\theta}$ for all k. A direct computation shows that $S_k(g) = \lambda\omega(2^{-k})$. Hence

$$\begin{aligned}\|R_k(g)\|_q &\gg \omega(2^{-k})\left(\int_{-\pi}^{\pi}|e^{i2^k x_1} - 1|^q dx_1\right)^{1/q} \\ &\gg \omega(2^{-k}).\end{aligned}$$

Thus (18) has been proved for the case $p \geq q$. For the case $p < q$, we define

$$g := \lambda\omega(2^{-k})\|\varphi\|_p^{-1}\varphi$$

where

$$\varphi := V_{2^{k+1},r(k+1)} - V_{2^k,r(k)}$$

and $r(k) = [\varepsilon 2^k]$ with $0 < \varepsilon < 1$ ($[a]$ denotes the integer part of a). Similarly, we can define a value of λ so that $g \in SB^{\omega}_{p,\theta}$ for all k and ε. Put

$$G := S_k(g) * v,$$

where $v := V_{2^{k-3}}$. Obviously, by definition $g * v = 0$, hence

$$R_k(g) * v = -G.$$

By Young's inequality and the inequality $\|v\|_1 \leq 3$, we get

$$\|G\|_q \leq \|R_k(g)\|_q\|v\|_1 \leq 3\|R_k(g)\|_q,$$

or equivalently,

$$\|R_k(g)\|_q \geq \|G\|_q/3. \tag{19}$$

By (3) we have $v_k * v = (3 \times 2^k)^{-d}v$ and, therefore,

$$G(x) = (3 \times 2^k)^{-d}\sum_{s \in Q_k} g(sh_k)v(x - sh_k).$$

On the other hand,

$$2^{-dk}\sum_{s\in Q_k} g(sh_k)v(x-sh_k) = \sum_{s\in \mathbf{Z}^d} G_s(x),$$

where $G_s := (e^{i2^k(s,x)}) * v$. This equality was proved in [4] for the univariate case. The multivariate case can be proved similarly. Denote by E the set of all $s \in \mathbf{Z}^d$ such that the set

$$P_s := \prod_{j=1}^{d}(|2^k s_j - 2^{k-2}|, |2^k s_j + 2^{k-2}|)$$

is contained in the set $[(1+\varepsilon)2^k, 2^{k+1}]^d$, and by F the set of all $s \in \mathbf{Z}^d$ such that

$$P_s \cap \{(2^k,(1+\varepsilon)2^k)^d \cup (2^{k+1},(1+\varepsilon)2^{k+1})^d\} = \emptyset.$$

It is easy to verify that, for any $s \in E$,

$$G_s(x) = \lambda\omega(2^{-k})\|\varphi\|_p^{-1} v(x).$$

We have

$$\begin{aligned}
\|G\|_q &\geq \sum_{s\in E}\|G_s\|_q - \sum_{s\in F}\|G_s\|_q \\
&\geq |E|\lambda\omega(2^{-k})\|\varphi\|_p^{-1}\|v\|_q - |F|\lambda\omega(2^{-k})\|\varphi\|_p^{-1}\|\varphi\|_1\|v\|_q \\
&= \lambda\omega(2^{-k})\|\varphi\|_p^{-1}\|v\|_q(|E| - |F|\ \|\varphi\|_1),
\end{aligned}$$

where $|A|$ denotes the number of elements of the set A. By simple computations we get the estimates $|E| \geq (1-\varepsilon) - 2$ and $|F| \leq 3\varepsilon + 4$, and $\|\varphi\|_1 \leq C|\ln\varepsilon|$. Therefore, we can find an ε so that

$$|E| - |F|\ \|\varphi\|_1 \geq C_\varepsilon.$$

Thus, we get

$$\|G\|_q \gg \omega(2^{-k})\|\varphi\|_p^{-1}\|v\|_q.$$

We have $\|v\|_q \asymp 2^{d(1-1/q)k}$ and $\|\varphi\|_p \asymp 2^{d(1-1/p)k}$. Hence

$$\|G\|_q \gg \omega(2^{-k})2^{(d/p-d/q)k}.$$

This estimate and (19) prove (18) the case $p < q$. □

Proof of Theorem 1. The upper bound follows from Theorem 2. We will establish the lower bound. Denote by $d_n(W,X)$ the Kolmogorov n-width of the set W in the normed space X (see, e.g., [8] for the definition). By definition we have

$$R_n(W,X) \geq d_n(W,X). \tag{20}$$

For $d_n(SB^\omega_{p,\theta}, L_q)$, under the assumption of Theorem 1 there holds the following asymptotical order

$$d_n(SB^\omega_{p,\theta}, L_q) \asymp \omega(d/n)n^{(d/p-d/q)_+}. \tag{21}$$

This was proved in [6] for the univariate classical Besov-Hölder class $SB^{\omega}_{p,\infty}$. The general case can be proved in a similar way. Combining (20) with (21) gives the lower bound. □

References

[1] R. DeVore and G. Lorentz, *Contructive Approximation*, Springer-Verlag, 1993.

[2] Dinh Dung, *Best multivariate approximations by trigonometric polynomials with frequencies from hyperbolic crosses*, J. Approximation Theory **91** (1997), 205–225.

[3] Dinh Dung (Din' Zung), *On recovery and one-sided approximation of periodic functions of several variables*, Dokl. Akad. SSSR **313** (1990), 787-790.

[4] Dinh Dung, *On interpolation recovery for periodic functions*, In: Functional Analysis and Related Topics (Ed. S. Koshi), World Scientific, Singapore 1991, pp. 224–233.

[5] Dinh Dung, *On optimal recovery of multivariate periodic functions*, In: Harmonic Analysis (Satellite Conference Proceedings, Ed. S. Igary), Springer-Verlag 1991, Tokyo-Berlin, pp. 96–105.

[6] Dinh Dung, *Approximation of functions of several variables by means of harmonic analysis*, Doctor Dissertation, Moscow, 1985 (in Russian).

[7] V. Temlyakov, *Approximate recovery of periodic functions of several variables*, Mat. Sb. **128** (1985), 256–268.

[8] V. Temlyakov, *Approximation of Periodic Functions*, Nova Science Publishers, New York, 1993.

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