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# THE NORMALIZED DUALITY MAPPING AND TWO RELATED CHARACTERISTIC PROPERTIES OF A UNIFORMLY CONVEX BANACH SPACE

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Abstract. This paper is devoted to the study of some properties of the normalized duality mapping and two related characteristic properties of a uniformly convex Banach space. In particular, a theorem due to J. Prüß is extended. Based on this extention of the theorem of Prüß, some new results on the continuity of the metric projection onto a family of closed convex sets in uniformly convex Banach spaces are obtained.

## 1. Introduction

The concept of duality mapping was introduced by Beurling and Livingston [4, p. 407] in a geometric form. A slightly extended version of the concept was proposed by Asplund [1] who showed how the duality mappings can be characterized via the subdifferential of convex functions. It is well known that the geometric properties of a Banach space $X$ correspond to the analytic properties of the duality mapping $J : X \to 2^{X^*}$ (see Definition 2.1 below). For instance, $X$ is strictly convex if and only if $J$ is strictly monotone; $X^*$ is uniformly convex if and only if $J$ is single-valued and uniformly continuous on any bounded subset of $X$.

The duality mapping has many other applications. For example, in some recent papers (see, for instance, [6], [11], [12]), it has been used as the main tool for studying the continuity of the metric projection.

J. Prüß [14] proved that $X$ is uniformly convex if and only if $J$ is, in some sense, uniformly strictly monotone. Thanks to this property of $J$, the author obtained several interesting results on accretive operators in a uniformly convex space.

Together with $J$, the normalized duality mapping $J_p$, $1 < p < +\infty$, is also considered (see [12]). In several cases where we have to work with problems in uniformly convex Banach spaces, like $L_p(\Omega)$ and ${W_p}^m(\Omega)$ (with $p \neq 2$), $J_p$ is more suitable than $J$. This is because in the spaces we often have to deal with functions which are expressed via certain exponents of degree $p$. On the other

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hand, the metric projection in such spaces can be defined via $J_p$, hence analytical properties of $J_p$ lead us to a better understanding of the behavior of the metric projection. So it is of interest to study various analytical properties of $J_p$.

It is worthy noting that the uniform convexity of a Banach space can be characterized fully via the normalized duality mapping $J_p$, in a similar way as it was characterized via the duality mapping $J$.

The aim of this paper is to obtain such characterizations and apply them to study the continuity of the metric projection onto a family of closed convex sets in a uniformly convex Banach space.

In Section 2, after recalling the definition of the normalized duality mapping $J_p$, we derive from [1] a formula for computing the map (Proposition 2.1) and show how the metric projection can be characterized via $J_p$ (Proposition 2.2). We also obtain the first characteristic property of uniformly convex Banach spaces by using $J_p$ (Proposition 2.3). In Section 3 we establish an extended version of a theorem due to J. Prüß [14], which gives another characteristic property of uniformly convex Banach spaces. Section 4 is devoted to the study of the metric projection onto a family of closed convex sets in a uniformly convex Banach space. In particular, it is shown that Lemma 1.1 from [15] on the continuity of the metric projection in Hilbert spaces can be extended to the case of the functions spaces $L_p(\Omega,\mu),\ p>1$.

## 2. Normalized duality mapping $J_p$

Throughout, $X$ denotes a normed space or a Banach space with the dual $X^*$. Symbol $\overline{B}(0,1)$ stands for the closed unit ball in $X$. Let $K$ be closed convex set in $X$ and $y\in X$. We denote by $P_K(y)$ the projection of $y$ onto $K$, that is $P_K(y)\in K$ and

$$\|y-P_K(y)\| = d(y,K) := \inf_{z\in K}\|y-z\|.$$

The set

$$N_K(x)=\begin{cases}\{x^*\in X^*:\langle x^*,y-x\rangle\le 0\ \ \forall y\in K\} & \text{if}\ \ x\in K\\ \emptyset & \text{if}\ \ x\notin K\end{cases}$$

is called the normal cone to $K$ at $x$. For a convex function $\varphi: X\to R\cup\{+\infty\}$, the set

$$\partial\varphi(x) := \{x^*\in X:\varphi(z)-\varphi(x)\ge\langle x^*,z-x\rangle\ \ \forall z\in X\}$$

is called the subdifferential of $\varphi$ at $x$. For each $p>1$, we set

$$\varphi_p(x)=\frac{1}{p}\|x\|^p.$$

**Definition 2.1.** [12] The set-valued map $J_p: X\to 2^{X^*}$ defined by setting $J_p(x)=\partial\varphi_p(x)$ for all $x\in X$, is called the normalized duality mapping of $X$. In the special case where $p=2$, the map $J_2$ is denoted by $J$.

**Remark 2.1.** For every $x \in X$ we have $J_p(x) \neq \emptyset$ because $\varphi_p(x)$ is a continuous convex function on $X$. In particular, the effective domain

$$D(J_p) := \{x : J_p(x) \neq \emptyset\}$$

coincides with $X$ (see [8]). Moreover, for every $x \in X$, $J_p(x)$ is a nonempty, convex, weakly* compact set.

For computing the set $J_p(x)$ one can use the following fact.

**Proposition 2.1.** (See [1, Theorem 1]) *For each $x \in X$, it holds*

$$J_p(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|\|x^*\|, \quad \|x^*\| = \|x\|^{p-1}\}. \tag{2.1}$$

Setting $\phi(t) := t^{p-1}$ $(t \geq 0)$, we see at once that

$$\Phi(t) := \int_0^t \phi(s)ds = \frac{1}{p}t^p \quad (t \geq 0)$$

is the primitive function of $\phi$. Hence, using the arguments for proving Theorem 1 from [1] we can establish formula (2.1). It is worth pointing out that duality mappings in the sense of [4] and [1] are single-valued functions while, in general, $J_p$ is a set-valued mapping.

For the convenience of the reader, we provide here a direct proof of formula (2.1).

*Proof of Proposition 2.1.* For any $x \in X$ we set

$$A(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|\|x^*\|, \quad \|x^*\| = \|x\|^{p-1}\}. \tag{2.2}$$

We have to show that $J_p(x) = A(x)$. We first consider the case $x = 0$. From the inequality $p > 1$ it follows that

$$J_p(0) = \partial\varphi_p(0) = \{x^* : \langle x^*, y\rangle \leq \frac{1}{p}\|y\|^p \ \forall y \in X\} = \{0\}.$$

Besides, it is obvious that $A(0) = \{0\}$. Hence $J_p(0) = A(0)$.

Next, we consider the case $x \neq 0$. Taking any $x^* \in J_p(x)$ we have

$$\frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p \geq \langle x^*, y - x\rangle \tag{2.3}$$

for all $y \in X$. Substituting $y = \lambda x, \lambda > 1$, into (2.3) yields

$$\frac{1}{p}\frac{\lambda^p - 1}{\lambda - 1}\|x\|^p \geq \langle x^*, x\rangle.$$

Letting $\lambda \to 1$ and noting that $\lim\limits_{\lambda \to 1}\left(\frac{\lambda^p - 1}{\lambda - 1}\right) = p$, we obtain

$$\|x\|^p \geq \langle x^*, x\rangle. \tag{2.4}$$

Similarly, substituting $y = (1 - \lambda)x$, $\lambda > 0$, into (2.3) and letting $\lambda \to 0$ we get

$$\|x\|^p \leq \langle x^*, x\rangle. \tag{2.5}$$

Combining (2.4) with (2.5) yields $\|x\|^p = \langle x^*, x\rangle$ and hence $\|x^*\| \geq \|x\|^{p-1}$. On the other hand, for any $y$ satisfying $\|y\| \leq \|x\|$, by (2.3) we have $0 \geq \langle x^*, y - x\rangle$. Therefore $\langle x^*, x\rangle \geq \langle x^*, y\rangle$. Consequently, $\|x\|^p \geq \langle x^*, y\rangle$. This implies

$$\|x\|^{p-1} \geq \langle x^*, \frac{y}{\|x\|}\rangle.$$

Hence $\|x\|^{p-1} \geq \langle x^*, z\rangle$ for all $z$ satisfying $\|z\| \leq 1$. It follows that $\|x\|^{p-1} \geq \|x^*\|$. From what has already been said, we deduce that $\|x\|^{p-1} = \|x^*\|$. We have thus proved that $J_p(x) \subset A(x)$. To prove the reverse inclusion, we fix any $x^* \in A(x)$. Consider the function $f(t) =$ -lnt, $t > 0$. Since $f''(t) = \frac{1}{t^2} > 0$ for all $t > 0$, $f(t)$ is a convex function. Therefore

$$\text{-ln}(\frac{1}{p}\|y\|^p + \frac{p-1}{p}\|x\|^p) \leq -\frac{1}{p}\text{ln}(\|y\|^p) - \frac{p-1}{p}\ \text{ln}(\|x\|^p)$$

for all $y \in X\backslash\{0\}$. This implies

$$\frac{1}{p}\|y\|^p + \frac{p-1}{p}\|x\|^p \geq \|y\|\|x\|^{p-1}$$

for all $y \in X\backslash\{0\}$. Hence

$$\begin{aligned}\frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p &\geq \|y\|\|x\|^{p-1} - \|x\|^p \\ &\geq \|y\|\|x^*\| - \langle x^*, x\rangle \\ &\geq \langle x^*, y - x\rangle\end{aligned}$$

for all $y \in X$. Thus $\varphi_p(y) - \varphi_p(x) \geq \langle x^*, y - x\rangle$ for all $y \in X$. This shows that $x^* \in J_p(x)$. The inclusion $A(x) \subset J_p(x)$ has been established, and the proof is complete. □

**Remark 2.2.** From (2.1) it follows that $J_p(tx) = t^{p-1}J_p(x)$ for every $p > 1$, $x \in X$ and $t > 0$. From (2.1) it also follows that $J_p(x) = \|x\|^{p-2}J(x)$ for every $p > 1$ and $x \in X$.

**Remark 2.3.** The formulation and the proof of the Asplund theorem [1, Theorem 1] given in [9, p. 249, Theorem 8.1.12] are inaccurate. The inaccuracy happens because the necessary use of a primitive function in [1, pp. 200-201] was omitted in [9].

The metric projection in Banach spaces can be characterized by using the normalized duality mapping $J_p$.

**Proposition 2.2.** *Suppose that $X$ is a Banach space, $K$ is a nonempty closed convex set in $X$, and $y \in X$. Then $x = P_K(y)$ if and only if*

$$0 \in J_p(x - y) + N_K(x). \tag{2.6}$$

*Proof.* It is clear that $x = P_K(y)$ if and only if $x$ is a solution of the problem

$$\begin{cases} f(z) := \frac{1}{p}\|z-y\|^p + i_K(z) \to \inf \\ \text{subject to } z \in X, \end{cases} \tag{2.7}$$

where $i_K(z)$ is the indicator function of $K$, i.e.,

$$i_K(z) = \begin{cases} 0 & \text{if } z \in K \\ +\infty & \text{if } z \notin K. \end{cases}$$

Since $f(z)$ is a convex function, $x$ is a solution of (2.7) if and only if $0 \in \partial f(x)$. By the Moreau-Rockafellar theorem (see [8, p. 200, Theorem 1]), the inclusion $0 \in \partial f(x)$ is equivalent to the following one:

$$0 \in J_p(x-y) + N_K(x).$$

The proof is complete. □

**Definition 2.2.** [16, p. 256] A Banach space $X$ is called strictly convex if for all $x, y \in X$, $\|x\| = \|y\| = 1$ and $x \neq y$, and for all $\lambda \in (0,1)$ it holds $\|\lambda x + (1-\lambda)y\| < 1$. A Banach space $X$ is called uniformly convex if for every $\varepsilon$, $0 < \varepsilon \leq 2$, there exists a $\delta > 0$ for which $\|x\| = 1$, $\|y\| = 1$ and $\|x-y\| \geq \varepsilon$ imply $\|x+y\| \leq 2(1-\delta)$.

**Lemma 2.1.** [3, p. 42, Proposition 2.13] *A Banach space $X$ is strictly convex if and only if the following equivalent properties hold:*

(a) *If $\|x+y\| = \|x\| + \|y\|$ and $x \neq 0$, there exists $t \geq 0$ such that $y = tx$.*

(b) *If $\|x\| = \|y\| = 1$ and $x \neq y$ then $\left\|\dfrac{x+y}{2}\right\| < 1$.*

**Proposition 2.3.** (See [9, p. 251, Theorem 8.1.18]) *Let $X$ be a Banach space such that the dual $X^*$ is a strictly convex Banach space. Then, for every $p > 1$, $J_p$ is a single-valued map.*

*Proof.* Taking any $x \in X$ and consider the set $J_p(x)$. If $x = 0$ then $J_p(0) = \{0\}$. Suppose that $x \neq 0$ and $x_1^*, x_2^* \in J_p(x)$. Then $\|x_1^*\| = \|x_2^*\| = \|x\|^{p-1}$ and

$$\begin{aligned} 2\|x_1^*\|\|x\| = 2\|x\|^p &= \langle x_1^* + x_2^*, x\rangle \\ &\leq \|x_1^* + x_2^*\|\|x\|. \end{aligned}$$

Hence $2\|x_1^*\| \leq \|x_1^* + x_2^*\|$. From this it follows that $\|x_1^*\| + \|x_2^*\| \leq \|x_1^* + x_2^*\|$. This implies that $\|x_1^*\| + \|x_2^*\| = \|x_1^* + x_2^*\|$. By the strict convexity of $X^*$ and the first assertion of Lemma 2.1 we get $x_1^* = x_2^*$, as desired. □

It turns out that the uniform convexity of $X^*$ can be characterized via the uniform continuity of the map $J_p(\cdot)$.

**Theorem 2.1.** *The dual space $X^*$ of a Banach space $X$ is uniformly convex if and only if $J_p$ is a single-valued map which is uniformly continuous on each bounded subset of $X$.*

To prove this result we shall need the following

**Lemma 2.2.** [5, p. 36, Theorem 1] *Let $X$ be a Banach space. Then $X^*$ is uniformly convex if and only if the norm of $X$ is uniformly Fréchet differentiable, i.e., the limit* $\lim_{\lambda\to 0}\dfrac{\|x+\lambda y\|-\|x\|}{\lambda}$ *exists uniformly for all $x, y \in S(X)$, where* $S(X) = \{x \in X : \|x\| = 1\}$.

*Proof of the Theorem 2.1.* (This proof is based on some arguments similar to those of the proof of Theorem 1 in [5, p. 36]).

*Necessity.* Assume that $X^*$ is uniformly convex. Then $X^*$ is strictly convex. By Proposition 2.3, $J_p$ is a single-valued map. It remains to show that $J_p$ is uniformly continuous on each bounded subset of $X$. According to Remark 2.2, we have

$$J_p(x) = \|x\|^{p-2}J(x) \quad \text{for every } \ x \in X.$$

By Proposition 32.22 of [17, p. 861], $J$ is uniformly continuous on bounded subset of $X$. Hence $J_p$ is uniformly continuous on each bounded subset that lies outside some neighborhood of $x = 0$. On the other hand, $\|J_p(x)\| = \|x\|^{p-1}$. So $J_p(0) = 0$ and $J_p$ is continuous at $x = 0$. Combining these arguments we can conclude that $J_p$ is uniformly continuous on each bounded subset of $X$.

*Sufficiency.* For any $x, y \in S(X)$ and $\lambda > 0$, using Proposition 2.1 one has

$$\begin{aligned}
\frac{\langle J_p(x), y\rangle}{\|x\|^{p-1}} &= \frac{\langle J_p(x), \lambda y\rangle}{\lambda\|x\|^{p-1}} \\
&= \frac{\langle J_p(x), x\rangle - \|x\|^p + \langle J_p x, \lambda y\rangle}{\lambda\|x\|^{p-1}} \\
&= \frac{\langle J_p(x), x+\lambda y\rangle - \|x\|^p}{\lambda\|x\|^{p-1}} \\
&\le \frac{\|x\|^{p-1}\|x+\lambda y\| - \|x\|^p}{\lambda\|x\|^{p-1}} \\
&= \frac{\|x+\lambda y\| - \|x\|}{\lambda} \\
&= \frac{\|x+\lambda y\|^p - \|x\|\|x+\lambda y\|^{p-1}}{\lambda\|x+\lambda y\|^{p-1}} \\
&\le \frac{\langle J_p(x+\lambda y), x+\lambda y\rangle - |\langle J_p(x+\lambda y), x\rangle|}{\lambda\|x+\lambda y\|^{p-1}} \\
&= \frac{\lambda\langle J_p(x+\lambda y), y\rangle + \langle J_p(x+\lambda y), x\rangle - |\langle J_p(x+\lambda y), x\rangle|}{\lambda\|x+\lambda y\|^{p-1}} \\
&\le \frac{\langle J_p(x+\lambda y), y\rangle}{\|x+\lambda y\|^{p-1}}.
\end{aligned}$$

Hence, for all $x, y \in S(X)$ and $\lambda > 0$ it holds

$$\frac{\langle J_p x, y\rangle}{\|x\|^{p-1}} \le \frac{\|x+\lambda y\| - \|x\|}{\lambda} \le \frac{\langle J_p(x+\lambda y), y\rangle}{\|x+\lambda y\|^{p-1}}.$$

Using the uniform continuity of $J_p$ on each bounded subset of $X$, from the last property one can obtain that

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} = \frac{\langle J_p x, y \rangle}{\|x\|^{p-1}} \cdot$$

According to Lemma 2.2, $X^*$ is uniformly convex. The proof is complete. □

## 3. Generalization of a theorem of J. Prüß

Our aim in this section is to establish a generalized version of Theorem 1 of [14].

**Definition 3.1.** [2, p. 708] A function $\omega : [0, +\infty) \to [0, +\infty)$ is said to be firm if $\omega(0) = 0$ and $\omega(\rho) > 0$ for all $\rho > 0$. If $\omega$ is nondecreasing and firm then it is called a gauge. The set of gauges $\omega : [0, +\infty) \to [0, +\infty)$ is denoted by $G$.

**Theorem 3.1.** (Theorem of Prüß, see [14, Theorem 1]) *A Banach space $X$ is uniformly convex if and only if for every $\gamma > 0$ there is a function $\omega_\gamma \in G$ such that*

$$\langle x^* - y^*, x - y \rangle \geq \omega_\gamma(\|x - y\|)\|x - y\| \tag{3.1}$$

*for all $x, y \in \overline{B}(0, \gamma)$, $x^* \in J(x)$ and $y^* \in J(y)$.*

One may ask whether the conclusion of the above theorem is still true if instead of $J(\cdot)$ one considers the normalized duality mapping $J_p(\cdot)$.

Our main result in this section can be formulated as follows.

**Theorem 3.2.** *If a Banach space $X$ is uniformly convex then, for every $p \geq 2$ and for every $\gamma > 0$, there is a function $\omega_{p,\gamma} \in G$ such that*

$$\langle x^* - y^*, x - y \rangle \geq \omega_{p,\gamma}(\|x - y\|)\|x - y\| \tag{3.2}$$

*for all $x, y \in \overline{B}(0, \gamma)$, $x^* \in J_p(x)$ and $y^* \in J_p(y)$. Conversely, if for a fixed $p \geq 2$ and for all $\gamma > 0$ there exists an $\omega_{p,\gamma} \in G$ satisfying the property (3.2) then $X$ is a uniformly convex Banach space.*

**Remark 3.1**. Theorem 3.1 follows from Theorem 3.2 if we choose $p = 2$.

To prove Theorem 3.2 we shall need the following lemmas.

**Lemma 3.1.** (Bishop-Phelps Theorem, [5, p. 3]) *Let $A$ be a closed bounded convex set in a Banach space $X$. Then the collection of functionals from $X^*$ that achieve their maximum on $A$ is dense in $X^*$.*

**Lemma 3.2.** (James Theorem, [5, p. 12]) *A Banach space $X$ is reflexive if and only if for every $f \in X^*$, there exists $x \in X$ such that $f(x) = \|f\|$.*

**Lemma 3.3.** *For all $a \geq 0$, $b \geq 0$ and $p \geq 2$ one has*

$$a^p + b^p - ab^{p-1} - ba^{p-1} \geq |a - b|^p. \tag{3.3}$$

*Proof.* If $a = 0$ or $b = 0$ then (3.3) is trivial. We consider the case $a > 0$ and $b > 0$. By the symmetry we can assume that $a > b$. Dividing two sides of (3.3) by $a^p$ we obtain the following equivalent inequality

$$1 + \Big(\frac{b}{a}\Big)^p - \Big(\frac{b}{a}\Big)^{p-1} - \frac{b}{a} \geq \Big(1 - \frac{b}{a}\Big)^p. \tag{3.4}$$

For $x := \dfrac{b}{a}$, $x \in (0, 1)$, (3.4) is equivalent to

$$\begin{aligned} & 1 + x^p - x^{1-p} - x \geq (1 - x)^p \\ \Leftrightarrow\ & (1 - x)(1 - x^{p-1}) \geq (1 - x)^p \\ \Leftrightarrow\ & 1 - x^{p-1} \geq (1 - x)^{p-1} \\ \Leftrightarrow\ & 1 \geq x^{p-1} + (1 - x)^{p-1}. \end{aligned} \tag{3.5}$$

Noting that $1 = (x + (1 - x))^{p-1}$ and using the fact that $(u + v)^\alpha \geq u^\alpha + v^\alpha$ for all $u, v \geq 0$, $\alpha \geq 1$ (see [7, p. 32]), we can assert that (3.5) is true. The proof is complete. □

*Proof of Theorem 3.2.* (The proof scheme is similar to that of Theorem 1 of [14])
*Necessity.* Let $X$ be a uniformly convex Banach space and let $p \geq 2$. For each $\gamma > 0$ we consider the function $\omega_{p,\gamma}$ defined by

$$\omega_{p,\gamma}(0) = 0, \quad \omega_{p,\gamma}(\rho) = \omega_{p,\gamma}(2\gamma)$$

for all $\rho \geq 2\gamma$ and

$$\omega_{p,\gamma}(\rho) = \inf\Big\{\frac{\langle x^* - y^*, x - y\rangle}{\|x - y\|} : x, y \in \overline{B}(0, \gamma), \|x - y\| \geq \rho, \\ x^* \in J_p(x), y^* \in J_p(y)\Big\}$$

for all $0 < \rho \leq 2\gamma$. Since $\omega_{p,\gamma}$ is nondecreasing, in order to prove that $\omega_{p,\gamma}(\rho) \in G$ we have only to show that $\omega_{p,\gamma}(\rho) > 0$ for all $\rho \in (0, 2\gamma)$. Suppose on the contrary that $\omega_{p,\gamma}(\rho) = 0$ for some $\rho \in (0, 2\gamma)$. Then there are sequences $(x_n)$, $(y_n)$ of vectors from $\overline{B}(0, \gamma)$, $x_n^* \in J_p(x_n)$, $y_n^* \in J_p(y_n)$, such that $\|x_n - y_n\| \geq \rho$ and

$$\frac{\langle x_n^* - y_n^*, x_n - y_n\rangle}{\|x_n - y_n\|} \to 0.$$

This forces $\langle x_n^* - y_n^*, x_n - y_n\rangle \to 0$. On the other hand, from the inclusions $x_n^* \in J_p(x_n)$, $y_n^* \in J_p(y_n)$ and Lemma 3.3 it follows that

$$\begin{aligned} \langle x_n^* - y_n^*, x_n - y_n\rangle &= \|x_n\|^p + \|y_n\|^p - \langle x_n^*, y_n\rangle - \langle y_n^*, x_n\rangle \\ &\geq \|x_n\|^p + \|y_n\|^p - \|x_n^*\|\|y_n\| - \|y_n^*\|\|x_n\| \\ &= \|x_n\|^p + \|y_n\|^p - \|x_n\|^{p-1}\|y_n\| - \|y_n\|^{p-1}\|x_n\| \\ &\geq |\|x_n\| - \|y_n\||^p. \end{aligned}$$

This implies $|\|x_n\| - \|y_n\|| \to 0$. Since $\|x_n\| \leq \gamma$ and $\|y_n\| \leq \gamma$, without loss of generality we may assume that $\lim\limits_{n\to\infty} \|x_n\| = \lim\limits_{n\to\infty} \|y_n\| = a$ for some $a \geq 0$. Since

$\|x_n\| + \|y_n\| \geq \|x_n - y_n\| \geq \rho$ for every $n$, the case $a = 0$ cannot occur, so we have $a > 0$. Hence there is no loss of generality in assuming that $\|x_n\| > 0$, $\|y_n\| > 0$ for all $n$. We have

$$\begin{aligned}\Big\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\Big\| &= \Big\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|x_n\|} + \frac{y_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\Big\| \\ &\geq \Big\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|x_n\|}\Big\| - \Big\|\frac{y_n}{\|y_n\|} - \frac{y_n}{\|x_n\|}\Big\|.\end{aligned}$$

Therefore

$$\begin{aligned}&\Big\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|x_n\|}\Big\| + \Big\|\frac{y_n}{\|y_n\|} - \frac{y_n}{\|x_n\|}\Big\| \geq \Big\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|x_n\|}\Big\| \\ &\Rightarrow \Big\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\Big\| + \|y_n\|\Big|\frac{1}{\|y_n\|} - \frac{1}{\|x_n\|}\Big| \geq \frac{1}{\|x_n\|}\|x_n - y_n\| \\ &\Rightarrow \liminf_{n\to\infty}\left(\Big\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\Big\| + \|y_n\|\Big|\frac{1}{\|y_n\|} - \frac{1}{\|x_n\|}\Big|\right) \\ &\qquad \geq \liminf_{n\to\infty}\left(\frac{1}{\|x_n\|}\|x_n - y_n\|\right)\end{aligned}$$

$$\Rightarrow \liminf_{n\to\infty}\Big\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\Big\| \geq \frac{\rho}{a}. \tag{3.6}$$

Choose $\varepsilon_1 \in \left(0, \frac{\rho}{a}\right)$ and put $\varepsilon = \frac{\rho}{a} - \varepsilon_1$. By (3.6), there exists $n_0$ such that

$$\Big\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\Big\| > \varepsilon \tag{3.7}$$

for all $n \geq n_0$. Since

$$\begin{aligned}\|x_n + y_n\| &= \|x_n\|\Big\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|x_n\|}\Big\| \\ &= \|x_n\|\Big\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} - \frac{y_n}{\|y_n\|}\Big\| \\ &\leq \|x_n\|\Big\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\Big\| + \|x_n\|\Big\|\frac{y_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\Big\| \\ &= \|x_n\|\Big\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\Big\| + \|x_n\|\|y_n\|\Big|\frac{1}{\|x_n\|} - \frac{1}{\|y_n\|}\Big|,\end{aligned}$$

by (3.7) and by the uniform convexity of $X$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\begin{aligned}\limsup_{n\to\infty}\|x_n + y_n\| &\leq \limsup_{n\to\infty}\left(\|x_n\|\Big\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\Big\| + \|x_n\|\|y_n\|\Big|\frac{1}{\|x_n\|} - \frac{1}{\|y_n\|}\Big|\right) \\ &\leq \limsup_{n\to\infty} a\Big\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\Big\| \\ &\leq a.2(1-\delta). \qquad (3.8)\end{aligned}$$

Since $\|x_n\|^p + \|y_n\|^p - \langle y_n^*, x_n\rangle - \langle x_n^*, y_n\rangle = \langle x_n^* - y_n^*, x_n - y_n\rangle \to 0$,

$$\lim_{n\to\infty}(\langle y_n^*, x_n\rangle + \langle x_n^*, y_n\rangle) = 2a^p.$$

Combining this with (3.8), we have

$$\begin{aligned}
2a^p &= \lim_{n\to\infty}(\langle y_n^*, x_n\rangle + \langle x_n^*, y_n\rangle)\\
&= \limsup_{n\to\infty}(\langle y_n^*, x_n\rangle + \langle x_n^*, y_n\rangle + \langle x_n^*, x_n\rangle - \langle x_n^*, x_n\rangle)\\
&= \limsup_{n\to\infty}(\langle y_n^* - x_n^*, x_n\rangle + \langle x_n^*, x_n + y_n\rangle)\\
&\le \limsup_{n\to\infty}(\|y_n^*\|\|x_n\| - \|x_n\|^p) + \limsup_{n\to\infty}\|x_n^*\|\|x_n + y_n\|\\
&\le \limsup_{n\to\infty}(\|y_n\|^{p-1}\|x_n\| - \|x_n\|^p) + \limsup_{n\to\infty}\|x_n\|^{p-1}\|x_n + y_n\|\\
&= \limsup_{n\to\infty}\|x_n\|^{p-1}\|x_n + y_n\|\\
&\le a^{p-1}2a(1-\delta) = 2a^p(1-\delta),
\end{aligned}$$

a contradiction.

*Sufficiency.* Using Proposition 2.1 we can show that $J_p(X) := \cup_{x\in X} J_p(x)$ is the collection of functionals from $X^*$ that achieve their maximum on $\overline{B}_X$. By Lemma 3.1, $\overline{J_p(X)} = X^*$, where $\overline{J_p(X)}$ denotes the closure of $J_p(X)$. We shall show that the range $J_p(X)$ of $J_p$ is closed. Let $(x_n^*) \subset J_p(X)$ and $x_n^* \to x^*$. Then for each $n$ there exists $x_n \in X$ such that $x_n^* \in J_p(x_n)$. Since $(x_n^*)$ is bounded, so is $(x_n)$. Let $\gamma > 0$ be such that $\|x_n\| \le \gamma$ for every $n$. By our hypothesis, there exists $\omega_{p,\gamma} \in G$ such that

$$\begin{aligned}
\omega_{p,\gamma}(\|x_n - x_m\|)\|x_n - x_m\| &\le \langle x_n^* - x_m^*, x_n - x_m\rangle\\
&\le \|x_n^* - x_m^*\|\|x_n - x_m\|
\end{aligned}$$

for all $m$, $n$. Taking account of this fact, we deduce from the convergence of the sequence $(x_n^*)$ that $(x_n)$ is a Cauchy sequence. Therefore $x_n \to x$ for some $x \in X$. Since $x_n^* \in J_p(x_n)$ it follows that $x^* \in J_p(x)$. Thus $J_p(X) = \overline{J_p(X)} = X^*$. Consequently, for every $x^* \in X^*\backslash\{0\}$ there exists $x \in X\backslash\{0\}$ such that $x^* \in J_p(x)$. This implies that

$$\left\langle x^*, \frac{x}{\|x\|}\right\rangle = \frac{\|x\|^p}{\|x\|} = \|x\|^{p-1} = \|x^*\|.$$

By Lemma 3.2 we conclude that $X$ is reflexive. In this case we get $x^* \in J_p(x)$ if and only if $x \in J_q^*(x^*)$, where $q > 0$ satisfies the relation $\frac{1}{p} + \frac{1}{q} = 1$ and $J_q^*$ denotes the normalized mapping of $X^*$. Hence $J_p^{-1} = J_q^*$. Fix any $\varrho > 0$. Fix any $x^*,\ y^* \in \overline{B}^*(0,\varrho)$, $x \in J_q^*(x^*)$, $y \in J_q^*(y^*)$. Choose $\gamma$ so that $\gamma^{p-1} = \varrho$. Then $x^* \in J_p(x)$, $y^* \in J_p(y)$, and $x, y \in \overline{B}(0,\gamma)$. By (3.2),

$$\omega_{p,\gamma}(\|x - y\|) \le \|x^* - y^*\|.$$

This implies that the map $J_q^*$ is single-valued and uniformly continuous on each bounded subset of $X^*$. By Theorem 2.1, $X = \left(X^*\right)^*$ is uniformly convex. The proof is complete. ∎

In the forthcoming section, Theorem 3.2 will serve us as a tool for studying the continuity of a Hölder type of the metric projection in uniformly Banach spaces.

## 4. Continuity of the metric projection onto a family of closed convex sets

From now on, $(\Lambda, d)$ is a metric space and $K : \Lambda \to 2^X$ is a set-valued map with nonempty closed convex values. Let $(\bar{\lambda}, \bar{x}) \in \Lambda \times X$ be a point satisfying $\bar{x} \in K(\bar{\lambda})$. For each $y \in X$, the projection of $y$ onto $K(\lambda)$ is denoted by $P_{K(\lambda)}(y)$.

**Definition 4.1.** A set-valued map $K : \Lambda \to 2^X$ is said to be pseudo-Lipschitz around $(\bar{\lambda}, \bar{x})$ if there exist a positive constant $l$ and neighborhoods $U$, $V$ of $\bar{x}$ and $\bar{\lambda}$, respectively, such that

$$K(\lambda) \cap U \subset K(\lambda') + l d(\lambda, \lambda') \overline{B}(0,1) \tag{4.1}$$

for all $\lambda$, $\lambda' \in U$.

**Theorem 4.1.** *Let $X$ be a uniformly convex Banach space, and $p \geq 2$. Let $K : \Lambda \to 2^X$ be a set-valued map, which is pseudo-Lipschitz around $(\bar{\lambda}, \bar{x})$. Then there exist positive constant $l_1$, a neighborhood $U_1$ of $\bar{x}$, a neighborhood $V_1$ of $\bar{\lambda}$ and a function $\omega_{p,\gamma} \in G$ such that*

$$\omega_{p,\gamma}(\|P_{K(\lambda)}(y) - P_{K(\lambda')}(y)\|)\|P_{K(\lambda)}(y) - P_{K(\lambda')}(y)\| \leq l_1 d(\lambda, \lambda')$$

*for all $\lambda, \lambda' \in V_1$ and $y \in U_1$.*

*Proof.* The proof runs similarly as the proof of Theorem 3.1 in [10], where $\omega_\gamma$ and $J$ are replaced by $\omega_{p,\gamma}$ and $J_p$, respectively. Instead of using Theorem 3.1, which is due to J. Prüß, we use Theorem 3.2 proved in the preceding section. □

From Theorem 3.2 it follows that the function $\omega_{p,\gamma}$ can be computed by the formula

$$\omega_{p,\gamma}(t) = \begin{cases} 0 & \text{for } t = 0, \\ \inf\Big\{ \dfrac{\langle x_1^* - x_2^*, x_1 - x_2 \rangle}{\|x_1 - x_2\|} : x_1, x_2 \in B(0,\gamma), & \\ \|x_1 - x_2\| \geq t, x_1^* \in J_p(x_1), x_2^* \in J_p(x_2) \Big\} & \text{for } t \in (0, 2\gamma], \\ \omega_{p,\gamma}(2\gamma) & \text{for } t > 2\gamma. \end{cases} \tag{4.2}$$

In many situations, in order to apply Theorem 4.1 it is enough to have a lower estimate for the function $\omega_{p,\gamma}$. Such an estimate in the case $X = L_p(\Omega, \mu)$, $1 < p \leq 2$, has been obtained in [10]. Here and in the sequel, $X = L_p(\Omega, \mu)$ denotes the space of all the measurable functions $x(\cdot)$ defined on an open set $\Omega \subset R^n$ for which $\int_\Omega |x(s)|^p d\mu < \infty$, where $\mu$ stands for the Lebesgue measure in $R^n$. It is well known that for every $1 < p < +\infty$, this space and its dual $X^* = L_q(\Omega, \mu)$, $q = \dfrac{p}{p-1}$, are uniformly convex Banach spaces. We now want to use the normalized duality mapping $J_p$ to obtain an estimate for $\omega_{p,\gamma}$ in the case $X = L_p(\Omega, \mu)$ with $p \geq 2$.

**Lemma 4.1.** ([13, Corollary 2.1]) *Suppose that $x, y \in L_p(\Omega, \mu)$ and $1 < p < +\infty$. Then for every $t \in (0,1)$ one has*

(a) $\|tx+(1-t)y\|^2 \le t\|x\|^2+(1-t)\|y\|^2-\frac{p-1}{64}t(1-t)\|x-y\|^2$ *for all* $1 < p < 2$.

(b) $\|tx + (1-t)y\|^p \le t\|x\|^p + (1-t)\|y\|^p - \frac{1}{p2^p}(t(1-t)^p + t^p(1-t))\|x-y\|^p$ *for all* $p \ge 2$.

**Theorem 4.2.** *Suppose that $X = L_p(\Omega, \mu), 1 < p < +\infty$. Then the following properties hold:*
(a) *If $1 < p \le 2$ then*

$$\langle J_p(x_1) - J_p(x_2), x_1 - x_2\rangle \ge \frac{p-1}{64}\|x_1 - x_2\|^2$$

*for all $x_1, x_2 \in X$. Besides, for each $\gamma > 0$ there is a function $\omega_{2,\gamma}$ such that $\omega_{2,\gamma}(t) \ge \frac{p-1}{64}t$ for all $t \in [0, 2\gamma]$.*
(b) *If $p \ge 2$ then*

$$\langle J_p(x_1) - J_p(x_2), x_1 - x_2\rangle \ge \frac{1}{p^2 2^{p-1}}\|x_1 - x_2\|^p$$

*for all $x_1, x_2 \in X$. Besides, for each $\gamma > 0$ there exists a function $\omega_{p,\gamma}$ such that $\omega_{p,\gamma}(t) \ge \frac{1}{p^2 2^{p-1}}t^{p-1}$ for every $t \in [0, 2\gamma]$.*

*Proof.* The proof of $(a)$ runs similarly as the proof of Proposition 4.1 in [10]. It remains to prove $(b)$. Let $\varphi_p(x) = \frac{1}{p}\|x\|^p$, $p \ge 2$ Since $X = L_p(\Omega, \mu)$ and its dual $X^* = L_q(\Omega, \mu)$, where $q = \frac{p}{p-1}$, are uniformly convex Banach spaces. By Theorem 2.1, we have $\partial\varphi_p(x) = \{J_p(x)\}$. By Lemma 4.1 (b),

$$\varphi_p(y + t(x-y)) - \varphi_p(y) \le t(\varphi_p(x) - \varphi_p(y)) - \frac{1}{p^2 2^p}(t(1-t)^p + t^p(1-t))\|x-y\|^p.$$

Hence

$$\frac{1}{t}(\varphi_p(y + t(x-y)) - \varphi_p(y)) \le \varphi_p(x) - \varphi_p(y) - \frac{1}{p^2 2^p}((1-t)^p + t^{p-1}(1-t))\|x-y\|^p.$$

Letting $t \to 0$ we obtain

$$\varphi_p'(y; x-y) \le \varphi_p(x) - \varphi_p(y) - \frac{1}{p^2 2^p}\|x-y\|^p$$

for all $x, y \in X$. Here $\varphi_p'(y; x-y)$ denotes the directional derivative of $\varphi_p$ at $y$ in direction $x-y$. Since $\langle J_p(y), x-y\rangle = \varphi_p'(y; x-y)$, we have

$$\langle J_p(y), x-y\rangle \le \varphi_p(x) - \varphi_p(y) - \frac{1}{p^2 2^p}\|x-y\|^p$$

for all $x, y \in X$. Consequently, for all $x_1, x_2 \in X$ we have

$$\langle J_p(x_2), x_1 - x_2\rangle \leq \varphi_p(x_1) - \varphi_p(x_2) - \frac{1}{p^2 2^p}\|x_1 - x_2\|^p,$$
$$\langle J_p(x_1), x_2 - x_1\rangle \leq \varphi_p(x_2) - \varphi_p(x_1) - \frac{1}{p^2 2^p}\|x_2 - x_1\|^p.$$

Combining these inequalities we obtain

$$\langle J_p(x_2) - J_p(x_1), x_2 - x_1\rangle \geq \frac{2}{p^2 2^p}\|x_1 - x_2\|^p.$$

For each $\gamma > 0$ and $t \in (0, 2\gamma)$, using (4.2) we have

$$\begin{aligned} \omega_{p,\gamma}(t) &= \inf\Big\{\frac{\langle x^* - y^*, x - y\rangle}{\|x - y\|}\big| x, y \in \overline{B}(0,\gamma), \|x - y\| \geq t, \\ &\qquad\qquad x^* \in J_p(x), y^* \in J_p(y)\Big\} \\ &\geq \inf\Big\{\frac{2}{p^2 2^p}\|x - y\|^{p-1}\big| x_1, x_2 \in \overline{B}(0,\gamma), \|x - y\| \geq t\Big\} \\ &\geq \frac{2}{p^2 2^p} t^{p-1}. \end{aligned}$$

The proof of property (b) is complete. □

The following theorem describes a Hölder continuity property of the metric projection onto a family of closed convex sets in the space $L_p(\Omega, \mu)$. This result shows that Lemma 1.1 from [15], which was obtained for the case of the metric projection onto closed convex sets in Hilbert spaces, can be extended to the case of the functions spaces $L_p(\Omega, \mu)$, $1 < p < +\infty$.

**Theorem 4.3.** *Suppose that $X = L_p(\Omega, \mu)$, $1 < p < +\infty$, and $K : \Lambda \to 2^X$ is a set-valued map, which is pseudo-Lipschitz around $(\bar{\lambda}, \bar{x})$. Then the following assertions hold:*

(a) *If $1 < p \leq 2$ then there exist a positive constant $l_0$, a neighborhood $U_1$ of $\bar{x}$, a neighborhood $V_1$ of $\bar{\lambda}$ such that*

$$\|P_{K(\lambda)}(y) - P_{K(\lambda')}(y)\| \leq l_0 d(\lambda, \lambda')^{\frac{1}{2}}$$

*for all $\lambda, \lambda' \in V_1$ and $y \in U_1$.*

(b) *If $p \geq 2$ then there exist a positive constant $l_1$, a neighborhood $U_1'$ of $\bar{x}$, a neighborhood $V_1'$ of $\bar{\lambda}$ such that*

$$\|P_{K(\lambda)}(y) - P_{K(\lambda')}(y)\| \leq l_1 d(\lambda, \lambda')^{\frac{1}{p}}$$

*for all $\lambda, \lambda' \in V_1'$ and $y \in U_1'$.*

*Proof.* Assertion $(a)$ is just the content of Proposition 4.2 in [10], so it remains to prove $(b)$. By Theorems 4.1 and 4.2, there exist a constant $\bar{l} > 0$ and neighborhoods $U_1'$ and $V_1'$ of $\bar{x}$ and $\bar{\lambda}$, respectively, such that

$$\frac{1}{p^2 2^{p-1}} \|P_{K(\lambda)}(y) - P_{K(\lambda')}(y)\|^p \leq \bar{l} d(\lambda, \lambda')$$

for all $\lambda, \lambda' \in V_1'$ and $y \in U_1'$. Hence

$$\|P_{K(\lambda)}(y) - P_{K(\lambda')}(y)\| \leq (\bar{l} p^2 2^{p-1})^{\frac{1}{p}} d(\lambda, \lambda')^{\frac{1}{p}}.$$

Setting $l_1 = (\bar{l} p^2 2^{p-1})^{\frac{1}{p}}$ we obtain the desired conclusion. □

**Remark 4.1.** The referee of this paper observed that using Lemma 2.3 of a recent paper by D. N. Bessis, Yu. S. Ledyaev and R. B. Vinter ("Dualization of the Euler and Hamiltonian inclusions", Nonlinear Analysis Vol. 43, 2001, pp. 861–882) one can replace the pseudo-Lipschitz property in Theorems 4.1 and 4.3 by a Lipschitz property. The lemma is stated as follows:

*Suppose that $K : R^m \to 2^{R^n}$ is a set-valued map with convex values. Here the norms in $R^m$ and $R^n$ are not necessarily Euclidean. Let $K$ be pseudo-Lipschitz around a point $(\bar{\lambda}, \bar{x}) \in R^m \times R^n$ satisfying $\bar{x} \in K(\bar{\lambda})$, i.e. there exist $k > 0$, $\varepsilon_0 > 0$ and $\beta_0 > 0$ such that*

$$K(\lambda) \cap (\bar{x} + \varepsilon_0 \bar{B}_{R^n}) \subset K(\lambda') + k\|\lambda - \lambda'\| \bar{B}_{R^n}$$

*for all $\lambda$, $\lambda'$ in $\bar{\lambda} + \beta_0 B_{R^m}$, where $B_{R^m}$ and $\bar{B}_{R^n}$ denote the open unit ball in $R^m$ and the closed unit ball in $R^n$, respectively. Then for any $\varepsilon \in (0, \varepsilon_0]$ and for any $\beta \in \left(0, \dfrac{\varepsilon}{4k}\right)$, the set-valued map*

$$K_\varepsilon(\lambda) := K(\lambda) \cap (\bar{x} + \varepsilon \bar{B}_{R^n})$$

*is Lipschitz continuous with constant $5k$ on the ball $\bar{\lambda} + \beta B_{R^m}$, that is*

$$K_\varepsilon(\lambda) \subset K_\varepsilon(\lambda') + 5k\|\lambda - \lambda'\| \bar{B}_{R^n}$$

*for all $\lambda$, $\lambda'$ in $\bar{\lambda} + \beta B_{R^m}$."*

We don't know whether the statement is still valid if one replaces $R^m$ by a metric space $\Lambda$ and $R^n$ by a Banach space $X$. However, it is likely that the results in Theorems 4.1 and 4.3 can be established under a local Lipschitz property similar to the one stated in the above lemma.

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