ACTA MATHEMATICA VIETNAMICA
Volume 27, Number 1, 2002, pp. 41-52

AUTOMATA WITH A TIME-VARIANT STRUCTURE AND SUPPLY-DEMAND THEOREMS

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Abstract. We consider automata with a time-variant structure. In these automata not only the function of state transition may be time-variant, but the set of states itself may be also time-variant. We show that there are a lot of supply-demand theorems for the automata. Some applications of these theorems for different processing systems are investigated.

1. Introduction

A natural way to generalize the notion of a finite automaton is to allow the structure of the automaton to be time-variant. The automata with a time-variant structure have been investigated by some authors, for example, by Agasandjan and Salomaa for finite automata with a time-variant structure [1, 2, 3], by Turakainen for probabilistic automata with a time-variant structure controlled by finite automata [4], by P. D. Dieu and P. T. An for probabilistic automata with a time-variant structure [6, 7].

In this work a concept of automata with a time-variant structure in a rather general sense is developed. In these automata, not only the function of state transition and the set of final states may be time-variant, but the set of states itself may be also time-variant, the number of states may increase along with the time. An idea of this automata was appeared in [6]. The new model has more flexible possibilities in simulating processing systems such as adaptive and learning systems.

This paper is concentrated on the investigation of the capacity of automata with a time-variant structure and its special subclasses. In order to study the automata with a time-variant structure we propose a new tool: the supply-demand theorems. They describe the relation between state growth speed of an automaton (a supply) and (non-equivalent) word growth speed of the language which is accepted by this automaton (a demand). Applying the supply-demand theorems for different processing systems: finite automaton (FA), finite automaton with

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Received September 18, 2000.

1991 *Mathematics Subject Classification.* 68Q90.

Key words and phrases. Supply-demand theorem, automaton with a time-variant structure (TVA), automaton with a deterministic time-variant structure (DTVA), Petri net (PN), Petri net with a time-variant structure (TVPN).

This work was supported by the National Basic Research Program in Natural Sciences.

a time-variant structure (FTVA), $\varphi(t)$-automaton with a time-variant structure (φ-TVA), Petri net (PN), Petri net with a time-variant structure (TVPN), we get the necessary conditions for the classes of the languges accepted by these systems but on an united point of view.

The definitions of automaton with a time-variant structure and of acceptable language are introduced in Section 2. Section 3 deals with the notion of representative complexity of a language. Section 4 is devoted to supply-demand theorems of automata with a time-variant structure. Finally, in Section 5 some applications of these theorems for different processing systems are considered.

2. NOTATIONS AND DEFINITIONS

For a finite alphabet Σ, Σ^* (resp. Σ^r, $\Sigma^{\leq r}$) denotes the set of all words (resp. of all words of length r, of length at most r) on the alphabet Σ. The empty word is denoted by Λ. For any word $\omega \in \Sigma^*$, $l(\omega)$ denotes the length of ω. Every subset $L \subseteq \Sigma^*$ is called a language over the alphabet Σ. Let N be the set of all non-negative integers and $N^+ = N\backslash\{0\}$.

Definition 1. An *automaton with a time-variant structure* (abbreviated TVA) is given by a list

$$A = (I, s_0, S_t, \delta_t, F_t),$$

where

I is a non-empty finite alphabet of inputs;

$\forall t \in N$, S_t is a finite set of states at time t;

$s_0 \in S_0$, s_0 is the initial state;

$\forall t \in N$, $\delta_t : S_t \times I \to S_{t+1}$ is the function of state transition at time t;

$\forall t \in N$, $F_t \subseteq S_t$, F_t is a set of final states at time t.

We can extend the function $\delta_t : S_t \times I^* \to S$, where $S = \cup S_t$, $t \in N$, by induction as follows.

Let $s \in S_t$, $x \in I^*$, $a \in I$, then

$$\begin{cases} \delta_t(s, \Lambda) & = s, \\ \delta_t(s, xa) & = \delta_{t+l(x)}(\delta_t(s, x), a). \end{cases}$$

The *language acceptable by an automaton with a time-variant structure* A is the set

$$L(A) = \big\{x \in I^* \mid \delta_0(s_0, x) \in F_{l(x)}\big\}.$$

Now we consider some important special cases of TVAs.

Definition 2. Let $A = (I, s_0, S_t, \delta_t, F_t)$, be an automaton with a time-variant structure. If A has the following properties:

(1) The map $\delta = \cup\delta_t$, $t \in N$ is *deterministic*, i.e., if $s \in S_{t_1} \cap S_{t_2}$, $t_1 \neq t_2$, then $\delta_{t_1}(s, a) = \delta_{t_2}(s, a)$, $\forall a \in I$;

(2) $F_t = F$, $\forall t \in N$;

then A is said to be an *automaton with a deterministic time-variant structure* (abbreviated DTVA).

Definition 3. Let $A = (I, s_0, S_t, \delta_t, F_t)$, be an automaton with a time-variant structure. If A has the following properties $\forall t \in N$:

(1) $S_t = S$,

(2) $\delta_t = \delta$,

(3) $F_t = F$,

then A is called an *automaton with a non time-variant structure*, or shortly, a *finite automaton* (abbreviated FA) and A is also given by $A = (I, s_0, S, \delta, F)$.

3. Representative complexities of a language

Let $L \subseteq \Sigma^*$. We define three relations $E_r(\mathrm{mod}L)$ in Σ^r, $E_{\le r}(\mathrm{mod}L)$ in $\Sigma^{\le r}$ and $E_{<\infty}(\mathrm{mod}L)$ in Σ^* as follows:

$$x_1 E_r x_2(\mathrm{mod}L) \Leftrightarrow \forall \omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L, \quad \forall x_1, x_2 \in \Sigma^r.$$

$$x_1 E_{\le r} x_2(\mathrm{mod}L) \Leftrightarrow \forall \omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L, \quad \forall x_1, x_2 \in \Sigma^{\le r}.$$

$$x_1 E_{<\infty} x_2(\mathrm{mod}L) \Leftrightarrow \forall \omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L, \quad \forall x_1, x_2 \in \Sigma^*.$$

It is easy to show that the relations $E_r(\mathrm{mod}L)$, $E_{\le r}(\mathrm{mod}L)$, $E_{<\infty}(\mathrm{mod}L)$ are reflexive, symmetric and transitive. Therefore, they are equivalence relations.

We define

$$H_L(r) = \mathrm{Rank}\, E_r(\mathrm{mod}L),$$
$$G_L(r) = \mathrm{Rank}\, E_{\le r}(\mathrm{mod}L),$$
$$K_L = \mathrm{Rank}\, E_{<\infty}(\mathrm{mod}L).$$

They are considered to be the representative complexity characteristics of the language L over Σ^r, over $\Sigma^{\le r}$ and over Σ^*, respectively.

First we give some of their simple properties:

(1) $H_L(r) \le G_L(r) \le K_L, \quad \forall r \in N$,

(2) $1 \le H_L(r) \le G_L(r) \le Exp(r)$,

where $\mathrm{Exp}(r)$ denotes some exponential function of r.

Now we estimate $H_L(r)$, $G_L(r)$, K_L for some languages.

Example 1. Let $\Sigma = \{a, b\}$ and

$$L_1 = \{a^n b^n \mid n \in N^+\}.$$

Denote $W = \{a, a^2, \cdots, a^n, \cdots\}$. We have $W \subset \Sigma^*$, for any $C = \mathrm{const}$ we get $|W| > C$ and $a^i \overline{E}_{<\infty} a^j(\mathrm{mod}L_1)$ with $i \ne j$. Therefore $K_{L_1} \ge |W| > C$.

Example 2. Let $|\Sigma| = k \ge 2$, $c \notin \Sigma$ and

$$L_2 = \{xcx \mid x \in \Sigma^+\}.$$

It can verify that if $x_1, x_2 \in \Sigma^{\leq r}$, $x_1 \neq x_2$ then $x_1 \overline{E}_{\leq r} x_2 \,(\mathrm{mod}\, L_2)$. Therefore

$$G_{L_2}(r) = |\Sigma^{\leq r}| = \frac{k(k^r - 1)}{(k-1)} \cdot$$

Example 3. Let $\Sigma = \{0, 1\}$, $c \notin \Sigma$, $k \geq 2$ and

$$L_{3,k} = \{xcx \mid x \in \Sigma^* \text{ , } |x|_1 = k\},$$

where $|x|_1$ denotes the number of occurences of 1 in x.

Denote

$$W_{r,k} = \big\{x \mid x \in \Sigma^*; l(x) = r; |x|_1 = k\big\},$$

It is easy to verify that

$$|W_{r,k}| \,=\, C_r^k = \frac{r!}{k!(r-k)!} = P_k(r),$$

where $P_k(r)$ denotes a polynominal of degree k.

For any $x_1, x_2 \in W_{r,k}$ with $x_1 \neq x_2$, by choosing $\omega = cx_1$ we have $x_1\omega = x_1cx_1 \in L_{3,k}$ whereas $x_2\omega = x_2cx_1 \notin L_{3,k}$, that is $x_1 \overline{E}_r x_2 \,(\mathrm{mod}\, L_{3,k})$. This means that

$$H_{L_{3,k}}(r) \,\geq\, |W_{r,k}| \,=\, P_k(r).$$

4. Supply-demand theorems for tva

First we introduce the notion of growth functions of an automaton with a time-variant structure.

Definition 4. Let $A = (I, s_0, S_t, \delta_t, F_t)$ be an automaton with a time-variant structure. Set $S_{\leq r} = \cup S_t$, $t \leq r$. The *growth functions of* A are defined by:

$$\begin{aligned} h_A(r) &= |S_r|, \\ g_A(r) &= |S_{\leq r}|. \end{aligned}$$

In particular, if A is a finite automaton $A = (I, s_0, S, \delta, F)$, then the growth function of A is defined by:

$$k_A = |S| = \text{const.}$$

Remark that k_A is a constant. Nervetheless, we call it a function because in this case, $h_A(r) = g_A(r) = k_A$.

There are nice relations between the growth functions of an automaton with a time-variant structure and the representative complexities of the language which is accepted by this automaton. These relations are said to be the *supply-demand theorems.*

Theorem 1. (The supply-demand theorem for TVA). *Let A be an automaton with a time-variant structure and $L = L(A)$. Then for any $r \in N^+$,*

$$H_L(r) \quad \leq \quad h_A(r).$$

Proof. Let $A = (I, s_0, S_t, \delta_t, F_t)$ and $L = L(A)$. We shall prove that

$$H_L(r) \ \leq \ h_A(r) \quad \forall r \in N^+.$$

To prove this, we assume the contrary, i.e., $\exists r \in N^+ : H_L(r) > h_A(r)$. Therefore, there are $x, y \in I^r$ such that $x\overline{E}_r y \,(\operatorname{mod} L)$ and $\delta_0(s_0, x) = \delta_0(s_0, y)$. Since $l(x) = l(y) = r$, it follows that $\forall \omega \in I^*$:

$$\delta_{l(x)}(\delta_0(s_0, x), \omega) = \delta_{l(y)}(\delta_0(s_0, y), \omega),$$

and

$$F_{l(x)+l(\omega)} = F_{l(y)+l(\omega)}.$$

We obtain

$$x\omega \in L \leftrightarrow y\omega \in L.$$

It means that $xE_r y(\operatorname{mod} L)$. This conflicts with hypothesis $x\overline{E}_r y \,(\operatorname{mod} L)$. Therefore,

$$H_L(r) \ \leq \ h_A(r) \quad \forall r \in N^+.$$

□

Theorem 2. (The supply-demand theorem for DTVA). *Let A be an automaton with a deterministic time-variant structure and $L = L(A)$. Then for any $r \in N^+$,*

(1) $H_L(r) \ \leq \ h_A(r)$,

(2) $G_L(r) \ \leq \ g_A(r)$.

Proof. Since each DTVA is an TVA, (1) is immediate. Now we prove (2).

Let $A = (I, s_0, S_t, \delta_t, F_t)$ be an DTVA where $\delta_0 : S_0 \times I \to S_1$. We extend $\delta_0 : S_0 \times I^{\leq r} \to S_{\leq r}$ as follows:

$$\begin{cases} \delta_0(s, \Lambda) & = s, \quad \forall s \in S_0, \\ \delta_0(s, xa) & = \delta_{l(x)}(\delta_0(s, x), a), \quad \forall s \in S_0, \ \forall x \in I^{\leq r-1}, \ \forall a \in I. \end{cases}$$

Assume to the contrary that $\exists r \in N^+ : G_L(r) > g_A(r)$. Then there exist $x, y \in I^{\leq r}$ such that $x\overline{E}_{\leq r} y \,(\operatorname{mod} L)$ but $\delta_0(s_0, x) = \delta_0(s_0, y)$. Since A is deterministic, it follows that $\forall \omega \in I^*$:

$$\delta_{l(x)}(\delta_0(s_0, x), \omega) = \delta_{l(y)}(\delta_0(s_0, y), \omega),$$

$$\delta_{l(x)}(\delta_0(s_0, x), \omega) \in F \leftrightarrow \delta_{l(y)}(\delta_0(s_0, y), \omega) \in F,$$

$$x\omega \in L \leftrightarrow y\omega \in L.$$

It means that $xE_{\leq r} y \,(\operatorname{mod} L)$ which contradicts the hypothesis $x\overline{E}_{\leq r} y \,(\operatorname{mod} L)$. Therefore,

$$G_L(r) \ \leq \ g_A(r) \quad \forall r \in N^+.$$

□

Theorem 3. (The supply-demand theorem for FA). *Let A be an finite automaton and $L = L(A)$. Then for any $r \in N^+$,*

(1) $H_L(r) \;\le\; h_A(r), \quad \forall r \in N^+,$

(2) $G_L(r) \;\le\; g_A(r), \quad \forall r \in N^+,$

(3) $K_L \;\le\; k_A.$

Proof. Since each FA is an $DTVA$, (1) and (2) are obvious. Now we prove (3).

Let $A = (I, s_0, S, \delta, F)$ be an FA where $\delta : S \times I \to S$. We extend $\delta : S \times I^* \to S$ as follows:

$$\begin{cases} \delta(s, \Lambda) &= s, \quad \forall s \in S, \\ \delta(s, xa) &= \delta(\delta(s, x), a), \quad \forall s \in S,\ \forall x \in I^*,\ \forall a \in I. \end{cases}$$

Assume to the contrary that $K_L > k_A = |S|$. Then there exist $x, y \in I^*$ such that $x\overline{E}_{<\infty}y \,(\operatorname{mod} L)$ but $\delta(s_0, x) = \delta(s_0, y)$. It follows that $\forall \omega \in I^*$:

$$\delta(\delta(s_0, x), \omega) = \delta(\delta(s_0, y), \omega),$$
$$\delta(\delta(s_0, x), \omega) \in F \leftrightarrow \delta(\delta(s_0, y), \omega) \in F,$$
$$x\omega \in L \leftrightarrow y\omega \in L.$$

We obtain $xE_{<\infty}y \,(\operatorname{mod} L)$ which contradicts the initial hypothesis $x\overline{E}_{<\infty}y \,(\operatorname{mod} L)$. Therefore, $K_L \;\le\; k_A$. □

5. Somes applications of supply-demand theorems for different processing systems

We consider alternately the following processing systems:

Finite automaton (FA) (See Section 2)

The language acceptable by an FA is called an FA-language. The set of all FA-languages is denoted by $\mathcal{L}(FA)$.

Corollary 1. *Let $L \in \mathcal{L}(FA)$. Then there exists a constant C such that*

$$K_L \le C.$$

Proof. Let $L = L(A)$ where $A = (I, s_0, S, \delta, F)$ is an FA. In this case we have $k_A = |S| = C = \text{const}$. Applying Theorem 3, we obtain

$$K_L \le k_A = C.$$

It follows that $K_L \le C$. □

Example 4. Let $\Sigma = \{a, b\}$ and:

$$L_1 = \{a^n b^n \mid n \in N^+\}.$$

In Example 1, we have proved that $K_{L_1} > C$ for all $C = \text{const}$. According to Corollary 1, $L_1 \notin \mathcal{L}(FA)$.

Remark 1. Myhill had proved that the condition in Corollary 1 is also sufficient, i.e.,

$$L \in \mathcal{L}(FA) \leftrightarrow \exists C = \text{const} \ : \ K_L \leq C.$$

(See [15, 16]).

Finite automaton with a time-variant structure (FTVA) (See [1, 2, 3])

A *finite automaton with a time-variant structure* is an TVA $A = (I, s_0, S_t, \delta_t, F_t)$ with $\forall t \in N$, $S_t = S$, $|S| = C = \text{const}$.

The language acceptable by an FTVA is called an FTVA-language. The set of all FTVA-languages is denoted by $\mathcal{L}(FTVA)$.

Corollary 2. *Let $L \in \mathcal{L}(FTVA)$. Then there exists a contant C such that*

$$H_L(r) \leq C, \quad \forall r \in N^+.$$

Proof. Let $L = L(A)$ with $A = (I, s_0, S, \delta_t, F_t)$. Since $h_A(r) = |S_r| = |S| = C$, using Theorem 1, we have $\forall r \in N^+$

$$H_L(r) \leq h_A(r) = C.$$

It follows that $H_L(r) \leq C, \ \forall r \in N^+$. $\square$

Example 5. Let $|\Sigma| = k \geq 2$ and

$$L_5 = \left\{ xx^R \mid x \in \Sigma^* \right\},$$

where x^R is the inverse image of x. It is easy to show that if $x_1, x_2 \in \Sigma^r$, $x_1 \neq x_2$ then $x_1 \overline{E}_r x_2 \pmod{L_5}$. Therefore, $H_{L_5}(r) = |\Sigma^r| = k^r$. According to Corollary 2, it follows that $L_5 \notin \mathcal{L}(FTVA)$.

Remark 2. Agasandjan and Salomaa had proved that the condition in Corollary 2 is also sufficient, i.e.,

$$L \in \mathcal{L}(FTVA) \leftrightarrow H_L(r) \leq C, \quad \forall r \in N^+$$

(See [1, 3]).

φ-automaton with a time-variant structure ($\varphi - TVA$) (See [6])

Let $\varphi(t)$ be a function from N into N. An *φ-automaton with a time-variant structure* is an TVA $A = (I, s_0, S_t, \delta_t, F_t)$, with $|S_t| = \varphi(t)$, $\forall t \in N$.

The language acceptable by an φ-TVA is called an φ-TVA language. The set of all φ-TVA languages is denoted by $\mathcal{L}(\varphi - TVA)$.

Corollary 3. *Let $L \in \mathcal{L}(\varphi - TVA)$. Then*

$$H_L(r) \leq \varphi(r), \quad \forall r \in N^+.$$

Proof. Let $L = L(A)$ where $A = (I, s_0, S_t, \delta_t, F_t)$ with $|S_t| = \varphi(t)$. In this case, we have $h_A(r) = |S_r| = \varphi(r)$. According to Theorem 1, we obtain $\forall r \in N^+$

$$H_L(r) \leq h_A(r) = \varphi(r).$$

It follows that $H_L(r) \leq \varphi(r), \ \forall r \in N^+$. $\square$

Example 6. Let $\Sigma = \{0, 1\}$, $c \notin \Sigma$, $k \geq 2$ and

$$L_{3,k} = \{xcx \mid x \in \Sigma^* \ , \ |x|_1 = k\},$$

where $|x|_1$ denotes the number of occurences of 1 in x. In Example 3, we have proved that

$$H_{L_{3,k}}(r) \ \geq \ P_k(r),$$

where $P_k(r)$ is a polynominal of degree k. Now if we choose $\varphi(r)$ such that $\varphi(r) = O(P_{k-1}(r))$, then $\exists r \in N^+ : \ H_{L_{3,k}}(r) > \varphi(r)$. According to Corollary 3, we obtain $H_{L_{3,k}}(r) \notin \mathcal{L}(\varphi - TVA)$ with $\varphi(r) = O(P_{k-1}(r))$.

Remark 3. P. D. Dieu and P. T. An had proved that the condition in Corollary 3 is also sufficient, i.e.,

$$L \in \mathcal{L}(\varphi - TVA) \ \leftrightarrow \ H_L(r) \leq \varphi(r), \quad \forall r \in N^+.$$

(See [6]).

(Free-labeled) Petri net (PN). (See [8, 9, 12, 13, 14])

A *(free-labeled) Petri net* $\mathcal{N}$ is given by a list

$$\mathcal{N} = (P, T, I, O, \mu_0, M_f),$$

where

$P = \{p_1, \ldots, p_n\}$ is a finite set of *places*;

$T = \{\tau_1, \ldots, \tau_m\}$ is a finite set of *transitions*, $P \cap T = \emptyset$;

$I : P \times T \to N$ is the *input function*;

$O : T \times P \to N$ is the *output function*;

$\mu_0 : P \to N$ is the *initial marking*;

$M_f = \{\mu_{f_1}, \ldots, \mu_{f_k}\}$ is a finite set of *final marking.*

A *marking* μ (global configuration) of a Petri net $\mathcal{N}$ is a function

$$\mu : P \to N.$$

The marking μ can also be defined as an n-vector $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_i = \mu(p_i)$ and $|P| = n$.

A transition $\tau \in T$ is said to be *firable at the marking* μ if

$$\forall p \in P : \mu(p) \geq I(p, \tau).$$

Let τ be firable at μ and if τ fires, then the Petri net $\mathcal{N}$ shall change its state from marking μ to a new marking μ' which is defined as follows:

$$\forall p \in P : \mu'(p) = \mu(p) - I(p, \tau) + O(\tau, p).$$

We set $\delta(\mu, \tau) = \mu'$ and the function δ is said to be the *function of state transition* of the net.

A *firing sequence* can be defined as a sequence of transitions such that the firing of each its prefix will be led to a marking at which the following transition will be firable. By $\mathcal{F}_{\mathcal{N}}$ we denote the set of all firing sequences of the net $\mathcal{N}$.

We extend the function δ for a firing sequence by induction as follows:

Let $x \in T^*, \tau \in T, \mu$ be a marking, at which $x\tau$ is a firing sequence, then

$$\begin{cases} \delta(\mu, \Lambda) & = \mu, \\ \delta(\mu, x\tau) & = \delta(\delta(\mu, x), \tau). \end{cases}$$

The *language acceptable* by the Petri net $\mathcal{N}$ is the set

$$L(\mathcal{N}) = \big\{ \in T^* \mid (x \in \mathcal{F}_{\mathcal{N}}) \wedge (\delta(\mu_0, x) \in M_f)\big\}.$$

The language acceptable by a Petri net is called an PN-language. The set of all PN-languages is denoted by $\mathcal{L}(PN)$.

Corollary 4. *Let $L \in \mathcal{L}(PN)$. Then, there exist $k \in N$ and a polynominal P_k of degree k such that*

$$G_L(r) \leq P_k(r), \quad \forall r \in N^+.$$

Proof. Let $L = L(\mathcal{N})$ with $\mathcal{N} = (P, T, I, O, \mu_0, M_f)$. We denote M_r (resp. $M_{\leq r}$) the set of all reachable markings of $\mathcal{N}$ by firing r transitions (resp. at most r transitions) and $k = \min\{|P|, |T|\}$. The following result had been established in [10]:

Let $\mathcal{N}$ be a Petri net. Then,there exists a polynominal P_k of degree k such that

$$|M_{\leq r}| \ \leq \ P_k(r), \quad \forall r \in N^+.$$

Now from the Petri net $\mathcal{N}$, we contruct an TVA A as follows:

$$A = (T, \mu_0, M_t, \delta_t, M_f),$$

with $\delta_t : M_t \times T \rightarrow M_{t+1}$ is the function of marking transition of $\mathcal{N}$ after firing t transitions. Now δ_t becomes the function of state transition of TVA A at the time t. Since $\delta = \cup\delta_t$, $t \in N$ is the function of marking transition of net $\mathcal{N}$, therefore, δ is deterministic. It follows that A is an DTVA.

It is easy to verify that $L = L(\mathcal{N}) = L(A)$, so $L \in \mathcal{L}(DTVA)$ and

$$g_A(r) = |M_{\leq r}| \leq P_k(r).$$

Applying Theorem 2 for DTVA A, we obtain

$$G_L(r) \leq g_A(r) \leq P_k(r) \quad \forall r \in N^+.$$

It follows that $G_L(r) \leq P_k(r) \ \ \forall r \in N^+$. □

Example 7. Let $|\Sigma| = k \geq 2$, $c \notin \Sigma$ and

$$L_2 = \{xcx \mid x \in \Sigma^+\}.$$

In Example 2, we have shown that

$$G_{L_2}(r) = |\Sigma^{\leq r}| = \frac{k(k^r - 1)}{(k-1)} \cdot$$

According to Corollary 4, it follows that $L_2 \notin \mathcal{L}(PN)$.

Remark 4. P. T. An and P. V. Thao had proved that the condition in Corollary 4 is not sufficient

We consider the following languages:

$$L' = \{a^n b^n \mid n > 1\},$$
$$L = (L')^+.$$

In [11] we have shown that $G_L(r) \le G_{L'}(r) \le P_5(r)$, but $L \notin \mathcal{L}(PN)$.

(Free-labeled) Petri net with a time-variant structure (TVPN)

In this part we introduce a notion of Petri net with a time-variant structure.

A *(free-labeled) Petri net with a time-variant structure* $\mathcal{N}$ is given by a list

$$\mathcal{N} = (P, T, I_t, O_t, \mu_0, M_{f,t}),$$

where

$P = \{p_1, \dots, p_n\}$ is a finite set of *places*;

$T = \{\tau_1, \dots, \tau_m\}$ is a finite set of *transitions*, $P \cap T = \emptyset$;

$\forall t \in N,\ I_t : P \times T \to N$ is the *input function at the time* t;

$\forall t \in N,\ O_t : T \times P \to N$ is the *output function at the time* t, with the condition

$$\sup_{t \in N} \left(\max_{i,j} \mid O_t(\tau_j, p_i) - I_t(p_i, \tau_j) \mid \right) \le l = \text{const},$$

where $0 \le i \le n;\ 0 \le j \le m.$

$\mu_0 : P \to N$ is the *initial marking*;

$\forall t \in N,\ M_{f,t}$ is a finite set of *final marking* at the time t.

For any $t \in N$ *a marking* μ_t of a Petri net $\mathcal{N}$ at the time t is a function

$$\mu_t : P \to N.$$

A transition $\tau \in T$ is said to be *firable at the marking* μ_t if

$$\forall p \in P : \mu_t(p) \ge I_t(p, \tau).$$

Let τ be firable at μ_t and if τ fires, then the Petri net $\mathcal{N}$ shall change its state from marking μ_t to a new marking μ_{t+1} which is defined as follows:

$$\forall p \in P : \mu_{t+1}(p) = \mu_t(p) - I_t(p, \tau) + O_t(\tau, p).$$

We set $\delta_t(\mu, \tau) = \mu_{t+1}$ and the function δ_t is said to be *function of state transition of the net at the time* t.

We extend the function δ_t for a firing sequence by induction as follows:

Let $x \in T^*$, $\tau \in T$, μ_t be a marking, at which $x\tau$ is a firing sequence, then

$$\begin{cases} \delta_t(\mu_t, \Lambda) & = \mu_t, \\ \delta_t(\mu_t, x\tau) & = \delta_{t+l(x)}(\delta_t(\mu_t, x), \tau). \end{cases}$$

The *language acceptable* by a Petri net with a time-variant structure $\mathcal{N}$ is the set:

$$L(\mathcal{N}) = \left\{ x \in T^* \mid (x \in \mathcal{F}_{\mathcal{N}}) \land (\delta_0(\mu_0, x) \in M_{f,l(x)}) \right\},$$

The language acceptable by an TVPN is called an TVPN-language. The set of all TVPN-languages is denoted by $\mathcal{L}(TVPN)$.

Corollary 5. *Let* $L \in \mathcal{L}(TVPN)$. *Then, there exist* $k \in N$ *and a polynominal* P_k *of degree* k *such that*

$$H_L(r) \leq P_k(r), \quad \forall r \in N^+.$$

Proof. Let $L = L(\mathcal{N})$ with $\mathcal{N} = (P, T, I_t, O_t, \mu_0, M_{f,t})$. We denote M_r the set of all reachable markings of $\mathcal{N}$ by firing r transitions and $k = \min\{|P|, |T|\}$. Similarly as in [10] we can prove the following: Let $\mathcal{N}$ be a Petri net with a time-variant structure. Then, there exists a polynominal P_k of degree k such that

$$|M_r| \leq P_k(r), \quad \forall r \in N^+.$$

From the Petri net $\mathcal{N}$, we contruct an TVA A as follows:

$$A = (T, \mu_0, M_t, \delta_t, M_{f,t}).$$

We remark that A is in general not deterministic.

It is easy to verify that $L = L(\mathcal{N}) = L(A)$, so $L \in \mathcal{L}(TVA)$ and $h_A(r) = |M_r| \leq P_k(r)$.

Applying Theorem 1 we obtain

$$H_L(r) \leq h_A(r) \leq P_k(r), \quad \forall r \in N^+.$$

Therefore, $H_L(r) \leq P_k(r)$, $\forall r \in N^+$. □

Example 8. Let $|\Sigma| = k \geq 2$ and:

$$L_5 = \{xx^R \mid x \in \Sigma^*\},$$

where x^R is the inverse image of x. In Example 5 we have shown that $H_{L_5}(r) = |\Sigma^r| = k^r$. According to Corollary 5, it follows that $L_5 \notin \mathcal{L}(TVPN)$.

Remark 5. It is an open problem whether the condition in Corollary 5 is sufficient or not. There are reasons to believe that the answer could be negative.

Acknowledgement

I would like to thank my colleagues at the seminar Mathematical Foundation of Computer Science of Hanoi Institute of Mathematics for useful discussions this work.

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