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ON THE CONTINUITY OF VECTOR CONVEX MULTIVALUED FUNCTIONS

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Abstract. The well-known Banach Steinhaus Theorem is extended to the case of convex and concave functions and its applications are shown to find necessary and sufficient conditions for the C-continuity of vector convex functions. Relations between upper and lower C-continuities are also obtained.

1. Introduction

Let X and Y be topological Hausdorff spaces and $f : X \to Y$ a given single valued function. As usually, we say that f is continuous at a point $x_0 \in X$ if for any open subset V in Y containing $f(x_0)$ there is an open subset U containing x_0 such that $f(U) \subset V$. In the case when $F : X \to 2^Y$ is a multivalued function (in this paper we also say that F is a multivalued mapping), one defines the continuity of F in the sense of Berge [4]: F is said to be upper semicontinuous at x_0 if for any open subset V with $F(x_0) \subset V$ one can find an open subset U of X containing x_0 such that $F(x) \subset V$ holds for all $x \in U$. And, F is said to be lower semicontinuous at x_0 if for any open subset V with $F(x_0) \cap V \neq \emptyset$ there is an open subset U containing x_0 with $F(x) \cap V \neq \emptyset$ for all $x \in U$.

In the case $Y = R$, the space of real numbers, and $f : X \to R$, one says that f is upper (lower) semicontinuous at x_0 if for any $\varepsilon > 0$ there is a neighborhood U of x_0 with $f(x) \leq f(x_0) + \varepsilon$ $(f(x) \geq f(x_0) - \varepsilon$, respectively) for all $x \in U$. These notions can be also formulated for vector (singlevalued and multivalued) mappings in the case when Y is a topological locally convex space with a cone C.

Convex functions have been studied for some time by Hölder [5], Jensen [6], Minkowski [8] and many others. They play very important roles in convex analysis, one of the most beautiful and most developed branches of mathematics, and are used much in optimization, operation research, economics, engineering, etc. Some nice properties of convex functions have been investigated in the books of Rockafellar [10], Aubin and Ekeland [1], Aubin and Frankowska [2]. These concepts of functions and their properties are also extended to vector (singlevalued and multivalued) mappings (see, for example, [7]) and they also play important

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role in the theory of vector optimization, vector equilibrium problems etc, (see, for example, [2], [7], [11]).

The purpose of this paper is to study some other interesting properties of lower (upper) C-convex, C-concave, lower (upper) C-continuous mappings and some relations between them. The paper is organized as follows. In Section 2 we introduce the notions of C-continuities, C-convexity of vector mappings. In Section 3 we extend the well-known Banach-Steinhaus Theorem [3] to the family of convex, lower semicontinuous (concave upper semicontinuous) functions. As a corollary we can show that if X is a barrel space and $f : X \to R$ is convex lower semicontinuous on some neighborhood U_0 of $x_0 \in X$ and $f(x) < +\infty$ for all $x \in X$, then f is continuous at x_0.

Section 4 is devoted to the C-continuities of vector multivalued mappings. We give necessary and sufficient conditions for the upper (lower) C-continuity, sufficient conditions for an upper (lower) C-convex and upper (lower) C- continuous mapping to become weak upper (lower) C-continuous. Further, we show some relations between the upper C-continuity and lower C-continuity of multivalued mappings.

2. Preliminaries

Let X be a topological locally convex space, $D \subset X$ be a convex set. By R we denote the space of real numbers with the usual topology and $\overline{R} = R \cup \{\pm\infty\}$. We recall the following definitions.

Definition 2.1. (a) A function $f : D \to \overline{R}$ is called a convex function if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

holds for all $x, y \in \operatorname{dom} f = \{x \in D \;/\; f(x) < +\infty\}$ and $\alpha \in [0,1]$.

(b) A function $f : D \to \overline{R}$ is called a concave function if $-f$ is convex.

Throughout this paper, without loss of generality, any neighborhood of the origin in a topological convex space is supposed to be convex open symmetric. We introduce the following definitions.

Definition 2.2. Let $\{f_\alpha,\ \alpha \in I\}$ be a family of functions on D, where I is a nonempty parameter set. We say that this family is upper equisemicontinuous at $x_0 \in D$ if for every $\varepsilon > 0$, there is a neighborhood U of x_0 in X such that

$$f_\alpha(x) \leq f_\alpha(x_0) + \varepsilon$$

for all $x \in U \cap D$ and $\alpha \in I$. Analogically, we say that this family is lower equisemicontinuous at $x_0 \in D$ if the family $\{-f_\alpha, \alpha \in I\}$ is upper equisemicontinuous at x_0.

Further, let Y be another topological locally convex space with a cone C and F a multivalued mapping from D to Y (denoted by $F : D \to 2^Y$) which means that $F(x)$ is a set in Y for each $x \in D$. We denote the set of all $x \in D$ such that $F(x) \neq \emptyset$ by $\operatorname{dom} F$.

Definition 2.3. (a) F is upper C-continuous (lower C-continuous) at $x_0 \in D$ if for each neighborhood V of the origin in Y, there is a neighborhood U of x_0 in X such that

$$F(x) \subset F(x_0) + V + C$$
$$(F(x_0) \subset F(x) + V - C, \text{ respectively})$$

holds for all $x \in U \cap \operatorname{dom} F$.

(b) F is C-continuous at x_0 if it is upper and lower C-continuous at that point; and F is upper (respectively, lower,...) C-continuous on D if it is upper (respectively, lower...) C-continuous at every point of D.

(c) We say that F is weak upper (lower) C-continuous at x_0 if the neighborhood U of x_0 as above is in the weak topology of X.

Proposition 2.1. (a) *If $F(x_0)$ is a compact set in Y, then F is upper C-continuous at x_0 if and only if for any open set G with $F(x_0) \subset G + C$ there is a neighborhood U of x_0 such that*

$$F(x) \subset G + C,$$

holds for all $x \in U$ dom F.

(b) *If $F(x_0)$ is a compact set in Y, then F is lower C-continuous at x_0 if and only if for any $y \in F(x_0)$ and neighborhood V of the origin in Y there is a neighborhood U of x_0 such that*

$$F(x) \cap (y + V + C) \neq \emptyset$$

holds for all $x \in U$ dom F.

It is also equivalent to: For any open set G with $F(x_0) \cap (G + C) \neq \emptyset$, there is a neighborhood U of x_0 such that

$$F(x) \cap (G + C) \neq \emptyset$$

holds for all $x \in U \cap \operatorname{dom} F$.

Proof. (a) Assume that F is upper C-continuous at x_0 and G is an open set with $F(x_0) \subset G + C$. Since $F(x_0)$ is a compact set, there exists a neighborhood V_0 of the origin in Y such that $F(x_0) + V_0 \subset G + C$. For a given neighborhood V of the origin in Y there is a neighborhood U of x_0 such that.

$$F(x) \subset F(x_0) + V_0 \cap V + C \quad \text{for all} \quad x \in U \cap \operatorname{dom} F.$$

It follows that

$$F(x) \subset G + C \quad \text{for all} \quad x \in U \cap \operatorname{dom} F.$$

Suppose now that for any open set G with $F(x_0) \subset G + C$ there is a neighborhood U of x_0 such that

$$F(x) \subset G + C \quad \text{for all} \quad x \in U \cap \operatorname{dom} F.$$

Let V be an arbitrary neighborhood of the origin in Y. It is clear that $G = F(x_0) + V$ is also a open set and $F(x_0) \subset G + C$. One can find a neighborhood U of x_0 such that

$$F(x) \subset G + C \quad \text{for all} \ \ x \in U \cap \ \operatorname{dom} F.$$

It follows that

$$F(x) \subset F(x_0) + V + C \quad \text{for all} \ \ x \in U \cap \ \operatorname{dom} F.$$

This means that F is upper C-continuous at x_0.

(b) Assume first that F is lower C-continuous at x_0. For given neighborhood V of the origin in Y one can find a neighborhood U of x_0 in X such that

$$F(x_0) \subset F(x) + V - C \quad \text{for all} \ \ x \in U \cap \ \operatorname{dom} F.$$

This implies that for any $y \in F(x_0)$ and neighborhood V of the origin in Y

$$F(x) \cap (y + V + C) \neq \emptyset \quad \text{for all} \ \ x \in U \cap \ \operatorname{dom} F.$$

Suppose now that for any $y \in F(x_0)$ and neighborhood V of the origin in Y there is a neighborhood U_y of x_0 such that

$$F(x) \cap (y + V + C) \neq \emptyset \quad \text{for all} \ \ x \in U_y \cap \ \operatorname{dom} F.$$

It is clear that

$$F(x_0) \subset \bigcup \Big\{ y + \frac{V}{2} | \ \ y \in F(x_0) \Big\}.$$

Since $F(x_0)$ is compact, we conclude that $F(x_0) \subset \bigcup\limits_{i=1}^{n} \{y_i + \frac{V}{2}\}$ for some $y_1, \ldots, y_n \in F(x_0)$. Therefore, one can find neighborhoods U_{y_i} of x_0, $i = 1, \ldots, n$, such that

$$F(x) \cap \Big(y_i + \frac{V}{2} + C \Big) \neq \emptyset \quad \text{for all} \ \ x \in U_{y_i} \cap \ \operatorname{dom} F.$$

Putting $U = \bigcap\limits_{i=1}^{n} U_{y_i}$, we claim that

$$F(x_0) \subset F(x) + V - C \quad \text{for all} \ \ x \in U \cap \ \operatorname{dom} F.$$

Indeed, let $y \in F(x_0)$. We have $y \in y_i + \dfrac{V}{2}$ for some $i = 1, 2, ..., n$ and

$$F(x) \cap \Big(y_i + \frac{V}{2} + C \Big) \neq \emptyset \quad \text{for all} \ \ x \in U.$$

It follows that

$$y \in F(x) + V - C,$$

and hence,

$$F(x_0) \subset F(x) + V - C \quad \text{for all} \ \ x \in U \cap \ \operatorname{dom} F.$$

This means that F is lower C-continuous at x_0.

Now, let G be an open set with $F(x_0) \cap (G + C) \neq \emptyset$. Take $y \in F(x_0) \cap (G + C)$, $y = y_1 + C$ with $y_1 \in G$ and $c \in C$, we conclude that there is a neighborhood V

of the origin in Y such that $y \in y_1 + c + V \subset G + C$. Therefore, there exists a neighborbood U of x_0 such that

$$F(x) \cap (y + V + C) \neq \emptyset \quad \text{for all} \;\; x \in U \cap \; \text{dom}\, F.$$

Consequently,

$$F(x) \cap (G + C) \neq \emptyset \quad \text{for all} \;\; x \in U \cap \; \text{dom}\, F.$$

Let $y \in F(x_0)$ and V be a neighborhood of the origin in Y. Then

$$F(x_0) \cap (y + V + C) \neq \emptyset$$

with $y + V$ open. Hence, there exists a neighborhood U of x_0 in Y such that.

$$F(x) \cap (y + V + C) \neq \emptyset \quad \text{for all} \;\; x \in U \cap \; \text{dom}\, F.$$

This completes the proof. □

Remark 1. (a) If $C = \{0\}$ and $F(x_0)$ is compact, the upper $\{0\}$-continuity and the lower $\{0\}$-continuity of F at x_0 in Definition 2.3 coincide with the ones introduced by Berge in [4]. Moreover, if F is upper $\{0\}$-continuous and lower $\{0\}$-continuous at x_0 simultaneously, then it is continuous in the Hausdorff distance at x_0 provided that Y is a norm space.

(b) If F is single-valued, then the upper C-continuity and the lower C- continuity of F at x_0 coincide and we say that F is C-continuous at x_0.

(c) If $Y = R$ and $C = R_+ = \{x \in R \;/\; x \geq 0\}$ (or $C = R_- = \{x \in R \;/ x \;\leq 0\}$ and F is C-continuous at x_0, then F is lower semicontinuous (upper semicontinuous, respectively) at x_0 in the usual sense.

Definition 2.4. (a) F is said to be upper (lower) C-convex if

$$\alpha F(x) + (1-\alpha)F(y) \subset F(\alpha x + (1-\alpha)y) + C$$
$$(F(\alpha x + (1-\alpha)y) \subset \alpha F(x) + (1-\alpha)F(y) - C, \text{ respectively})$$

holds for all $x, y \in$ dom F and $\alpha \in [0, 1]$.

(b) F is said to be upper (lower) C-concave if $-F$ is upper(lower, respectively) C-convex.

Remark 2. (a) If $C = \{0\}$, then the lower $\{0\}$-convexity and the lower $\{0\}$-concavity (the upper $\{0\}$-convexity and the upper $\{0\}$- concavity) of F coincide and F is said to be lower sublinear (upper sublinear, respectively).

(b) If F is single-valued, then the lower C-convexity and the upper C- convexity (the lower C-concavity and the upper C-concavity) of F coincide and it is said to be C-convex (C-concave, respectively).

Let Y' denote the topological dual space of Y and

$$C' = \big\{\xi \in Y' | \langle \xi, y\rangle \geq 0, \text{ for all } y \in C\big\}.$$

It is called the polar cone of the cone C. For given $F : D \to 2^Y$ and $\xi \in C'$ we define functions $g_\xi,\ G_\xi : D \to \overline{R}$ by

$$g_\xi(x) = \inf_{y \in F(x)} \langle \xi, y \rangle \ ,\ x \in D$$

and

$$G_\xi(x) = \sup_{y \in F(x)} \langle \xi, y \rangle \ ,\ x \in D.$$

We have

Proposition 2.2. (a) *If F is an upper (a lower) C-convex mapping, then the function g_ξ (G_ξ, respectively) is convex.*

(b) *If F is an upper (a lower) C-concave mapping then the function G_ξ (g_ξ, respectively) is concave.*

Proof. The proofs of these assertions follow immediately from the definitions of the functions g_ξ, G_ξ and the upper, lower C-convexities of F. □

In the following proposition we assume that Y is a Banach space.

Proposition 2.3. (a) *If F is upper (lower) C-continuous at $x_0 \in domF$, then g_ξ (G_ξ, respectively) is lower semicontinuous at x_0.*

(b) *If F is upper (lower) $(-C)$-continuous at $x_0 \in dom\ F$, then g_ξ (G_ξ, respectively) is upper semicontinuous at x_0.*

Proof. We only prove the lower semicontinuity of g_ξ in the part a). (the proof of the other assertions proceeds similarly). Let $\varepsilon > 0$ be given. Since $\xi \in C'$, there is a neighborhood V of the origin in Y such that $\xi(V) \subset (-\varepsilon, \varepsilon)$. For F is upper C-continuous at x_0, it follows that there is a neighborhood U of x_0 in X such that

$$F(x) \subset F(x_0) + V + C \quad \text{for all } x \in U \cap D.$$

This implies

$$g_\xi(x) = \inf_{y \in F(x)} \langle \xi, y \rangle \ \geq \ \inf_{y \in F(x_0)} \langle \xi, y \rangle - \varepsilon = g_\xi(x_0) - \varepsilon$$

and hence, g_ξ is lower semicontinuous at x_0.

This completes the proof of the proposition. □

3. The equisemicontinuity of convex and concave functions

In this section we prove some theorems on the equisemicontinuities of a families of functions. We recall that a barrel space is a topological locally convex space, in which any nonempty closed symmetric, convex and absorbing set is a neighborhood of the origin (see, for example [10]). We extend the well-known Banach-Steinhaus Theorem to families of convex and concave functions by the following theorems:

Theorem 3.1. *Assume that X is a barrel space, I is an index set and $f_\alpha : X \to \overline{R}$, $\alpha \in I$, is convex and lower semicontinuous on some neighborhood U_0 of $x_0 \in X$. In addition, suppose that for any $x \in X$ there is a constant $\gamma > 0$ such that $f_\alpha(x) \leq \gamma$, for all $\alpha \in I$. Then the family $\{f_\alpha,\ \alpha \in I\}$ is upper equisemicontinuous at x_0.*

Proof. By setting $\bar{f}_\alpha(x) = f_\alpha(x + x_0) - f_\alpha(x_0)$ if necessary, we may assume that $x_0 = 0$ and $f_\alpha(0) = 0$ for all $\alpha \in I$. For given $\varepsilon > 0$ we put

$$A_\alpha = \big\{x \in X \mid f_\alpha(x) \leq \varepsilon\big\}.$$

For $0 \in A_\alpha$ we conclude $A_\alpha \neq \emptyset$. Without loss of generality we may assume that U_0 is a closed convex symmetric neighborhood of the origin in X. Since A_α is a level set of the convex lower semicontinuous f_α, then $U_0 \cap A_\alpha$ is a closed convex.

Further, we put $U = \bigcap\limits_{\alpha \in I} U_0 \cap A_\alpha \cap (-A_\alpha)$. It follows that U is a nonempty closed, symmetric and convex set. We claim that U is absorbing. Indeed, let $x \in X$. By the hypotheses of the theorem there is a constant $\gamma > 0$ such that

$$f_\alpha(x) \leq \gamma$$

and

$$f_\alpha(-x) \leq \gamma \quad \text{for all } \alpha \in I.$$

We may assume $\gamma > \varepsilon$. Since

$$\begin{aligned} f_\alpha\big(\frac{\varepsilon}{\gamma}x\big) &= f_\alpha\big(\frac{\varepsilon}{\gamma}x + (1 - \frac{\varepsilon}{\gamma})0\big) \\ &\leq \frac{\varepsilon}{\gamma} f_\alpha(x) + (1 - \frac{\varepsilon}{\gamma}) f(0) = \frac{\varepsilon}{\gamma} f(x) \leq \varepsilon. \end{aligned}$$

This shows $\frac{\varepsilon}{\gamma}x \in A_\alpha$. Since U_0 is absorbing, there is a constant $\rho > 0$ such that $\frac{x}{\rho}$, $-\frac{x}{\rho} \in U_0$. For $\gamma_0 = \max\{\gamma, \rho\}$, we conclude $\frac{\varepsilon}{\gamma_0}x \in A_\alpha \cap U_0$. By a similar argument one obtains $\frac{-\varepsilon}{\gamma_0}x \in A_\alpha \cap U_0$ for all $\alpha \in I$, and then $\frac{\varepsilon}{\gamma_0}x \in U$. It means that U is absorbing. Remarking that X is a barrel space, we deduce that U is a neighborhood of the origin in X. For $x \in U$ we have

$$f_\alpha(x) \leq \varepsilon = f_\alpha(0) + \varepsilon \quad \text{for all } \alpha \in I.$$

Consequently, the family $\{f_\alpha,\ \alpha \in I\}$ is upper equisemicontinuous at the origin.

This completes the proof of the theorem. □

Corollary 3.1. *Assume that X is a barrel space, $f : X \to \overline{R}$ is convex and lower semicontinuous on some neighborhood U_0 of x_0 dom $f = X$. Then f is continuous at x_0.*

Proof. It follows immediately from Theorem 2.1 with $I = \{1\}$. □

Theorem 3.2. *Assume that X is a barrel space, I is an index set and $f_\alpha : X \to \overline{R}$, $\alpha \in I$, is concave and upper semicontinuous on some neighborhood U_0 of $x_0 \in X$. In addition, suppose that for any $x \in X$ there is a constant $\gamma > 0$ such that $f_\alpha(x) \geq -\gamma$ for all $\alpha \in I$. Then the family $\{f_\alpha \mid \alpha \in I\}$ is lower equisemicontinuous at x_0.*

Proof. The proof follows immediately from Theorem 2.1 with f_α replaced by $-f_\alpha$. □

4. The continuity of vector multivalued mappings

Throughout this section we assume that X is a topological locally convex space and Y is a Banach space, $D \subset X$ is a nonempty closed convex set and $C \subset Y$ is a convex cone with the polar cone C'. For $\xi \in C'$ let g_ξ, G_ξ be defined as in Section 3.

Theorem 4.1. *Let $F : D \to 2^Y$ and $x_0 \in dom\, F$ with $F(x_0) + C$ convex. Then F is upper C-continuous at x_0 if and only is the family $\{g_\xi \mid \xi \in C',\ \|\xi\| = 1\}$ is lower equisemicontinuous at x_0.*

Proof. We first assume that F is upper C-continuous at x_0. Let $\varepsilon > 0$ be given. By Banach-Steinhaus Theorem the family $\{\xi \in C' \mid \|\xi\| = 1\}$ is equicontinuous. Therefore there is a neighborhood V of the origin in Y such that $\xi(y) \in (-\varepsilon, \varepsilon)$ holds for all $y \in V$ and $\xi \in C'$, $\|\xi\| = 1$. Without loss of generality, we may assume that V is bounded. From the upper C-continuity of F at x_0 there exists a neighborhood U of x_0 in X such that.

$$F(x) \subset F(x_0) + V + C \quad \text{for all } x \in U \cap D.$$

It follows that

$$\begin{aligned} g_\xi(x) = \inf_{y \in F(x)} \langle \xi, y \rangle &\geq \inf_{y \in F(x_0)} \langle \xi, y \rangle + \inf_{y \in V} \langle \xi, y \rangle + \inf_{y \in C} \langle \xi, y \rangle \\ &\geq \inf_{y \in F(x_0)} \langle \xi, y \rangle - \varepsilon \\ &= g_\xi(x_0) - \varepsilon \end{aligned}$$

holds for all $x \in U \cap D$ and $\xi \in C'$, $\|\xi\| = 1$. This means that the family $\{g_\xi \mid \xi \in C',\ \|\xi\| = 1\}$ is lower equisemicontinuous at x_0.

Now, assume that this family is lower equisemicontinuous at x_0 . But, F is not upper C-continuous at x_0. This implies that there exists a neighborhood V of the origin in Y such that one can find a net $\{x_\alpha\}$ in X with $\lim x_\alpha = x_0$ and

$$F(x_\alpha) \not\subseteq F(x_0) + V + C.$$

Then, we take $y_\alpha \in F(x_\alpha)$ with

$$y_\alpha \notin F(x_0) + V + C.$$

Since the set $\mathrm{cl}(F(x_0)+\frac{V}{2}+C)$ is closed convex, applying a separation theorem, one can find some ξ_α from the topological dual of Y with unit norm such that

$$\xi_\alpha(y_\alpha) < \xi_\alpha(y)$$

for all $y \in F(x_0)+\frac{V}{2}+C$. This clearly implies $\xi_\alpha \in C'$ for all α.

It is clear that $\inf_{y\in F(x_0)} \langle \xi_\alpha, y\rangle > -\infty$. Therefore, for arbitrary $\delta > 0$ there exist $\bar{y}_\alpha \in F(x_0),\ \bar{v}_\alpha \in \frac{V}{2}$ and $\bar{c}_\alpha \in C$ such that

$$\begin{aligned}
\langle \xi_\alpha, \bar{y}_\alpha\rangle &\le \inf_{y\in F(x_0)} \langle \xi_\alpha, y\rangle + \frac{\delta}{3}\\
\langle \xi_\alpha, \bar{v}_\alpha\rangle &\le \inf_{v\in \frac{V}{2}} \langle \xi_\alpha, v\rangle + \frac{\delta}{3}\\
\langle \xi_\alpha, \bar{c}_\alpha\rangle &\le \inf_{c\in C} \langle \xi_\alpha, c\rangle + \frac{\delta}{3}\cdot
\end{aligned}$$

Hence, for $z_\alpha = \bar{y}_\alpha + \bar{v}_\alpha + \bar{c}_\alpha \in F(x_0)+\frac{V}{2}+C$, we have

$$\xi_\alpha(y_\alpha) < \xi_\alpha(z_\alpha) \le \inf_{y\in F(x_0)} \langle \xi_a, y\rangle + \inf_{v\in \frac{V}{2}} \langle \xi_\alpha, v\rangle + \inf_{c\in C} \langle \xi_\alpha, c\rangle + \delta.$$

Consequently,

$$(1) \qquad g_{\xi_\alpha}(x_\alpha) < g_{\xi_\alpha}(x_0) + \inf_{v\in \frac{V}{2}} \langle \xi_\alpha, v\rangle + \delta.$$

Since the family $\{\xi_\alpha \mid \xi_\alpha \in C',\ \|\xi_\alpha\| = 1\}$ is equisemicontinuous, we conclude that

$$\sup_\alpha \inf_{v\in \frac{V}{2}} \langle \xi_\alpha, v\rangle = \delta_0 < 0.$$

Consequently, (1) implies

$$g_{\xi_\alpha}(x_\alpha) < g_{\xi_\alpha}(x_0) + \delta_0 + \delta, \quad \text{for all } \alpha.$$

Since δ is arbitrary, we conclude

$$g_{\xi_\alpha}(x_\alpha) \le g_{\xi_\alpha}(x_0) + \delta_0.$$

Taking $\varepsilon = -\frac{\delta_0}{2}$, we obtain

$$(1) \qquad g_{\xi_\alpha}(x_\alpha) < g_{\xi_\alpha}(x_0) - \varepsilon \quad \text{for all } \alpha.$$

It contradicts the lower equisemicontinuity of the family $\{g_\xi \mid \xi \in C',\ \|\xi\| = 1\}$. This completes the proof of the theorem. □

Theorem 4.2. *Let $F : D \to 2^Y$ be a multivalued mapping with $F(x) - C$ convex for all $x \in D$. Then F is lower C-continuous at x_0 if and only if the family $\{G_\xi \mid \xi \in C',\ \|\xi\| = 1\}$ is lower equisemicontinuous at x_0.*

Proof. The proof of this theorem proceeds exactly as the one of Theorem 4.1 with g_ξ, inf, $\geq$, $-\varepsilon$ replaced by G_ξ, sup, $\leq$ and $+\varepsilon$ everywhere. □

The following theorems can be also proved by the same arguments of the proofs of Theorems 4.1 and 4.2.

Theorem 4.3. *Let $F : D \to 2^Y$ and $x_0 \in dom\, F$ with $F(x_0) - C$ convex. Then F is upper $(-C)$-continuous at x_0 if and only if the family $\{G_\xi \mid \xi \in C',\ \|\xi\| = 1\}$ is upper equisemicontinuous at x_0.*

Theorem 4.4. *Let $F : D \to 2^Y$ be such that $F(x) + C$ is convex for all $x \in D$. Then F is lower $(-C)$-continuous at $x_0 \in dom\, F$ if and only if the family $\{g_\xi \mid \xi \in C',\ \|\xi\| = 1\}$ is upper equisemicontinuous at x_0.*

Next, we recall that a set $B \subset Y$ generates the cone C and write $C = cone(B)$ if $C = \{tb \mid b \in B, t \geq 0\}$. If in addition, B does not contain the origin and for each $c \in C$, $c \neq 0$, there are unique $b \in B, t > 0$ such that $c = tb$, then we say that B is a base of C. Moreover, if B is a polyhedron, i.e. $B = \text{conv}\{y_1, y_2, \ldots, y_n\}$ for some $y_1, y_2, \ldots, y_n \in Y$, we say that C is a polyhedral cone.

Theorem 4.5. *Let D, X, Y be as above and let C be a convex cone with C' a polyhedral cone. Assume that $F : D \to 2^Y$ is upper C-convex and upper C-continuous on $dom\, F$ with $F(x) + C$ convex for all $x \in D$. Then F is weak upper C-continuous on $dom\, F$.*

Proof. Assume that

$$C' = \text{cone } (\text{conv } \{\xi_1, \ldots, \xi_n\}).$$

It is clear that for $i = 1, \ldots, n$, g_{ξ_i} is a convex and lower semicontinuous from D to $\overline{R}$. Therefore, it is weak lower semicontinuous from D to $\overline{R}$.

Suppose, that $x_0 \in \text{dom}\, F$. We show that F is weak upper C-continuous at x_0. Indeed, for given $\varepsilon > 0$ and $i = 1, \ldots, n$, we can find a neighborhood U_i of x_0 in the weak topology of X such that

$$g_{\xi_i}(x) \geq g_{\xi_i}(x_0) - \beta_0 \varepsilon, \text{ for all } x \in U_i \cap D,$$

where $\beta_0 = \min \Big\{ \big\| \sum_{i=1}^n \lambda_i \xi_i \big\| \quad \big| \ \sum_{i=1}^n \lambda_i = 1 \Big\}$. Remarking that $0 \notin conv\{\xi_1, \ldots, \xi_n\}$ we conlude that $\beta_0 > 0$. Putting $U = \bigcap_{i=1}^n U_i$ we obtain

$$g_{\xi_i}(x) \geq g_{\xi_i}(x_0) - \beta_0 \varepsilon \ \text{ for all } x \in U \cap D \text{ and } \ i = 1, \ldots, n.$$

This shows that the family $\{g_{\xi_i} \mid i = 1, \ldots, n\}$ is weak lower equisemicontinuous at x_0. Now, we claim that

$$g_\xi(x) \geq g_\xi(x_0) - \varepsilon \ \text{ for all } x \in U \cap D \ \text{ and } \xi \in C',\ \|\xi\| = 1.$$

Indeed, for $\xi \in C'$, $\|\xi\| = 1$ we can write $\xi = \beta \sum_{i=1}^{n} \lambda_i \xi_i$ for some $\beta > 0$. We have

$$1 = \|\xi\| = \beta \Big\| \sum_{i=1}^{n} \lambda_i \xi_i \Big\|.$$

Therefore

$$\beta = \frac{1}{\Big\| \sum_{i=1}^{n} \lambda_i \xi_i \Big\|} \leq \frac{1}{\beta_0}$$

or, $\beta\beta_0 \leq 1$. Since

$$\begin{aligned} g_\xi(x) &= \inf_{y \in F(x)} \langle \xi, y \rangle \\ &= \inf_{y \in F(x)} \langle \beta \sum_{i=1}^{n} \lambda_i \xi_i, y \rangle \\ &= \beta \sum_{i=1}^{n} \lambda_i \inf_{y \in F(x)} \langle \xi_i, y \rangle \\ &\geq \beta \sum_{i=1}^{n} \lambda_i \Big(\inf_{y \in F(x_0)} \langle \xi_i, y \rangle - \beta_0 \varepsilon \Big) \\ &= \inf_{y \in F(x_0)} \langle \beta \sum_{i=1}^{n} \lambda_i \xi_i, y \rangle - \beta\beta_0 \varepsilon \\ &\geq g_\xi(x_0) - \varepsilon \quad \text{for all } x \in U \cap D,\ \xi \in C',\ \|\xi\| = 1. \end{aligned}$$

Consequently, the family $\{g_\xi \mid \xi \in C', \|\xi\| = 1\}$ is weak lower equisemicontinuous at x_0. Applying Theorem 4.1 we conclude that F is weak upper C-continuous at x_0. This completes the proof of the theorem. □

Similarly, we have

Theorem 4.6. *Let $F : D \to 2^Y$ be a lower $(-C)$-continuous and upper C-concave mapping with $F(x) + C$ convex for all $x \in D$. Then F is weak lower $(-C)$-continuous on $\mathrm{dom}\, F$.*

Theorem 4.7. *Let X and Y be barrel spaces and $F : X \to 2^Y$ be upper C-convex and upper C-continuous on some neighborhood U_0 of $x_0 \in \mathrm{dom}\, F$. In addition, assume that $F(x) + C$ is convex for all $x \in D$ and for any $x \in X$ and any bounded neighborhood V of the origin in Y there is a constant $\gamma > 0$ such that $F(x) \cap (\gamma V - C) \neq \emptyset$. Then F is lower $(-C)$-continuous at x_0.*

Proof. By part (a) of Propositions 2.2 and 2.3, for any $\xi \in C'$ $\|\xi\| = 1$, g_ξ is a convex lower semicontinuous function on the neighborhood U_0 of x_0. Since for any $x \in X$ and any bounded neighborhood V of the origin in Y there is a constant $\gamma > 0$ such that $F(x) \cap (\gamma V - C) \neq \emptyset$, we conclude that.

$$g_\xi(x) = \inf_{y\in F(x)}\langle \xi, y\rangle \ \leq \sup_{y\in\gamma V - C}\langle \xi, y\rangle \ \leq \gamma \sup_{y\in V}\langle \xi, y\rangle \ = K < +\infty$$

for all $\xi \in C'$, $\|\xi\| = 1$. Applying Theorem 3.1, we conclude that the family $\{g_\xi \mid \xi \in C',\ \|\xi\| = 1\}$ is upper equisemicontinuous at x_0. Then, from Theorem 4.4 it follows that F is lower $(-C)$-continuous at x_0. □

The proof of the following theorems proceeds similarly as the one of Theorem 4.7.

Theorem 4.8. *Let X and Y be barrel spaces and $F : X \to 2^Y$ be lower C-convex and lower C-continuous on some neighborhood U_0 of $x_0 \in dom F$. In addition, assume that $F(x) - C$ convex for all $x \in D$ and for any $x \in X$ and any bounded neighbborhood V of the origin in Y there is a constant $\gamma > 0$ such that $F(x) \subset \gamma V - C$. Then F is upper $(-C)$-continuous at x_0.*

Theorem 4.9. *Let X and Y be barrel spaces and $F : X \to 2^Y$ be upper C-concave and upper $(-C)$-continuous on some neighborbood U_0 of $x_0 \in dom F$. In addition, assume that $F(x) + C$ convex for all $x \in D$ and for any $x \in X$ and any bounded neighborhood V of the origin in Y there exists a constant $\gamma > 0$ such that $F(x) \cap (\gamma V + C) \neq \emptyset$. Then F is lower C-continuous at x_0.*

Theorem 4.10. *Let X and Y be barrel spaces and $F : X \to 2^Y$ be lower C-concave and lower $(-C)$-continuous on some neighborhood U_0 of $x_0 \in dom F$. In addition, assume that $F(x) - C$ is convex for all $x \in X$ and for any $x \in X$ and any bounded neighorhood U of the origin in Y there exists a constant $\gamma > 0$ such that*

$$F(x) \subset \gamma V + C.$$

Then F is upper C-continuous at x_0.

Corollary 4.1. *Let C have a closed convex bounded base and $f : X \to Y$ be a singlevalued C-convex and C-continuous on some neighborhood U_0 of $x_0 \in X$. In addition, assume that for any $x \in X$ and any neighborhood V of the origin in Y there is a constant $\gamma > 0$ such that $f(x) \in \gamma V - C$. Then f is continuous at x_0.*

Proof. Let W be a given neighborhood of the origin in Y. We claim that there is a neighborhood U of x_0 in X such that $f(x) \in f(x_0) + W$ holds for all $x \in U$. Indeed, applying Proposition 1.8 in [7] it follows that there exists another neighborhood V of the origin in Y such that.

$$(V + C) \cap (V - C) \subseteq W.$$

Since f is C-continuous at x_0, there exists a neighborhood U_1 of x_0 such that $f(x) \in f(x_0) + V + C$ holds for all $x \in U_1$. Using Theorem 4.7, we conclude that f is $(-C)$-continuous. Therefore, there is a neighborhood U_2 of x_0 such that

$$f(x_0) \in f(x) + V + C, \quad \text{for all } x \in U_2,$$

or

$$f(x) \in f(x_0) + V - C \quad \text{for all } x \in U_2.$$

Putting $U = U_1 \cap U_2$, we obtain for all $x \in U$

$$\begin{aligned} f(x) &\in (f(x_0) + V + C) \cap (f(x_0) + V - C) \\ &= f(x_0) + (V + C) \cap (V - C) \subset f(x_0) + W. \end{aligned}$$

□

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