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ON EXPLICIT VISCOSITY SOLUTIONS TO NONCONVEX-NONCONCAVE HAMILTON-JACOBI EQUATIONS

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Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

Abstract. We consider the Cauchy problem for Hamilton-Jacobi equations in the case where the Hamiltonian is supposed to be a sum of a convex and a concave function and to depend also on the unknown function. Hopf-Oleinik-Lax-type formulas for viscosity sub- and super-solutions are presented. A sharp estimate for the unique viscosity solution is established.

1. Introduction and main results

In this paper we study viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations of the form

$$u_t + H(u, D_x u) = 0 \quad \text{in} \quad (0,T)\times\mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^n, \tag{1.2}$$

where H, u_0 are continuous functions in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, respectively.

Barron, Jensen and Liu have recently developed the theory of quasiconvex duality aimed to solve some important problems in optimal control and Hamilton-Jacobi equations (see [1, 16, 17, 5] and the references therein). To do that they sucessefully used the level set technique. More precisely, assume that the Hamiltonian $H = H(\gamma, p), \gamma \in \mathbb{R}, p \in \mathbb{R}^n$, is nondecreasing in γ, convex and positively homogeneous of degree one in p, the viscosity solution v of problem (1.1)-(1.2) can be derived by

$$v(t,x) := \underset{y\in\mathbb{R}^n}{\overrightarrow{}\inf}\ \big\{H^{\#}\big(\frac{x-y}{t}\big) \vee u_0(y)\big\}, \tag{1.3}$$

where the quasiconvex dual $H^{\#}$ is defined by

$$H^{\#}(q) := \inf\{\gamma\in\mathbb{R} : H(\gamma,p) \ge (p,q), \forall p \in \mathbb{R}^n\},$$

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the notation $(.,.)$ stands for the ordinary scalar product on $\mathbb{R}^n$, and

$$a \vee b := \max\{a, b\}.$$

The formula (1.3) with bounded and Lipschitz continuous initial data u_0 was found by Barron, Jensen and Liu [6].

Bardi and Faggian [4] studied an interesting problem: what happens if the Hamiltonian $H = H(p)$, $p \in \mathbb{R}$ takes the form of being the sum of a convex and a concave function? Using the familiar notion of convex duality, they presented their Hopf-type estimates and formulas for the unique viscosity solution of the Cauchy problem. This research motivates us to expect an analogous result for Hamiltonians of the form $H = H(\gamma, p)$ using quasiconvex duality.

We consider here Problem (1.1)-(1.2) when the Hamiltonian $H(\gamma, p)$ is a nonconvex-nonconcave function in the variable p. A nonconvex-nonconcave function is meant to be the sum of a convex and a concave function. This kind of functions is known as d.c. functions and plays a very important role in global optimization (see Tuy [12]).

The following hypotheses are assumed in this note:

(**A**) The Hamiltonian $H(\gamma, p)$, $(\gamma, p) \in \mathbb{R} \times \mathbb{R}^n$, is a nonconvex-nonconcave function in p, i.e.,

$$H(\gamma, p) = H_1(\gamma, p) + H_2(\gamma, p), \quad (\gamma, p) \in \mathbb{R} \times \mathbb{R}^n,$$

where H_1, H_2 are continuous on $\mathbb{R}^{n+1}$, and for each fixed $\gamma \in \mathbb{R}$, $H_1(\gamma, p)$ is convex, $H_2(\gamma, p)$ is concave, $H_1(\gamma, p)$, $H_2(\gamma, p)$ are positively homogeneous of degree one in p; for each fixed $p \in \mathbb{R}^n$, $H_1(\gamma, p), H_2(\gamma, p)$ are nondecreasing in γ;

(**B**) The initial function u_0 is continuous in x.

The expected solutions of the problem (1.1)-(1.2) are:

$$(1.4) \qquad u_-(t, x) := \underset{z}{\overrightarrow{\sup}} \; \underset{y}{\overrightarrow{\inf}} \left\{ [H_1^{\#}(y) \vee u_0(x - t(y + z))] \wedge H_{2\#}(z) \right\}, \quad (t, x) \in (0, T) \times \mathbb{R}^n,$$

and

$$(1.5) \qquad u_+(t, x) := \underset{y}{\overrightarrow{\inf}} \; \underset{z}{\overrightarrow{\sup}} \left\{ H_1^{\#}(y) \vee [u_0(x - t(y + z)) \wedge H_{2\#}(z)] \right\}, \quad (t, x) \in (0, T) \times \mathbb{R}^n,$$

where the operations $\vee$, $^{\#}$ are defined as in (1.3) and the operations $\wedge$, $_{\#}$ act as

$$a \wedge b := \min\{a, b\}, \quad \text{and} \quad H_{\#}(q) = \sup\{\gamma \in \mathbb{R} : H(\gamma, p) \leq (p, q), \forall p \in \mathbb{R}^n\}.$$

We call (1.4) and (1.5) *Hopf-Oleinik-Lax-type formulas.* The following theorem is the main result of the paper.

Theorem 1.1. i) *The function* u_- *determined by* (1.4) *is a viscosity subsolution of the equation* (1.1) *and satisfies* (1.2), *i.e.,*

$$\lim_{\substack{\rightarrow \\ (t,x')\to(0,x)}} u_-(t,x') = u_0(x), \quad \forall x \in \mathbb{R}^n. \tag{1.6}$$

ii) *The function* u_+ *determined by* (1.5) *is a viscosity supersolution of the equation* (1.1) *and satisfies* (1.2), *i.e.,*

$$\lim_{\substack{\rightarrow \\ (t,x')\to(0,x)}} u_+(t,x') = u_0(x), \quad \forall x \in \mathbb{R}^n. \tag{1.7}$$

Relying on the results of Theorem 1.1, we can obtain the upper and lower bounds for the unique viscosity solution of the problem (1.1)-(1.2).

Corollary 1.1. *If, in addition,* $u_0 \in BUC(\mathbb{R}^n)$*, then Problem* (1.1)-(1.2) *admits a unique viscosity solution* u *in* $BUC([0,T] \times \mathbb{R}^n)$ *such that*

$$u_- \le u \le u_+, \quad \text{in} \quad [0,T] \times \mathbb{R}^n, \tag{1.8}$$

where u_- *and* u_+ *are defined in* (1.4) *and* (1.5) *respectively.*

Note that the two expressions in the brackets $\{.\}$ in (1.4) and (1.5) are, in general, not the same since the operations $\wedge$ and $\vee$ are not "commutative". However, for every fixed $(t,x) \in (0,T) \times \mathbb{R}^n$, the supremum in z and the infimum in y may be taken over convex sets in which these two expressions coincide. The min-max theorems then yield the coincidence of u_+ and u_- in many cases (see Tuy [12], for example). In these cases, the unique viscosity solution of Problem (1.1)-(1.2) can be easily computed.

By means of the above results, we can deduce several interesting conclusions: if $H_2 = 0$, then $u_+ = u_- = u$, u can be computed by the formula (1.3) and u is a viscosity solution for the initial data u_0, continuous in $\mathbb{R}^n$ (not necessarily bounded and Lipschitz continuous as in [6]). If $H_1 = 0$, then $u_- = u_+$ and we get a formula for viscosity solutions with a concave Hamiltonian. Actually, if $H_2 = 0$, then a direct calculation gives

$$H_{2\#}(z) = \begin{cases} +\infty & \text{if} \quad z = 0 \\ -\infty & \text{if} \quad z \neq 0. \end{cases}$$

The formulas (1.4) and (1.5) then yield

$$u(t,x) = u_-(t,x) = u_+(t,x), \forall (t,x) \in (0,T) \times \mathbb{R}^n.$$

We also note that the representation of generalized solutions of the Cauchy problem for some Hamilton-Jacobi equations with nonconvex-nonconcave initial data was obtained by Van, Hoang and Tsuji [13]. Barron, Jensen and Liu [7] presented their estimates for viscosity solutions of Problem (1.1)-(1.2) by a different method. As Bardi and Faggian [4], they relied on the Kruzkov double variables technique. As seen later on, we are to go directly from the formulas.

Finally, the reader is referred to [8, 2, 3, 10] for the general theory of viscosity solutions, to [1, 5-7, 11, 12, 16, 17] for the properties of convexity and quasiconvexity, and to [9, 4, 6, 7, 10, 13, 14, 15] for the Hopf-Oleinik-Lax-type formulas.

2. Proofs of Theorem 1.1 and Corollary 1.1

In order to prove Theorem 1.1, we need some properties of the quasiconvex duality [1, 16, 17, 5, 6]. Let a continuous function $H = H(\gamma, p)$, $(\gamma, p) \in \mathbb{R} \times \mathbb{R}^n$, be given.

Using the operations "$(.)^{\#}$", "$(.)_{\#}$", "$\wedge$" and "$\vee$" in Section 1, we set

$$H^{\#*}(\gamma, p) := \sup\{(p, q) : q \in \mathbb{R}^n, H^{\#}(q) \leq \gamma\}, \quad (\gamma, p) \in \mathbb{R} \times \mathbb{R}^n,$$

and

$$H_{\#*}(\gamma, p) := \inf\{(p, q) : q \in \mathbb{R}^n, H_{\#}(q) \geq \gamma\}, \quad (\gamma, p) \in \mathbb{R} \times \mathbb{R}^n.$$

Some basic features of this duality can be summarized in the following lemma.

Lemma 2.1. i) *Let H be nondecreasing in γ, convex and positively homogeneous of degree one in p. Then $H^{\#}$ is quasiconvex, lower semicontinuous and*

$$H^{\#}(z) \to +\infty \quad \text{as} \quad |z| \to \infty, \quad \text{and} \quad H^{\#*} = H.$$

Moreover, there exists $p^* \in \mathbb{R}^n$ such that

$$H^{\#}(p^*) = -\infty.$$

ii) *Let H be nondecreasing in γ, concave and positively homogeneous of degree one in p. Then $H_{\#}$ is quasiconcave, upper semicontinuous and*

$$H_{\#}(z) \to -\infty \quad \text{as} \quad |z| \to \infty, \quad \text{and} \quad H_{\#*} = H.$$

Moreover, there exists $q^* \in \mathbb{R}^n$ such that

$$H_{\#}(q^*) = +\infty.$$

Proof. i) The first assertion of i) was proved by Barron, Jensen and Liu [6]. Let us verify that there exists a $p^* \in \mathbb{R}^n$ such that $H^{\#}(p^*) = -\infty$. Assume the contrary, that

$$H^{\#}(z) > -\infty, \forall z \in \mathbb{R}^n.$$

Since $H^{\#} \to +\infty$ as $|z| \to \infty$, there exists $N > 0$ so that $H^{\#}(z) > 0$, for all $|z| > N$. Thus, we get

$$-\infty = \underset{z \in \mathbb{R}^n}{\to} \inf H^{\#}(z) = \underset{|z| \leq N}{\to} \inf H^{\#}(z). \tag{2.1}$$

Since $H^{\#}$ is lower semicontinuous, $H^{\#}(z) > -\infty, \forall z \in \mathbb{R}^n$, the function

$$h(z) := \min\{H^{\#}(z), 0\}, \quad z \in \mathbb{R}^n$$

is clearly finite and lower semicontinuous on $\mathbb{R}^n$. Hence,

$$\underset{|z| \leq N}{\to} \inf H^{\#}(z) \geq \underset{|z| \leq N}{\to} \inf h(z) := M > -\infty,$$

which contradicts (2.1). This contradiction proved the second part of i).

ii) Using $[-H(-\gamma,-p)]^{\#}(z) = -[H_{\#}(\gamma,p)](z)$, we symmetrically obtain ii). □

To investigate the functions u_-, u_+ we need two auxiliary functions determined by

$$v(t,x) := \underset{y\in\mathbb{R}^n}{\to} \inf \left\{H_1^{\#}\Big(\frac{x-y}{t}\Big) \vee u_0(y)\right\}, \quad (t,x)\in(0,T]\times\mathbb{R}^n, \tag{2.2}$$

$$w(t,x) := \underset{y\in\mathbb{R}^n}{\to} \sup \left\{H_{2\#}\Big(\frac{x-y}{t}\Big) \wedge u_0(y)\right\}, \quad (t,x)\in(0,T]\times\mathbb{R}^n. \tag{2.3}$$

The continuity of v, w can be verified by the following lemma.

Lemma 2.2. *The functions v, w are continuous on $[0,T]\times\mathbb{R}^n$ with*

$$v(0,x) := u_0(x), \qquad w(0,x) := u_0(x), \quad x\in\mathbb{R}^n.$$

Proof. We need only to the show that v is continuous on $[0,T]\times\mathbb{R}^n$. The argument for w would be similar.

It is convenient to rewrite the function v as

$$v(t,x) = \underset{z\in\mathbb{R}^n}{\to} \inf \left\{H_1^{\#}(z) \vee u_0(x-tz)\right\}, \quad \forall (t,x)\in(0,T]\times\mathbb{R}^n. \tag{2.4}$$

By virtue of Lemma 2.1 i), we can take a point $p^*\in\mathbb{R}^n$ such that $H_1^{\#}(p^*) = -\infty$ and keep it fixed. Let $r>0$ be arbitrarily chosen. Then for each $(t,x)\in(0,T]\times B(0;r)$,

$$v(t,x) \le H_1^{\#}(p^*)\vee u_0(x-tp^*) = u_0(x-tp^*) \le \underset{|y|\le r+T|p^*|}{\to} \max u_0(y) := K < +\infty.$$

Since $H_1^{\#}(z)\to+\infty$ as $|z|\to\infty$, there exists a constant $N>0$ such that

$$H_1^{\#}(z) > K, \quad \forall |z|\ge N.$$

Hence, the infimum in (2.4) has to be taken over the ball $\overline{B}(0;N)$ for all $(t,x)\in(0,T]\times B(0;r)$. Since the function $z\mapsto (H_1^{\#}(z)\vee u_0(x-tz))\wedge K$, $z\in\overline{B}(0;N)$ is finite (bounded) and lower semicontinuous on a compact set, it holds, for any $(t,x)\in(0,T]\times B(0;r)$,

$$\begin{aligned} v(t,x) &= \underset{|z|\le N}{\to} \inf \left\{H_1^{\#}(z)\vee u_0(x-tz)\right\} \wedge K \\ &= \underset{|z|\le N}{\to} \inf \left\{\big[H_1^{\#}(z)\vee u_0(x-tz)\big]\wedge K\right\} \\ &= \underset{|z|\le N}{\to} \min \left\{\big[H_1^{\#}(z)\vee u_0(x-tz)\big]\wedge K\right\} \\ &= \underset{|z|\le N}{\to} \min \{H_1^{\#}(z)\vee u_0(x-tz)\}. \end{aligned}$$

Thus, for every $(t,x)\in(0,T]\times B(0;r)$, the set

$$k(t,x) := \left\{y_0\in\mathbb{R}^n : H_1^{\#}(y_0)\vee u_0(x-ty_0) = \underset{z\in\mathbb{R}^n}{\to} \inf\{H_1^{\#}(z)\vee u_0(x-tz)\}\right\}$$

is not empty. Since r is arbitrary, we can extend the definition of $k(t,x)$ to the whole domain $(0,T]\times\mathbb{R}^n$. The above arguments show that

$$\|k(t,x)\| := \sup\{|y_0| : y_0 \in k(t,x)\} \le N, \quad (t,x)\in(0,T]\times B(0;r). \tag{2.5}$$

For any $(t,x),(t',x')\in(0,T]\times B(0;r)$, choosing $\xi\in k(t,x), |\xi|\le N$ (by virtue of (2.5)), we get

$$\begin{aligned} v(t',x')-v(t,x) &= \underset{z\in\mathbb{R}^n}{\rightarrow}\inf\ \Big\{H_1^{\#}(z)\vee u_0(x'-t'z)\Big\} - H_1^{\#}(\xi)\vee u_0(x-t\xi)\\ &\le H_1^{\#}(\xi)\vee u_0(x'-t'\xi) - H_1^{\#}(\xi)\vee u_0(x-t\xi)\\ &\le |u_0(x'-t'\xi)-u_0(x-t\xi)|. \end{aligned} \tag{2.6}$$

Exchanging (t,x) and (t',x'), we can select $\xi'\in k(t',x')$, $|\xi'|\le N$ so that

$$v(t,x)-v(t',x') \le |u_0(x'-t'\xi')-u_0(x-t\xi')|. \tag{2.7}$$

The estimates (2.6) and (2.7) yield

$$\underset{(t',x')\to(t,x)}{\rightarrow}\lim\ v(t',x') = v(t,x), \quad \forall (t,x)\in(0,T]\times B(x_0,r). \tag{2.1}$$

Since r is arbitrary, it follows that $u\in C((0,T]\times\mathbb{R}^n)$.

Next, let us verify that the function v is continuous until the boundary $\{0\}\times\mathbb{R}^n$, i.e.,

$$\underset{(t,x)\to(0,x_0)}{\rightarrow}\lim\ v(t,x) = u_0(x_0), \quad \forall x_0\in\mathbb{R}^n. \tag{2.8}$$

Indeed, by what was shown above one has, for some fixed $p^*\in\mathbb{R}^n$ at which $H_1^{\#}(p^*)=-\infty$,

$$v(t,x)\le H_1^{\#}(p^*)\vee u_0(x-tp^*) = u_0(x-tp^*), \quad \forall (t,x)\in(0,T]\times\mathbb{R}^n.$$

Consequently,

$$\underset{(t,x)\to(0,x_0)}{\rightarrow}\limsup v(t,x) \le \underset{(t,x)\to(0,x_0)}{\rightarrow}\lim u_0(x-tp^*) = u_0(x_0). \tag{2.9}$$

On the other hand, in view of (2.5) where $r>0$ is arbitrarily given, one has

$$v(t,x) = H_1^{\#}(\xi)\vee u_0(x-t\xi) \ge u_0(x-t\xi),$$

for every $(t,x)\in(0,T]\times B(0;r)$ with some fixed $\xi\in k(t,x)$, $|\xi|\le N$. Letting $(t,x)\to(0,x_0)$, we have

$$\underset{(t,x)\to(0,x_0)}{\rightarrow}\liminf\ v(t,x) \ge \underset{(t,x)\to(0,x_0)}{\rightarrow}\lim\ u_0(x-t\xi) = u_0(x_0). \tag{2.10}$$

The combination of (2.9) and (2.10) yields (2.8). The proof of Lemma 2.2 is complete. □

Proof of Theorem 1.1. i) First, we will show that u_- is continuous in $(0,T)\times\mathbb{R}^n$. Indeed, u_- can be rewritten as

$$u_-(t,x) = \underset{z}{\rightarrow}\sup\{v(t,x-tz)\wedge H_{2\#}(z)\},$$

where $v(t,x)$ is defined by (2.2). By virtue of Lemma 2.2, there is $q^* \in \mathbb{R}^n$, $H_{2\#}(q^*) = +\infty$. Hence, if $|x| \le M$ for some constant $M > 0$ then

$$\begin{aligned} u_-(t,x) &\ge v(t, x - tq^*) \wedge H_{2\#}(q^*) = v(t, x - tq^*) \\ &\ge_{s\in[0,T],|y|\le M+T|q^*|} \overset{\rightarrow}{} \min v(s,y) := K > -\infty. \end{aligned}$$

Also, there is $N > 0$ such that $H_{2\#}(z) < K \ \ \forall |z| > N$. Therefore,

$$u_-(t,x) = \underset{|z|\le N}{\rightarrow} \sup\{v(t, x - tz) \wedge H_{2\#}(z)\}, \quad \forall t \in [0,T], |x| \le M.$$

Since both v and $H_{2\#}$ are upper semicontinuous in the variable $z \in \mathbb{R}^n$, so is their minimum. Hence, the last expression becomes

$$u_-(t,x) = \underset{|z|\le N}{\rightarrow} \max\{v(t, x - tz) \wedge H_{2\#}(z)\}, \quad \forall t \in [0,T], |x| \le M. \tag{2.11}$$

By virtue of (2.11), let $|x| \le M, |x'| \le M$, and let, for some fixed $z_0 \in \mathbb{R}^n, |z_0| \le N$,

$$u_-(t,x) = v(t, x - tz_0) \wedge H_{2\#}(z_0).$$

Then

$$\begin{aligned} u_-(t,x) - u_-(t',x') &= v(t, x - tz_0) \wedge H_{2\#}(z_0) - \underset{|z|\le N}{\rightarrow} \max\{v(t', x' - t'z) \wedge H_{2\#}(z)\} \\ &\le v(t, x - tz_0) \wedge H_{2\#}(z_0) - v(t', x' - t'z_0) \wedge H_{2\#}(z_0) \\ &\le |v(t', x' - t'z_0) - v(t, x - tz_0)|. \end{aligned} \tag{2.12}$$

Interchanging (t', x') and (t,x) we get, for some $z_1, |z_1| \le N$,

$$u_-(t',x') - u_-(t,x) \le |v(t', x' - t'z_1) - v(t, x - tz_1)|. \tag{2.13}$$

The estimates (2.12), (2.13) and the continuity of v imply that u_- is continuous on $(0,T) \times \{x : |x| \le M\}$. Since M is arbitrarily chosen, the continuity in $(0,T) \times \mathbb{R}^n$ of u_- follows.

Next, we claim that for every $(t,x) \in (0,T) \times \mathbb{R}^n$, $0 < s < t$,

$$u_-(t,x) \le \underset{z}{\rightarrow} \inf\{H_1^{\#}\big(\frac{x - z}{t - s} - z_0\big) \vee u_-(s,z)\},$$

where $z_0 \in \mathbb{R}^n$ such that

$$u_-(t,x) = v(t, x - tz_0) \wedge H_{2\#}(z_0). \tag{2.14}$$

Actually, in view of (2.11), it holds

$$\begin{aligned} u_-(t,x) &= v(t, x - tz_0) \wedge H_{2\#}(z_0) \\ &\le \big[H_1^{\#}\big(\frac{x - y}{t} - z_0\big) \vee u_0(y)\big] \wedge H_{2\#}(z_0), \quad \forall y \in \mathbb{R}^n. \end{aligned}$$

Since $H_1^{\#}$ is quasiconvex, we have for each fixed $z \in \mathbb{R}^n$,

$$H_1^{\#}\big(\frac{x - y}{t} - z_0\big) \le H_1^{\#}\big(\frac{x - z}{t - s} - z_0\big) \vee H_1^{\#}\big(\frac{z - y}{s} - z_0\big), \quad \forall y \in \mathbb{R}^n.$$

Thus,

$$u_-(t,x) \le \big[H_1^{\#}\big(\frac{x-z}{t-s} - z_0\big) \vee H_1^{\#}\big(\frac{z-y}{s} - z_0\big) \vee u_0(y)\big] \wedge H_{2\#}(z_0), \quad \forall y \in \mathbb{R}^n.$$

By changing variable $p := (z-y)/s - z_0, \forall y \in \mathbb{R}^n$, we obtain from the last estimate

$$u_-(t,x) \le \big[H_1^{\#}\big(\frac{x-z}{t-s} - z_0\big) \vee \big(H_1^{\#}(p) \vee u_0(z - s(p+z_0))\big)\big] \wedge H_{2\#}(z_0), \quad \forall p \in \mathbb{R}^n.$$

Taking infimum in $p \in \mathbb{R}^n$ of both sides, we obtain

$$\begin{aligned} u_-(t,x) &\le \big[H_1^{\#}\big(\frac{x-z}{t-s} - z_0\big) \vee v(s, z - sz_0)\big] \wedge H_{2\#}(z_0) \\ &\le H_1^{\#}\big(\frac{x-z}{t-s} - z_0\big) \vee \big[v(s, z - sz_0) \wedge H_{2\#}(z_0)\big] \\ &\le H_1^{\#}\big(\frac{x-z}{t-s} - z_0\big) \vee u_-(s,z). \end{aligned}$$

Since z is arbitrary, the last inequality implies (2.14).

Next, the fact that u_- is a viscosity subsolution of the equation (1.1) will be proved as follows. Without loss of generality, we may assume that the maximum and the minimum in the definition of viscosity sub- and supersolutions are zero and global. Assume the contrary that u_- is not a viscosity subsolution. Then there exist a constant $\varepsilon_0 > 0$, and a point $(t_0, x_0) \in (0,T) \times \mathbb{R}^n$, a function $\varphi \in C^1$, such that $u_- - \varphi$ has zero as its maximum value at (t_0, x_0) and

$$\varphi_t(t_0,x_0) + H(u_-(t_0,x_0), D_x\varphi(t_0,x_0)) > \varepsilon_0.$$

Set $\gamma_0 := u_-(t_0,x_0)$. Since H is continuous, there exists a number $\delta > 0$, such that

$$\varphi_t(t_0,x_0) + H(\gamma_0 - \delta, D_x\varphi(t_0,x_0)) > \varepsilon_0.$$

Using $H_1^{\#*} = H_1$, $H_{2\#*} = H_2$ from Lemma 2.1, we have

$$\begin{aligned} \varphi_t(t_0,x_0) + &\underset{\{p: H_1^{\#}(p) \le \gamma_0 - \delta\}}{\overrightarrow{}} \sup(p, D_x\varphi(t_0,x_0)) \\ &+ \underset{\{q: H_{2\#}(q) \ge \gamma_0 - \delta\}}{\overrightarrow{}} \inf(q, D_x\varphi(t_0,x_0)) > \varepsilon_0. \end{aligned}$$

Thus there exists $p_0 \in \mathbb{R}^n$, with $H_1^{\#}(p_0) \le \gamma_0 - \delta$, such that

$$\varphi_t(t_0,x_0) + (p_0 + q, D_x\varphi(t_0,x_0)) > \varepsilon_0, \quad \forall q \in \mathbb{R}^n, H_{2\#}(q) \ge \gamma_0 - \delta. \tag{2.15}$$

On the other hand, let z_0 be selected and fixed at which the maximum in (2.11) corresponding to (t_0,x_0) is attained, i.e.,

$$\gamma_0 = u_-(t_0,x_0) = v(t_0, x_0 - t_0z_0) \wedge H_{2\#}(z_0) \le H_{2\#}(z_0).$$

By virtue of (2.14) for every $0 < s < t_0$, $\mu := t_0 - s > 0$,

$$\gamma_0 = u_-(t_0,x_0) \le \underset{z}{\overrightarrow{}} \inf\{H_1^{\#}\big(\frac{x_0 - z}{t_0 - s} - z_0\big) \vee u_-(s,z)\}.$$

Changing variable $p := (x_0 - z)/(t_0 - s) - z_0$, $\forall z \in \mathbb{R}^n$, and then replacing $s = t_0 - \mu$ in the right-hand side of the last inequality, we obtain

$$u_-(t_0, x_0) \leq \underset{p}{\rightarrow} \inf\{H_1^{\#}(p) \vee u_-(t_0 - \mu, x_0 - \mu(p + z_0))\}$$

$$\leq H_1^{\#}(p_0) \vee u_-(t_0 - \mu, x_0 - \mu(p_0 + z_0)). \tag{2.16}$$

Besides, since $u_-(t_0, x_0) - \delta \geq H_1^{\#}(p_0)$ and u_- is continuous in $(0, T) \times \mathbb{R}^n$, there exists $\mu_0 > 0$ such that

$$H_1^{\#}(p_0) < u_-(t_0 - \mu, x_0 - \mu(p_0 + z_0)), \quad 0 < \forall \mu < \mu_0.$$

This coupled with (2.16) gives

$$\varphi(t_0, x_0) = \gamma_0 \leq u_-(t_0 - \mu, x_0 - \mu(p_0 + z_0))$$
$$\leq \varphi(t_0 - \mu, x_0 - \mu(p_0 + z_0)), \quad 0 < \forall \mu < \mu_0.$$

Consequently,

$$\frac{\varphi(t_0 - \mu, x_0 - \mu(p_0 + z_0)) - \varphi(t_0, x_0)}{-\mu} \leq 0, \quad 0 < \forall \mu < \mu_0.$$

Letting $\mu \to 0$ in the last estimate, we see that

$$\varphi_t(t_0, x_0) + (p_0 + z_0, D_x \varphi(t_0, x_0)) \leq 0,$$

which contradicts (2.15) where z_0 plays the role of a $q \in \mathbb{R}^n$. This contradiction proves that u_- is a viscosity subsolution of the equation (1.1). It remains to prove (1.6). By Lemma 2.2, let $q^* \in \mathbb{R}^n$ be taken so that $H_{2\#}(q^*) = +\infty$. Then

$$u_-(t, x) \geq H_{2\#}(q^*) \wedge v(t, x - tq^*) = v(t, x - tq^*). \tag{2.17}$$

Besides, it follows from (2.11) that for every $|x| < M$, there exists $z_0 \in \mathbb{R}^n, |z_0| \leq N$, at which

$$u_-(t, x) = v(t, x - tz_0) \wedge H_{2\#}(z_0) \leq v(t, x - tz_0). \tag{2.18}$$

From (2.17) and (2.18), letting $t \to 0$ and using the continuity of v on $[0, T] \times \mathbb{R}^n$ with $v(0, x) = u_0(x)$, we obtain

$$u_-(0, x) = u_0(x), \quad |x| \leq M.$$

Since M is arbitrary, (1.6) follows. The part i) of Theorem 1.1 is thus completely proved.

ii) By a similar argument, we also get ii). Instead of (2.14), the following estimate is invoked

$$u_+(t, x) \geq \underset{z}{\rightarrow} \sup\left\{H_{2\#}\left(\frac{x - y}{t - s} - y_0\right) \wedge u_+(s, y)\right\},$$

where $y_0 \in \mathbb{R}^n$ is arbitrary so that

$$u_+(t, x) = w(t, x - ty_0) \vee H_1^{\#}(y_0).$$

□

Proof of Corollary 1.1. If $u_0 \in BUC(\mathbb{R}^n)$, then we can choose the constant N in (2.12) and (2.13) independent of $(t,x),(t',x') \in [0,T]\times\mathbb{R}^n$ so that these estimates still hold true. This implies that $u_-, u_+ \in BUC([0,T]\times\mathbb{R}^n)$. Hence, the conclusion follows from Theorem IV.1 of Barles [2]. □

Example 1. Consider the following Cauchy problem

$$u_t + |D_x u|\text{sh}u = 0 \quad \text{in} \quad (0,T)\times\mathbb{R}^n, \tag{2.19}$$

$$u(0,x) = u_0(x), \quad \text{in} \quad \mathbb{R}^n, \tag{2.20}$$

where sh$x$ is the *hyperbolic sine* function

$$\text{sh}x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}, \quad x \in \mathbb{R}.$$

The Hamiltonian $H(\gamma,p) = |p|\text{sh}\gamma$ can be written as

$$H = H_1 + H_2, \quad H_1(\gamma,p) := \frac{\mathrm{e}^\gamma|p|}{2}, \quad H_2(\gamma,p) := -\frac{\mathrm{e}^{-\gamma}|p|}{2}, \quad (\gamma,p)\in\mathbb{R}\times\mathbb{R}^n,$$

meetting the assumption (A). A direct calculation yields

$$H_1^{\#}(q) = \log 2|q|, \quad H_{2\#}(q) = -\log 2|q|, \quad q\in\mathbb{R}^n.$$

Hence, it is derived from the formulas (1.4) and (1.5) that

$$u_-(t,x) = \sup_{\vec{z}} \inf_{\vec{y}} \{[\log 2|y| \vee u_0(x - t(y+z))] \wedge (-\log 2|z|)\},$$

$$u_+(t,x) = \inf_{\vec{y}} \sup_{\vec{z}} \{\log 2|y| \vee [u_0(x - t(y+z)) \wedge (-\log 2|z|)]\},$$

$$(t,x)\in(0,T)\times\mathbb{R}^n.$$

Example 2. Let $f(x), x\in\mathbb{R}$, be an any continuous nondecreasing function. Our results can be applied to a Hamiltonian of the form

$$H(\gamma,p) := f(\gamma)|p|, \quad (\gamma,p)\in\mathbb{R}\times\mathbb{R}^n.$$

Actually, we need only to determine

$$H_1(\gamma,p) := \max\{f(\gamma),0\}|p|,$$

$$H_2(\gamma,p) := \min\{f(\gamma),0\}|p|, \quad (\gamma,p)\in\mathbb{R}\times\mathbb{R}^n.$$

Clearly, these functions satisfy the hypothesis (A).

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