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A COUNTEREXAMPLE ON THE CLOSEDNESS OF THE CONVEX HULL OF A CLOSED CONE IN R^n

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Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

Abstract. In this note, the following question, not discussed well in the literature, is considered:

Is the convex hull of A closed if A is a closed cone in R^n?

When $n = 1$ or 2, the answer to the above problem is affirmative. The main purpose of this brief note is to give a simple example showing that the answer is negative when $n \geq 3$.

A subset A of $\mathbb{R}^n$ is called a *cone* if $\lambda x \in A$ whenever $x \in A$ and $\lambda \geq 0$. Useful facts about cones can be found in [1]. We consider here the following question:

Is the convex hull of A closed if A is a closed cone in $\mathbb{R}^n$?

We will give an example to show that the answer to the question is negative when $n \geq 3$.

Let

$$A = \{(x, 0, 0)|x \in R\} \cup \{\alpha(x, \tanh x, 1)|\alpha \geq 0, x \in R\}$$

and

$$C(A) = \{(x, 0, 0)|x \in R\} \cup \{(x, y, z)|x \in R, -z < y < z,\ 0 < z\}.$$

Then A is a closed cone in R^3 but $co(A)$ is not closed where $co(A)$ denotes the convex hull of A. This statement will be verified by the following results.

Proposition 1. *A is a closed cone.*

Proof. It is clear that A is a cone. To show that A is closed, it suffices to prove that

$$cl(\{\alpha(x, \tanh x, 1)|\alpha \geq 0, x \in R\}) \subset A,$$

where $cl(K)$ denotes the closure of a set K in R^3. Let (x^*, y^*, z^*) be the limit of a sequence $\{\alpha_n(x_n, \tanh x_n, 1)\}$ in the set $\{\alpha(x, \tanh x, 1)|\alpha \geq 0, x \in R\}$, that is,

$$(x^*, y^*, z^*) = \lim_{n\to\infty} \alpha_n(x_n, \tanh x_n, 1).$$

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If $z^* = 0$, then $\alpha_n \to 0$ as $n \to \infty$ and so $y^* = 0$ because $|\tanh x_n| < 1$. Therefore, we have

$$\lim_{n\to\infty} \alpha_n(x_n, \tanh x_n, 1) = (x^*, 0, 0) \in A.$$

If $z^* \neq 0$, then $\alpha_n \to z^*$ as $n \to \infty$ and so

$$x_n \to \frac{x^*}{z^*} \quad \text{and} \quad \alpha_n \tanh x_n \to z^* \tanh \frac{x^*}{z^*} = y^* \quad \text{as} \quad n \to \infty.$$

Therfore, we have

$$(x^*, y^*, z^*) = z^* \Big(\frac{x^*}{z^*}, \tanh \frac{x^*}{z^*}, 1\Big) \in A$$

This completes the proof. $\square$

Proposition 2. *$C(A)$ is a convex set containing A but it is not closed.*

Proof. It is clear that $C(A)$ contains A and it is not closed. The convexity of $C(A)$ is clear from the following observations:

(i) If two points belong to $\{(x,0,0)|x \in R\}$, then the line segment connecting these given points is contained in $\{(x,0,0)|x \in R\}$.

(ii) If two points belong to $\{(x,y,z)|x \in R, -z < y < z,\ 0 < z\}$, then the line segment connecting these points is also contained in $\{(x,y,z)|x \in R, -z < y < z,\ 0 < z\}$.

(iii) If a point belongs to $\{(x,0,0)|x \in R\}$ and another point belongs to $\{(x,y,z)|x \in R, -z < y < z,\ 0 < z\}$, then the line segment connecting these points is also contained in $\{(x,y,z)|x \in R, -z < y < z,\ 0 < z\}$.

This completes the proof. $\square$

Proposition 3. *$co(A) = C(A)$.*

Proof. It is clear from Proposition 2 that $co(A) \subset C(A)$. It remains to prove that

$$C(A) \subset co(A).$$

It suffices to show that

$$\{(x,y,z)|x \in R, -z < y < z,\ 0 < z\} \subset co(A).$$

Let $(x,y,z) \in \{(x,y,z)|x \in R, -z < y < z,\ 0 < z\}$. We prove that there exists $x^* \in R$ such that

$$(x,y,z) \in \{\alpha(1,0,0) + \beta(x^*, \tanh x^*, 1)|\alpha \in R, \beta \geq 0\},$$

that is,

$$(x,y,z) \in co\Big(\{\alpha(1,0,0)|\alpha \in R\} \cup \{\beta(x^*, \tanh x^*, 1)|\beta \geq 0\}\Big).$$

Since $0 < z$, it is sufficient to check that

$$\Big(\frac{x}{z}, \frac{y}{z}, 1\Big) \in \{\alpha(1,0,0) + \beta(x^*, \tanh x^*, 1)|\alpha \in R, \beta \geq 0\}.$$

Since $-1 < \dfrac{y}{z} < 1$, there exists $x^* \in R$ such that $\tanh x^* = \dfrac{y}{z}$. And since

$$\langle \big(\frac{x}{z}, \frac{y}{z}, 1\big), (1,0,0) \times \big(x^*, \frac{y}{z}, 1\big)\rangle = \langle \big(\frac{x}{z}, \frac{y}{z}, 1\big), \big(0, -1, \frac{y}{z}\big)\rangle = 0,$$

(where $\langle\, ,\ \rangle$ and $\times$ denote the inner product and vector product in R^3, respectively) that is, $\left(\dfrac{x}{z}, \dfrac{y}{z}, 1\right), (1,0,0)$ and $\left(x^*, \dfrac{y}{z}, 1\right)$ are coplanar, we obtain

$$\left(\frac{x}{z}, \frac{y}{z}, 1\right) \in \{\alpha(1,0,0) + \beta(x^*, \tanh x^*, 1) | \alpha \in R, \beta \geq 0\},$$

which implies that

$$(x, y, z) \in co\Big(\{\alpha(1,0,0)|\alpha \in R\} \cup \{\beta(x^*, \tanh x^*, 1)|\beta \geq 0\}\Big) \subset co(A).$$

This completes the proof. $\square$

References

[1] R .T. Rockafellar, *Convex Analysis*, Princeton Mathematical Series vol 28, Princeton University Press, Princeton, NJ 1970

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