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# NECESSARY OPTIMALITY CONDITIONS IN PROBLEMS INVOLVING SET-VALUED MAPS WITH PARAMETERS

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*Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday*

Abstract. The Fritz John and Kuhn-Tucker necessary optimality conditions are proved for problem $\min F(x,u)$, s.t. $G(x,u) \subset -M$, $0 \in P(x,u)$ or s.t. $G(x,u) \cap (-M) \neq \emptyset$, $0 \in P(x,u)$, where $x$ is the state variable, $u$ is a parameter, $F$, $G$ and $P$ are multifunctions.

## 1. Introduction

Optimization problems with parameters, which appear not because of the problems being perturbed but as a second kind of variables and on which the imposed assumptions should differ from that imposed on the main variables are important and often met in applications. For instance, in control problems, the control variables should be considered separately from the state variables. Such parameterized optimization problems involving only single-valued functions are considered in [6], [8-12] with applications in deriving the Pontryagin maximum principle for control problems with state constraints.

On the other hand, optimization problems involving multi-valued functions become more and more the forms of the literature, see, e.g., [1-4], [13-21]. In [14], [15] the main results in [6], [11] are extended to multifunction optimization problems with an equality constraint still being single-valued. Such problems are of practical importance since in many situations the equality constraint represent equations, say differential equations, and initial conditions. However, in other cases, differential inclusions may replace the differential equations to describe the system under consideration. The aim of this note is to extend the results of [14], [15] to such cases. Namely, we consider the following problems.

Let $X$, $Y$, $Z$ and $W$ be Banach spaces, $Y$ and $Z$ being ordered by convex cones $K$ and $M$, respectively, containing the origin and with nonempty interiors. Let

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$U$ be a nonempty set. Let $F$, $G$ and $P$ be multifunctions of $X \times U$ into $Y$, $Z$ and $W$, respectively. Our problem is

$$\text{(P)} \qquad \begin{aligned} \min\ F(x,u),\\ G(x,u) \subset -M,\\ 0 \in P(x,u); \end{aligned}$$

or

$$(\tilde{P}) \qquad \begin{aligned} \min\ F(x,u),\\ G(x,u) \cap (-M) \neq \emptyset,\\ 0 \in P(x,u). \end{aligned}$$

Here "min" indicates a minimum or weak minimum. Recall that a multifunction $\mathcal{F} : X \rightsquigarrow Y$ is said to have a *(global) weak minimum* at $(x_0; f_0)$, where $f_0 \in \mathcal{F}(x_0)$, on a set $A \subset X$, if

$$\mathcal{F}(A) - f_0 \subset Y \setminus (-int\ K). \tag{1}$$

If there is a neighborhood $N$ of $x_0$ such that (1) holds with $\mathcal{F}(A)$ replaced by $\mathcal{F}(A \cap N)$, then $(x_0; f_0)$ is called a *local weak minimum* of $\mathcal{F}$. If in (1) $Y \setminus (-\text{int}\, K)$ is replaced by $Y \setminus ((-K) \setminus K)$, then we have the definition for *(global) minimum.*

Let $Z^\star$ be the topological dual to $Z$ and $M^\star$ the dual cone of $M$, i.e.

$$M^\star := \{\mu \in Z^\star :\ \langle \mu, z\rangle \geq 0,\ \forall z \in M\}.$$

In the sequel a feasible point $(x_0, u_0)$ and some $g_0 \in G(x_0, u_0) \cap (-M)$ will be fixed. Then we use the notations

$$M_0 := \{\gamma(z + g_0) : \gamma \in R_+, z \in M\},$$
$$M_0^\star := \{\mu \in M^\star :\ \langle \mu, g_0\rangle = 0\} = (M_0)^\star.$$

The graph of a multifunction $\mathcal{F} : X \rightsquigarrow Y$ is

$$gr\mathcal{F} := \{(x,y) \in X \times Y : y \in \mathcal{F}(x)\}$$

and the domain of $\mathcal{F}$ is

$$\text{dom}\,\mathcal{F} := \{x \in X : \mathcal{F}(x) \neq \emptyset\}.$$

Recall that the *Clarke derivative* of $\mathcal{F}$ at $(x_0, f_0) \in gr\mathcal{F}$, denoted by $D\mathcal{F}(x_0; f_0)$, is a multifunction of $X$ into $Y$ whose graph is

$$grD\mathcal{F}(x_0; f_0) = \Big\{(v,w) \in X \times Y : \forall (x_n, f_n) \to_{\mathcal{F}} (x_0, f_0), \forall t_n \to 0^+,$$
$$\exists (v_n, w_n) \to (v, w), \forall n, f_n + t_n w_n \in \mathcal{F}(x_n + t_n v_n)\Big\},$$

where $\to_{\mathcal{F}}$ means that $(x_n, f_n) \in gr\mathcal{F}$ and $(x_n, f_n) \to (x_0, f_0)$. Recall also that $D\mathcal{F}(x_0; f_0)$ is always a closed convex process, i.e. a multifunction whose graph is a nonempty closed convex cone.

We shall need the following directional differentiability and lower semicontinuity with respect to (w.r.t.) $x$.

**Definition 1** [14]. Let $X$ and $Y$ be Banach spaces, $Y$ being ordered by a convex cone $K$. A multifunction $\mathcal{F} : X \rightsquigarrow Y$ is called uniformly $K$-differentiable in the direction $\overline{x} \in X$ at $(x_0, f_0) \in gr\mathcal{F}$ if for each neighborhood $V$ of zero in $Y$ there is a neighborhood $N$ of $\overline{x}$ and a real $\gamma_0 > 0$ such that $\forall\gamma \in (0, \gamma_0)$, $\forall x \in N$, $\forall f \in \mathcal{F}(x_0 + \gamma x)$, $\forall f' \in D\mathcal{F}(x_0; f_0)\overline{x}$

$$\frac{1}{\gamma}(f - f_0) - f' \in V - K.$$

$\mathcal{F}$ is said to be uniformly $K$-differentiable at $(x_0; f_0)$ if this differentiability holds for all directions x in $\operatorname{dom} D\mathcal{F}(x_0; f_0)$.

**Definition 2** [14]. Let $X$, $Y$ and $\mathcal{F}$ be as in Definition 1. Let $x_0 \in \operatorname{dom}\mathcal{F}$ and $T \subset \mathcal{F}(x_0)$ be nonempty. Then, $\mathcal{F}$ is called $K$-strong lower semicontinuous ($K$-s.l.s.c.) with $T$ at $x_0$ if for each neighborhood $V$ of zero in $Y$, there is a neighborhood $N$ of $x_0$ such that $\forall x \in N, \exists f_x \in \mathcal{F}(x)$,

$$f_x - T \subset V - K.$$

If $\exists f_x \in \mathcal{F}(x)$ is replaced by $\forall f_x \in \mathcal{F}(x)$, then we have the definition of uniform $K$-lower semicontinuity ($K$-u.l.s.c.).

Note that if $x_0 \in \operatorname{int}\operatorname{dom}\mathcal{F}$, then $K$-u.l.s.c. implies $K$-s.l.s.c. and if $\mathcal{F}$ is $K$-s.l.s.c. with $\mathcal{F}(x_0)$, then $\mathcal{F}(.) + K$ is lower semicontinuous in the usual sense for multifunctions.

As for parameter $u$, the set $U$ is equipped with no structure. However, the following extensions of the usual convexlikeness [5] are needed.

**Definition 3.** Let $U$ be a set, $Y$ and $Z$ be vector spaces ordered by convex cones $K$ and $M$, respectively.

(i) A multifunction $\mathcal{F} : U \rightsquigarrow Y$ is said to be $K$-convexlike if $\forall u_i \in U$, $\forall f_i \in \mathcal{F}(u_i)$, $i = 1, 2$, $\forall\gamma \in [0, 1]$, $\exists u \in U$, $\exists f_u \in \mathcal{F}(u)$,

$$(1 - \gamma)f_1 + \gamma f_2 - f_u \in K.$$

(ii) $(\mathcal{F}, \mathcal{G}) : U \rightsquigarrow Y \times Z$ is said to be $K \times M$ convexlike strongly with respect to $\mathcal{G}$ if $\forall u_i \in U$, $\forall f_i \in \mathcal{F}(u_i)$, $\forall g_i \in \mathcal{G}(u_i)$, $i = 1, 2$, $\forall\gamma \in [0, 1]$, $\exists u \in U$, $\forall g_u \in \mathcal{G}(u)$, $\exists f_u \in \mathcal{F}(u)$,

$$(1 - \gamma)f_1 + \gamma f_2 - f_u \in K,$$
$$(1 - \gamma)g_1 + \gamma g_2 - g_u \in M.$$

Like the convex case, from the above properties for $i = 1, 2$ it follows that the properties are true for $i = 1, ....m$ for any natural number $m$.

Let $\Sigma^m := \big\{(\alpha_1, \alpha_2, ..., \alpha_m) : \sum_{i=1}^{m} \alpha_i \leq 1, \alpha_i \geq 0\big\}$.

**Definition 4.** A multifunction $(\mathcal{F}, \mathcal{P}) : X \times U \rightsquigarrow Y \times W$ is said to be weakly $K \times \{0\}$ -convexlike in $(U, \{u_0\})$ at $x_0$, strongly with respect to $\mathcal{P}$, if for any finite set $\{u_1, ..., u_m\} \subset U$, there is a neighborhood $N$ of $x_0$ such that $\forall x \in N$,

$\forall(\alpha_1,...,\alpha_m) \in \Sigma^m$, $\exists f_x^{u_i} \in F(x,u_i)$, $\forall p_x^{u_i} \in P(x,u_i)$, $i = 0,...,m$, $\exists u \in U$, $\exists f_x^u \in F(x,u)$, $\exists p_x^u \in P(x,u)$ such that

$$f_x^{u_0} + \sum_{i=1}^{m} \alpha_i (f_x^{u_i} - f_x^{u_0}) - f_x^u \in K,$$
$$p_x^{u_0} + \sum_{i=1}^{m} \alpha_i (p_x^{u_i} - p_x^{u_0}) - p_x^u = 0.$$

To work with inclusions we also need some notion for sections of a multifunction $\mathcal{F} : X \rightsquigarrow Y$.

Let $x_0 \in \operatorname{dom} \mathcal{F}$, $f_0 \in \mathcal{F}(x_0)$. A map $f : X \to Y$ defined in a neighborhood $N$ of $x_0$ is called a *regular local section* of $\mathcal{F}$ at $(x_0, f_0)$ if $f(x_0) = f_0$, $f(x) \in \mathcal{F}(x)$ for all $x \in N \bigcap \operatorname{dom} \mathcal{F}$ and $f'(x_0)X = Y$, where $f'$ denotes the Fre'chet derivative. $f$ is said to be a *subregular local section* if $f'(x_0)X = Y$ is replaced by $f'(x_0)X = D\mathcal{F}(x_0; f_0)X$ and $D\mathcal{F}(x_0; f_0)X$ has finite codimension.

## 2. Necessary optimality conditions

The following necessary optimality condition for local weak minima and local minima is an extension of Theorem 1 of [14].

**Theorem 1.** *Assume that $(x_0, u_0)$ is feasible for $(\tilde{P})$ and*

*$(i_1)$ for each $\overline{x} \in X$, $p'_{\overline{x}} \in D_x P(x_0, u_0; 0)\overline{x}$, $P(., u_0)$ has a subregular local section $p(., u_0)$ at $(x_0, 0)$ being continuously differentiable and such that $p'_{\overline{x}} = p'_x(x_0, u_0)\overline{x}$, where $D_x P(x_0, u_0; p_0) := DP(., u_0)(x_0; p_0)$ with $p_0 \in P(x_0, u_0)$;*

*$(i_2)$ For each $u \neq u_0$ and $p \in P(x_0, u)$, $P(., u)$ has a continuously differentiable local section $p(., u)$ at $(x_0, p)$;*

*(ii) $F(., u_0)$ and $G(., u_0)$ are uniformly $K$-differentiable at $(x_0, u_0; f_0)$ and uniformly $M$-differentiable at $(x_0, u_0; g_0)$, respectively, where $f_0 \in F(x_0, u_0)$, $g_0 \in G(x_0, u_0) \bigcap (-M)$;*

*(iii) for each $u \neq u_0$, $F(., u)$ ($G(., u)$, respectively) is $K$-s.l.s.c. with $F(x_0, u)$ ($M$-s.l.s.c. with $G(x_0, u)$) at $x_0$. Moreover, $F(., u_0)$ ($G(., u_0)$) is $K$-s.l.s.c. with $f_0$ ($M$-s.l.s.c. with $g_0$, respectively) at $x_0$;*

*(iv) for each $x$ in a neighborhood of $x_0$, $(F, G, P)(x, .)$ is $K \times M \times \{0\}$-convexlike.*

*If $(x_0, u_0; f_0)$ is a local weak minimum of $(\tilde{P})$, then there exists*

$$(\lambda_0, \mu_0, \nu_0) \in K^\star \times M_0^\star \times W^\star \setminus \{0\}$$

*such that, for all $(x, u) \in X \times U$,*

$$\begin{aligned} &\langle \lambda_0, D_x F(x_0, u_0; f_0)x + F(x_0, u) - f_0 \rangle + \langle \mu_0, D_x G(x_0, u_0; g_0)x + G(x_0, u) - g_0 \rangle \\ &\qquad + \langle \nu_0, D_x P(x_0, u_0; 0)x + P(x_0, u) \rangle \subset R^+. \end{aligned} \tag{2}$$

*Proof.* For $x$ outside the intersection of the domains of the three Clarke derivatives on the left-hand side of (2), this side is empty and (2) holds. So we may assume that $x$ belongs to this intersection. However, for simplicity we write $x \in X$.

Let

$$L = D_x P(x_0, u_0; 0)X, \quad B = L + P(x_0, U).$$

Then $B$ is convex, since $P(x_0, U)$ is convex by the convexlikeness assumed in (iv). Aff$(B)$ is a subspace with finite codimension. If Aff$(B) \neq W$ then there exists $\nu_0 \in W^* \setminus \{0\}$ such that, for all $(x, u) \in X \times U$,

$$\langle \nu_0, D_x P(x_0, u_0; 0)x + P(x_0, u)\rangle = 0$$

and (2) is satisfied with $\lambda_0 = 0$, $\mu_0 = 0$.

Now assume that Aff$(B) = W$ and let $\Pi$ stand for the canonical projection of $W$ onto $W/L$. Since $\Pi(B)$ is convex and Aff$(\Pi(B)) = W/L$, one has $\operatorname{int} \Pi(B) \neq \emptyset$. As $\Pi^{-1}(\Pi(B)) = B$, $\operatorname{int} B \neq \emptyset$. If $0 \notin \operatorname{int} B$, 0 can be separated from $B$ by some $\nu_0 \in W^\star \setminus \{0\}$, i.e., for all $(x, u) \in X \times U$,

$$\langle \nu_0, D_x P(x_0, u_0; 0)x + P(x_0, u)\rangle \geq 0$$

and (2) holds again with $\lambda_0 = 0$, $\mu_0 = 0$.

It remains the case $0 \in \operatorname{int} B$. Consider the set $C$ of all $(y, z, w) \in Y \times Z \times W$ such that $\exists (x, u) \in X \times U$, $\exists f'_x \in D_x F(x_0, u_0; f_0)x$, $\exists g'_x \in D_x G(x_0, u_0; g_0)x$, $\exists p'_x \in D_x P(x_0, u_0; 0)x$, $\exists f^u_{x_0} \in F(x_0, u)$, $\exists g^u_{x_0} \in G(x_0, u)$, $\exists p^u_{x_0} \in P(x_0, u)$,

$$f'_x + f^u_{x_0} - f_0 - y \in -\operatorname{int} K,$$
$$g'_x + g^u_{x_0} - g_0 - z \in -\operatorname{int} M,$$
$$p'_x + p^u_{x_0} - w = 0.$$

Similarly as in [14], by the convexity of the Clarke derivatives and the $K \times M \times \{0\}$-convexlikeness assumed in (iv) one sees the convexity of $C$.

By the uniform directional differentiability of $F(., u_0)$ and $G(., u_0)$ stated in (ii) and by a similar argument as that of Lemma 2.2 of [15] it is not hard to check that $intC \neq \emptyset$. If

$$C \bigcap \{(-\operatorname{int} K) \times (-\operatorname{int} M_0^{\star\star}) \times \{0\}\} = \emptyset, \tag{3}$$

then by a standard separation theorem we obtain (2). Therefore, it is sufficient now to prove (3). Suppose to the contrary the existence of $(\hat{x}, \hat{u}) \in X \times U$, $f'_{\hat{x}} \in D_x F(x_0, u_0; f_0)\hat{x}$, $g'_{\hat{x}} \in D_x G(x_0, u_0; g_0)\hat{x}$, $p'_{\hat{x}} \in D_x P(x_0, u_0; 0)\hat{x}$, $f^{\hat{u}}_{x_0} \in F(x_0, \hat{u})$, $g^{\hat{u}}_{x_0} \in G(x_0, \hat{u})$ and $p^{\hat{u}}_{x_0} \in P(x_0, \hat{u})$ such that

$$f'_{\hat{x}} + f^{\hat{u}}_{x_0} - f_0 \in -\operatorname{int} K, \tag{4}$$

$$g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 \in -\operatorname{int} M_0^{\star\star}, \tag{5}$$

$$p'_{\hat{x}} + p^{\hat{u}}_{x_0} = 0. \tag{6}$$

With there $\hat{x}$ and $p'_{\hat{x}}$, by $(i_1)$ one has a subregular local section $p(., u_0)$ at $(x_0, 0)$ such that $p'_{\hat{x}} = p'_x(x_0, u_0)\hat{x}$.

Since $0 \in \operatorname{int}\Pi(B)$ and $\operatorname{Aff}(\Pi(B)) = W/L$ is finite dimensional, there exist $z_1, ..., z_m \in \Pi(B)$ such that $\operatorname{Aff}\{z_1, ..., z_m\} = W/L$ and $z_1 + \cdots + z_m = 0$. By the definition of $B$ there are $x_1 \in X$, $u_i \in U$, $p_{x_0}^{u_i} \in P(x_0, u_i)$, $i = 1, ..., m$, such that

$$\Pi(p_{x_0}^{u_i}) = z_i, \tag{7}$$

$$p'_x(x_0, u_0)x_1 + \sum_{i=1}^{m} p_{x_0}^{u_i} = 0, \tag{8}$$

$$\operatorname{Aff}(L \bigcup \{P(x_0, u_i) : i = 1, ..., m\}) = W. \tag{9}$$

By $(i_2)$ there are local sections $p(., u_i)$ such that $p(x_0, u_i) = p_{x_0}^{u_i}$.

In virtue of (5), there is $\delta > 0$ such that

$$g'_{\hat{x}} + g_{x_0}^{\hat{u}} - g_0 + \delta B_Z - M \subset -\operatorname{int} M_0^{\star\star}, \tag{10}$$

where $B_Z$ is the open unit ball in $Z$.

The uniform $M$-differentiability of $G(., u_0)$ stated in (ii) implies the existence of a neighborhood $N_1$ of $\hat{x}$ and of $t_1 > 0$ such that $\forall z \in N_1$, $\forall t \in (0, t_1)$, $\forall g_z \in G(x_0 + tz, u_0)$,

$$g_z \in g_0 + tg'_{\hat{x}} + t\frac{\delta}{4}B_Z - M. \tag{11}$$

Choose $\varepsilon_1$ such that $\hat{x} + \varepsilon_1 x_1 \in N_1$. Take arbitrarily $g_{x_0}^{u_i} \in G(x_0, u_i)$, $i = 1, ..., m$, and $\varepsilon_2$ such that $\varepsilon_2 \sum\limits_{i=1}^{m}(g_{x_0}^{u_i} - g_0) \in \dfrac{\delta}{4}B_Z$. Set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and

$$\mathcal{P}(x, \alpha_0, \alpha_1, ..., \alpha_m) = p(x, u_0) + \alpha_0(p(x, \hat{u}) - p(x, u_0)) + \varepsilon \sum_{i=1}^{m} \alpha_i(p(x, u_i)) - p(x, u_0)).$$

We have

$$\begin{aligned}
\mathcal{P}(x_0, 0, ..., 0) &= 0,\\
\mathcal{P}'(x_0, 0, ..., 0)(x, \alpha_0, \alpha_1, ..., \alpha_m) &= p'_x(x_0, u_0)x + \alpha_0 p(x_0, \hat{u}) + \varepsilon \sum_{i=1}^{m} \alpha_i p(x_0, u_i),\\
\mathcal{P}'(x_0, 0, ..., 0)(\hat{x} + \varepsilon x_1, 1, ..., 1) &= p'_x(x_0, u_0)(\hat{x} + \varepsilon x_1) + p(x_0, \hat{u}) + \varepsilon \sum_{i=1}^{m} p(x_0, u_i)\\
&= p'_x(x_0, u_0)\hat{x} + p(x_0, \hat{u}) + \varepsilon(p'_x(x_0, u_0)x_1\\
&\quad + \sum_{i=1}^{m} p(x_0, u_i)) = 0.
\end{aligned}$$

So $(\hat{x} + \epsilon x_1, 1, ..., 1) \in \operatorname{Ker}\mathcal{P}'(x_0, 0, ..., 0)$. Hence, by the Lusternik theorem, there are mappings $t \to \overline{x}(t), t \to \alpha_0(t), ..., t \to \alpha_m(t)$ of some interval $[0, t_0]$ into $R$ such that $\overline{x}(t) \to 0$, $\alpha_i(t) \to 0$, $i = 1, ..., m$, as $t \to 0^+$, and

$$\mathcal{P}(t(\hat{x} + \varepsilon x_1 + \overline{x}(t)), t(1 + \alpha_0(t), ..., t(1 + \alpha_m(t))) = 0. \tag{12}$$

By setting $x(t) := x_0 + t(\hat{x} + \varepsilon x_1 + \overline{x}(t))$, (12) becomes

$$p(x(t), u_0) + t(1 + \alpha_0(t))(p(x(t), \hat{u}) - p(x(t), u_0)) + \varepsilon \sum_{i=1}^{m} t(1 + \alpha_i(t))(p(x(t), u_i) - p(x(t), u_0)) = 0.$$

By virtue of the strong lower semi continuity assumed in $(iii)$, for given $x(t)$ and $\hat{u}$ there are $g^{\hat{u}}_{x(t)} \in G(x(t), \hat{u})$, $g^{u_0}_{x(t)} \in G(x(t), u_0)$, $f^{\hat{u}}_{x(t)} \in F(x(t), \hat{u})$ and $f^{u_0}_{x(t)} \in F(x(t), u_0)$ such that, for all $t$ and $\varepsilon$ small enough,

$$g^{\hat{u}}_{x(t)} - g^{\hat{u}}_{x_0} \in \varepsilon B_Z - M, \tag{13}$$

$$-g^{u_0}_{x(t)} + g_0 \in \varepsilon B_Z - M, \tag{14}$$

$$f^{\hat{u}}_{x(t)} - f^{\hat{u}}_{x_0} \in \varepsilon B_Y - K, \tag{15}$$

$$-f^{\hat{u}}_{x(t)} + f_0 \in \varepsilon B_Y - K. \tag{16}$$

The convexlikeness supposed in $(iv)$ now yields $u(t) \in U$, $f_t \in F(x(t), u(t))$, and $g_t \in G(x(t), u(t))$ such that

(17)

$$f^{u_0}_{x(t)} + t(1 + \alpha_0(t))(f^{\hat{u}}_{x(t)} - f^{u_0}_{x(t)}) + \varepsilon \sum_{i=1}^{m} t(1 + \alpha_i(t))(f^{u_i}_{x(t)} - f^{u_0}_{x(t)}) - f_t \in K,$$

(18)

$$g^{u_0}_{x(t)} + t(1 + \alpha_0(t))(g^{\hat{u}}_{x(t)} - g^{u_0}_{x(t)}) + \varepsilon \sum_{i=1}^{m} t(1 + \alpha_i(t))(g^{u_i}_{x(t)} - g^{u_0}_{x(t)}) - g_t \in M,$$

$$0 \in P(x(t), u(t)). \tag{19}$$

At this point, a contradiction to the minimality of $(x_0, u_0; f_0)$ will be achieved if we can show that for all $t$ small enough, $g_t \in -M$ and $f_t - f_0 \in -\operatorname{int} K$. We argue in detail for $g_t$ since the reasoning for $f_t$ is similar.

We consider the terms in (18). By (13), (14) and since $\alpha_i(t)$ is small for small $t, i = 0, ..., m$, we have

$$\begin{aligned} t(1 + \alpha_0(t))(g^{\hat{u}}_{x(t)} - g^{u_0}_{x(t)}) &= t(1 + \alpha_0(t))(g^{\hat{u}}_{x(t)} - g^{\hat{u}}_{x_0} - g^{u_0}_{x(t)} + g_0) \\ &\quad + t(g^{\hat{u}}_{x_0} - g_0) + t\alpha_0(t)(g^{\hat{u}}_{x_0} - g_0) \\ &\subset t(1 + \alpha_0(t))(2\epsilon B_Z - M) + t(g^{\hat{u}}_{x_0} - g_0) + t\epsilon B_Z \\ &\subset t(g^{\hat{u}}_{x_0} - g_0 + \frac{\delta}{4} B_Z - M) \end{aligned}$$

for $t$ and $\varepsilon$ small enough. Similarly, for $i = 1, ..., m$,

$$t(1 + \alpha_i(t))(g^{u_i}_{x(t)} - g^{u_0}_{x(t)}) \subset t(g^{u_i}_{x_0} - g_0 + \frac{\delta}{4m} B_Z - M). \tag{20}$$

Combining (11) with $z = \hat{x} + \varepsilon x_1 + \overline{x}(t)$ and (20) shows that the left-hand side of (18) belongs to the left-hand side of the inclusion

$$g_0 + t(g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 + \varepsilon \sum_{i=1}^{m} (g^{u_i}_{x_0} - g_0) + \frac{\delta}{4} B_Z + \frac{\delta}{4} B_Z + \frac{\delta}{4} B_Z - M) - g_t$$
$$\subset g_0 + t(g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 + \delta B_Z - M) - g_t.$$

Therefore, for some $b^t_\delta \in \delta B_Z$ and $m^t \in M$, (18) becomes

$$g_0 + t(g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 + b^t_\delta - m^t) - g_t \in M. \tag{21}$$

To verify that $g_t \in -M$ for all sufficiently small $t > 0$, we suppose to the contrary that $\exists t_n \to 0^+$, $\exists \mu_n \in M^*$, $\|\mu_n\| = 1$, (then assume that $\mu_n$ tends *weakly to $\overline{\mu} \in M^*$), $\langle \mu_n, g_{t_n} \rangle \geq 0$. We claim that $\overline{\mu} \in M^*_0$. Indeed, if $\overline{\mu} \notin M^*_0$ there would be $\beta > 0$ such that $\langle \overline{\mu}, g_0 \rangle < -\beta$. On the other hand, (21) implies

$$\langle \mu_n, g_{t_n} \rangle \leq \langle \mu_n, g_0 \rangle + t_n \langle \mu_n, g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 + b^{t_n}_\delta - m^{t_n} \rangle. \tag{22}$$

Hence, $\langle \mu_n, g_{t_n} \rangle < 0$ for $t_n$ small enough, which is a contradiction. As $\overline{\mu} \in M^\star_0$, (10) gives, for large $n$,

$$t_n \langle \mu_n, g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 + b^{t_n}_\delta - m^{t_n} \rangle < 0,$$

and then (22) is impossible due to the supposed nonnegativity of $\langle \mu_n, g_{t_n} \rangle$.

The above reasoning for $g_t$ applied to $f_t$ shows that

$$f_t - f_0 \in t(f'_{\hat{x}} + f^{\hat{u}}_{x_0} - f_0 + b^t_\delta - k^t) - K \subset -K - \operatorname{int} K = -int\ K.$$

Thus a contradiction to the minimality of $(x_0, u_0; f_0)$ has been obtained and the proof is complete. □

With a constraint qualification of the Slater type added to the assumptions of Theorem 1 we obtain a Kuhn-Tucker necessary condition as follows.

**Theorem 2.** *Assume additionally to the assumptions of Theorem 1 that* $D_x P(x_0, u_0; 0)X + P(x_0, U)$ *contains a neighborhood of zero in* $W$ *and that there are* $(\tilde{x}, \tilde{u}) \in X \times U$, $g'_{\tilde{x}} \in D_x G(x_0, u_0; g_0)\tilde{x}$, $p'_{\tilde{x}} \in D_x P(x_0, u_0; 0)\tilde{x}$, $g^{\tilde{u}}_{x_0} \in G(x_0, \tilde{u})$ *and* $p^{\tilde{u}}_{x_0} \in P(x_0, \tilde{u})$ *such that*

$$g'_{\tilde{x}} + g^{\tilde{u}}_{x_0} - g_0 \in -int\, M^{**}_0,$$
$$p'_{\tilde{x}} + p^{\tilde{u}}_{x_0} = 0.$$

*Then,* $\lambda_0 \neq 0$.

*Proof.* The same as in [14]. □

**Remark 1.** Theorem 1 extends Theorem 1 of [14] to the case where $P : X \times U \rightsquigarrow W$ is a multifunction. Moreover, it is an improvement, even when applying to the case of $P$ being single-valued. Namely, in (i), $p(., u_0)$ is assumed subregular, not necessarily regular. However, in (iv), $(F, G, P)(x, .)$ needs to be assumed $K \times M \times \{0\}$-convexlike (in $U$, not only in $(U, \{u_0\})$). This strengthening in turn can be relaxed if we replace the strong semicontinuity in (iii) by the uniform semicontinuity as follows.

**Theorem 1'.** *Let the assumptions* $(i_1)$, $(i_2)$ *and* $(ii)$ *be as in Theorem 1. Assume further*

$(iii')$ *for each* $u \neq u_0$, $F(.,u)(G(.,u)$, *respectively) is* $K$ *-u.l.s.c. with* $F(x_0,u)(M$*-u.l.s.c. with* $G(x_0,u))$ *at* $x_0$. *Moreover,* $F(.,u_0)$ $(G(.,u_0))$ *is* $-K$*-u.l.s.c. with* $f_0$ $(-M$*-u.l.s.c. with* $g_0$, *respectively) at* $x_0$;

$(iv')$ $(F,G,P)(x_0,.)$ *is* $K \times M \times \{0\}$*-convexlike. Moreover,* $(F,G,P)(.,.)$ *is weakly* $K \times M \times \{0\}$*-convexlike in* $(U,\{u_0\})$ *at* $x_0$, *strongly w.r.t.* $P$.

*Then the conclusion of Theorem 1 is still valid.*

*Proof.* The proof is similar to that of Theorem 1. □

Passing to problem $(P)$, it is not hard to verify that Theorems 1, 2 and 1' still hold if the convexlikeness and weak convexlikeness assumed in (iv) and (iv') are strong w.r.t. $G$ (following Definition 3 (ii)).

In order to apply the Fritz John and Kuhn-Tucker conditions to optimal control problems with state constraints to derive the Pontryagin maximum principle, the assumed convexlikeness is replaced in [6], [11] by what we call approximate convexlikeness. This result is extended to multifunction optimization problems with single-valued equality constraints in [15]. Now we prove the corresponding extension to $(P)$ and $(\tilde{P})$. This time, a detailed presentation is devoted to $(P)$ instead of $(\tilde{P})$.

**Definition 5.** Problem $(P)$, or the composite multifunction $(F,G,P)$, is said to be approximate-convexlike at $(x_0,u_0)$ if there exists a section $\overline{p}(.,.)$ of $P(.,.)$ such that for each $u \in U$, $\overline{p}(.,u)$ is continuously differentiable at $x_0$ and $\overline{p}(x_0,u_0) = 0$, and that for each finite set $\{u_1,...,u_s\} \subset U$, for each $\delta > 0$, there are $\varepsilon > 0$, a neighborhood $V$ of $x_0$, a mapping $v : V \times \varepsilon\Sigma^s \longrightarrow U$, $e \in K$, $q \in M$ such that for all $x, x' \in V$, there are $f_x^{u_i} \in F(x,u_i)$, $g_x^{u_i} \in G(x,u_i)$, $i = 0,1,...,s$, so that for all $\alpha, \alpha' \in \epsilon \sum^s$, $g_x \in G(x,v(x,\alpha))$, there exists $f_x \in F(x,v(x,\alpha))$ such that, for all $\overline{p}' \in D_xP(x_0,u_0;0)(x-x')$ and $\overline{p}_{x_0}^{u_i} \in P(x_0,u_i)$,

$$v(x,0) = u_0,$$

(23) $$f_x^{u_0} + \sum_{i=1}^{s} \alpha_i(f_x^{u_i} - f_x^{u_0}) + \delta(\|x-x_0\| + \sum_{i=1}^{s} \alpha_i)e - f_x \in K,$$

(24) $$g_x^{u_0} + \sum_{i=1}^{s} \alpha_i(g_x^{u_i} - g_x^{u_0}) + \delta(\|x-x_0\| + \sum_{i=1}^{s} \alpha_i)q - g_x \in M,$$

$$\big\|\overline{p}(x,v(x,\alpha))-\overline{p}(x',v(x',\alpha'))-\overline{p}'+\sum_{i=1}^{s}(\alpha_i-\alpha_i')\overline{p}_{x_0}^{u_i}\big\|$$

$$\leq \delta(\|x-x'\|+\sum_{i=1}^{s}|\alpha_i-\alpha_i'|). \tag{25}$$

Now assume that $(x_0,u_0)$ is feasible for $(P)$ and $f_0\in F(x_0,u_0)$, $g_0\in G(x_0,u_0)$.

**Theorem 3.** *Assume* $(i_1)$, $(ii)$ *and* $(iii')$ *as in Theorem 1'. Assume further*

*(iv") $(F,G,P)$ is approximate convexlike at $(x_0,u_0)$.*

*Then the conclusion of Theorem 1' still holds.*

*Proof.* Set

$$L=D_xP(x_0,u_0;0)X, \quad B=L+\mathrm{convp}(x_0,U).$$

If $\mathrm{Aff}(B)\neq W$, $0\notin \mathrm{int}\,B$, the conclusion is obtained exactly as in Theorem 1.

For the case $0\in\mathrm{int}\,B$, define the set $C$ of all $(y,z,w)\in Y\times Z\times W$ such that $\exists x\in X$, $\exists\,\{u_1,...,u_m\}\subset U$, $\exists\gamma_i>0$, $\exists(f_i,g_i,p_i)\in(F,G,P)(x_0,u_i)$, $i=1,...,m$, $\exists f_x'\in D_xF(x_0,u_0;f_0)x$, $\exists g_x'\in D_xG(x_0,u_0;g_0)x$, $\exists p_x'\in D_xP(x_0,u_0;0)x$,

$$f_x'+\sum_{i=1}^{m}\gamma_i(f_i-f_0)-y\in-\mathrm{int}\,K, \tag{26}$$

$$g_x'+\sum_{i=1}^{m}\gamma_i(g_i-g_0)-z\in-\mathrm{int}\,M, \tag{27}$$

$$p_x'+\sum_{i=1}^{m}\gamma_ip_i-w=0. \tag{28}$$

Similarly as in [15], $C$ is seen to be a convex set with nonempty interior. If (3) holds, the proof will be complete by applying a standard separation theorem.

Now suppose, incontrary to (3), that there are $x'\in X$, $u_{01},...,u_{0m_0}\in U$, $\gamma_{01}>0,...,\gamma_{0m_0}>0$, $(f_j^0,g_j^0,p_j^0)\in(F,G,P)(x_0,u_{0j})$, $j=1,...,m_0$, $f'\in D_xF(x_0,u_0;f_0)x'$, $g'\in D_xG(x_0,u_0;g_0)x'$, $p'\in D_xP(x_0,u_0;0)x'$, such that

$$f'+\sum_{j=1}^{m_0}\gamma_{0j}(f_j^0-f_0)\in-\mathrm{int}\,K,$$

$$g'+\sum_{j=1}^{m_0}\gamma_{0j}(g_j^0-g_0)\in-\mathrm{int}\,M_0^{**},$$

$$p'+\sum_{j=1}^{m_0}\gamma_{0j}p_j^0=0.$$

Since $\mathrm{Aff}(\Pi(B))=W/L$ is finite dimensional, there exist $z_1,...,z_k\in\Pi(B)$ such that $\mathrm{Aff}\{z_1,...,z_k\}=W/L$ and $z_1+...+z_k=0$. The definition of $B$ yields $x_1\in X$, $\gamma_{11}>0,...,\gamma_{1m_1}>0,...,\gamma_{k1}>0,...,\gamma_{km_k}>0$, $u_{11},...,u_{1m_1},...,u_{k1},...,u_{km_k}\in U$,

$p'_{x_1} \in D_xP(x_0,u_0;0)x_1$, and $P_{qi} \in P(x_0,u_{qi})$, $q=1,...,k$, and $i=1,...,m_0$ such that

$$\Pi\Big(\sum_{i=1}^{m_q}\gamma_{qi}p_{qi}\Big) = z_q\,, \quad q=1,...,k,$$

$$p'_{x_1} + \sum_{q=1}^{k}\sum_{i=1}^{m_q}\gamma_{qi}p_{qi} = 0, \tag{29}$$

$$\mathrm{Aff}(L\bigcup\{p_{qi} : q=1,...,k, i=1,...,m_q\}) = W. \tag{30}$$

By Lemma 1.2 of [15] the conditions (26), (27), (28) show that for all $(f_i^q, g_i^q) \in Y\times Z$, all small $\theta > 0$, there are $\overline{f}' \in D_xF(x_0,u_0;f_0)(x'+\theta x_1)$, $\overline{g}' \in D_xG(x_0,u_0;g_0)(x'+\theta x_1)$, and for $\overline{p}' = p' + \theta p'_{x_1} \in D_xP(x_0,u_0;0)(x'+\theta x_1)$,

$$\overline{f}' + \sum_{j=1}^{m_0}\gamma_{0j}(f_j^0 - f_0) + \sum_{q=1}^{k}\Big(\sum_{i=1}^{m_q}\theta\gamma_{qi}(f_i^q - f_0)\Big) \in -\mathrm{int}\,K, \tag{31}$$

$$\overline{g}' + \sum_{j=1}^{m_0}\gamma_{0j}(g_j^0 - g_0) + \sum_{q=1}^{k}\Big(\sum_{i=1}^{m_q}\theta\gamma_{qi}(g_i^q - g_0)\Big) \in -\mathrm{int}\,M_0^{**}, \tag{32}$$

$$\overline{p}' + \sum_{j=1}^{m_0}\gamma_{0j}p_j^0 + \sum_{q=1}^{k}\sum_{i=1}^{m_q}\theta\gamma_{qi}p_{qi} = 0. \tag{33}$$

Setting $\overline{x} = x'+\theta x_1$, $\overline{u}_1 = u_{01},...,\overline{u}_{m_0} = u_{0m_0}$, $\overline{u}_{m_0+1} = u_{11},..,\overline{u}_s = \overline{u}_{m_0+m_1+..+m_k} = u_{km_k}$, $\overline{\alpha}_1 = \gamma_{01},...,\overline{\alpha}_{m_0} = \gamma_{0m_0}$, $\overline{\alpha}_{m_0+1} = \theta\gamma_{11},...,\overline{\alpha}_s = \theta\gamma_{km_k}$, $\overline{f}_1 = f_1^0$, $\overline{f}_{m_0} = f_{m_0}^0$, $\overline{f}_{m_0+1} = f_1^1,...,\overline{f}_s = f_{m_k}^k$ and similarly for $\overline{g}_1,...,\overline{g}_s$, and $\overline{p}_1,...,\overline{p}_s$. (31), (32), (33) together with (29), and (30) become

$$\overline{f}' + \sum_{j=1}^{s}\overline{\alpha}_j(\overline{f}_j - f_0) \in -\mathrm{int}\,K, \tag{34}$$

$$\overline{g}' + \sum_{j=1}^{s}\overline{\alpha}_j(\overline{g}_j - g_0) \in -\mathrm{int}\,M_0^{**}, \tag{35}$$

$$\overline{p'} + \sum_{j=1}^{s}\overline{\alpha}_j\overline{p}_j = 0, \tag{36}$$

$$\mathrm{Aff}\big(L\bigcup\{\overline{p}_j : j=1,...,s\}\big) = W. \tag{37}$$

Taking into account assumption $(iv")$, for these $\overline{u}_1,...,\overline{u}_s$ and some $\delta > 0$ we have (23)-(25) (with $\overline{u}_1,...,\overline{u}_s$ in the place of $u_1,...,u_s$).

In accordance with assumption $(i_1)$, for $\overline{x}$ and $\overline{p}'$ there is a local section $p(.,u_0)$ with $p(x_0,u_0) = 0$ and $\overline{p}' = p'_x(x_0,u_0)\overline{x}$. Define a mapping $\mathcal{P}$ and a bounded linear mapping $A$ of a neighborhood of $(x_0, 0) \in X \times R^s$ into $W$ by

$$\mathcal{P}(x,\alpha) = \overline{p}(x, v(x,\alpha^+)) + \sum_{j=1}^{s} \alpha_j^- \overline{p}_j,$$

$$A(x,\alpha) = p'_x(x_0,u_0)x + \sum_{j=1}^{s} \alpha_j \overline{p}_j,$$

where $\alpha_j^+ := \max\{\alpha_j, 0\}$; $\alpha_j^- := \alpha_j - \alpha_j^+$, $\alpha^+ := (\alpha_1^+, ..., \alpha_s^+)$. By (25), for all $(x,\alpha)$ and $(x', \alpha^{'})$ in $V \times \epsilon \sum^s$ we have

$$\|\mathcal{P}(x,\alpha) - \mathcal{P}(x',\alpha^{'}) - A(x,\alpha) + A(x',\alpha^{'})\|$$

$$= \Big\|\overline{p}(x, v(x,\alpha^+)) - \overline{p}(x', v(x', \alpha^{'+})) - p'_x(x_0,u_0)(x - x') - \sum_{j=1}^{s}(\alpha_j^+ - \alpha_j^{'+})\overline{p}_j\Big\|$$

(38)

$$\le \delta\Big(\|x - x'\| + \sum_{j=1}^{s} |\alpha_j - \alpha_j^{'}|\Big).$$

By (37), $A(X \times R^s) = W$. Let $\overline{A} : (X \times R^s)/\mathrm{Ker}\, A \to W$ be the one-to-one mapping corresponding to $A$. If $\delta$ is chosen so small that $\delta\|\overline{A}^{-1}\| < 1/2$, then by the Ioffe-Tihomirov generalization of the Lusternik theorem [6,p.34] (or [7] for more general setting), there exist $\overline{t} > 0, k > 0$ and a mapping $t \to (x(t), \alpha(t))$ of $[0, \overline{t}]$ into $X \times R^s$ such that, for all $t \in [0, \overline{t}]$,

$$\mathcal{P}(x_0 + t\overline{x} + x(t), t\overline{\alpha} + \alpha(t)) = 0,$$

$$\|x(t)\| + \sum_{j=1}^{s} |\alpha_j(t)| \le k\|\mathcal{P}(x_0 + t\overline{x}, t\overline{\alpha})\|$$

$$= k\|\mathcal{P}(x_0 + t\overline{x}, t\overline{\alpha}) - \mathcal{P}(x_0, 0) - A(x_0 + t\overline{x}, t\overline{\alpha}) - A(x_0, 0)\|$$

$$\le tk\delta(\|\overline{x}\| + \sum_{j=1}^{s} \overline{\alpha}_j), \tag{39}$$

Therefore, $x(t)$ and $u(t)$ tend to 0 as $t$ does. If $\delta$ is also satisfied

$$k\delta\Big(\|\overline{x}\| + \sum_{j=1}^{s} \overline{\alpha}_j\Big) < \min\{\overline{\alpha}_1, ..., \overline{\alpha}_s\},$$

then (39) implies that

$$\|x(t)\| + \sum_{j=1}^{s} |\alpha_j(t)| < t \min\{\overline{\alpha}_1, ..., \overline{\alpha}_s\}.$$

Hence $t\overline{\alpha}_j + \alpha_j(t) > 0$ for all $j$ and all small $t$. Consequently,

$$\mathcal{P}(x, t\overline{\alpha} + \alpha(t)) = \overline{p}(x, v(x, t\overline{\alpha} + \alpha(t))) \in P(x, v(x, t\overline{\alpha} + \alpha(t))).$$

Setting

$$\overline{x}(t) = x_0 + t\overline{x} + x(t)\ ,\ \overline{u}(t) = v(\overline{x}(t), t\overline{\alpha} + \alpha(t)),$$

we have

$$0 = \overline{p}(\overline{x}(t), \overline{u}(t)) \in P(\overline{x}(t), \overline{u}(t)).$$

Now the same lines in the last part of the proof of Theorem 2.1 in [15] indicates that for all $t$ small enough, $G(\overline{x}(t), \overline{u}(t)) \subset -M$ and there is $\overline{f}_t \in F(\overline{x}(t), \overline{u}(t))$ such that $\overline{f}_t - f_0 \in -\text{int}\, K$. By this, a contradiction to the minimality of $(x_0, u_0; f_0)$ is obtained and the proof of the theorem is complete. □

Examining the above proof it is easy to see that for problem $(\tilde{P})$, Theorem 3 is still valid even with the relaxation in ($iv$") that we only require the existenxe of $g_x \in G(x, v(x, \alpha))$.

## 3. Examples

The assumptions of the theorems presented in this paper have complicated formulations (they look so even in the corresponding results of [6], [11] for the single-valued case), but they are weak and not hard to check as shown by the following examples.

**Example 1** (Illustration of Theorem 1). Let $X = W = C^1_{[0,1]}$, $Y = Z = U = R$ and $K = M = R_+$. Let $f : X \to R$ be continuously differentiable in a neghborhood of $x_0$ satisfying $x_0 \neq const$ and $x_0(0) = 0$. Consider the following optimization problem involving a differential inclusion

$$\min\Big(\int_0^1 f(x(t))dt + u^2\Big), \tag{40}$$

$$\|x - x_0\|^2(|u|[0,1] - 1) \subset -R_+, \tag{41}$$

$$\dot{x} \in \Big\{\frac{\dot{x}_0 + 2a\dot{x}_0x_0 - a\dot{x}_0x}{ax_0 + 1} : a \in R\Big\}, \quad x(0) = 0. \tag{42}$$

It is easy to see that if there is a neighborhood $V$ of $x_0$ such that $f(x) \geq f(x_0)$ for all $x \in V$, then $(x_0, u_0) := (x_0, 0)$ is a local minimum of the problem (40)-(42). To reduce this problem to a problem of the type $(P)$ observe that (42) is equivalent to

$$0 \in \big\{a\dot{x}x_0 + \dot{x} - \dot{x}_0 - 2a\dot{x}_0x_0 + a\dot{x}_0x :\ a \in R\big\}, \quad x(0) = 0$$

or, what is the same,

$$0 \in \Big\{\frac{d}{dt}[(x - x_0)(ax_0 + 1)] :\ a \in R\Big\}, \quad x(0) = 0.$$

Integrating this shows that (42) is equivalent to

$$0 \in \{(x - x_0)(ax_0 + 1) : a \in R\},$$

which is of the form $0 \in P(x, u)$ (but $P$ does not depend on $u$).

Taking $f_0 = \int\limits_0^1 f(x(t))dt$ and $g_0 = 0$ we verify that all the assumptions of the necessary optimality condition for $(P)$ corresponding to Theorem 1 for $(\tilde{P})$, are satisfied.

Since $F$ is single-valued, assumptions $(ii)$ and $(iii)$ for $F$ are clear. As for $G$, an argument similar to that of the Example in [14] will do. Passing to $(i_1)$ let $\overline{x} \in X$ and $p'_{\overline{x}} \in D_x P(x_0, 0; 0)\overline{x}$. Then the section

$$P(x, u_0) = \begin{cases} x - x_0 & \text{if } \overline{x} = 0, \\ \dfrac{p'_{\overline{x}}}{\overline{x}}(x - x_0) & \text{if } \overline{x} \neq 0, \end{cases}$$

apparently meets $(i_1)$. For $(i_2)$, observing that $P(x_0, u) = \{0\}$ for each $u \in U$, the section $P(x, u) = x - x_0$ is seen to be suitable. Finally we check $(iv)$ (with the strong- w.r.t.$-G$ convexlikeness). $\forall x \in C^1_{[0,1]}$, $\forall\ u_1 \in U$, $\forall u_2 \in U$, $\forall \gamma \in [0, 1]$, taking $u = 0$, $f_u = \int\limits_0^1 f(x(t))dt$ and $g_x = -\|x - x_0\|^2$ we have

$$\begin{aligned} &(1-\gamma)\Big(\int_0^1 f(x(t))dt + u_1^2\Big) + \gamma\Big(\int_0^1 f(x(t))dt + u_2^2\Big) - \int_0^1 f(x(t))dt \\ &= (1-\gamma)u_1^2 + \gamma u_2^2 \in R_+, \\ &(1-\gamma)\|x - x_0\|^2(|u_1|[0,1] - 1) + \gamma\|x - x_0\|^2(|u_2|[0,1] - 1) + \|x - x_0\|^2 \\ &= (1-\gamma)\|x - x_0\|^2|u_1|[0,1] + \gamma\|x - x_0\|^2|u_2|[0,1] \subset R_+, \\ &(1-\gamma)P(x, u_1) + \gamma P(x, u_2) = P(x, u). \end{aligned}$$

Thus, $(iv)$ is fulfilled.

**Example 2** (Illustration of Theorem 3). Let $X = Y = Z = W = U = R$ and $K = M = R_+$. Let $\mathcal{F} : X \rightsquigarrow Y$, $v : X \to Y$, $w : U \to Y$ and $P : U \rightsquigarrow W$ be given multifunctions or (single-valued) mapping. Let $x_0 \in X$ and $u_0 \in U$. The problem under consideration is

$$\begin{aligned} &\min[(x - x_0)^2\mathcal{F}(u) + v(x) + w(u)], \\ &(x - x_0)^2(|u_1|[0,1] + u_0) \subset -R_+, \\ &0 \in (x - x_0)P(u). \end{aligned}$$

Let the following technical assumptions be satisfied.

(a) $\mathcal{F}(u)$ is bounded for all $u \in U$. $\mathcal{F}(.)$ has a section $f(.)$ such that $\sup\{|f(u)| : u \in N(u_0)\} := A$ is finite, where $N(u_0)$ is a neighborhood of $u_0$.

(b) $v(.)$ is differentiable at $x_0$ and $w(.)$ is linear.

(c) $1 \in P(u)$ for all $u \in U$ and $P(u_0) = \{1\}$.

Now we verify the assumptions of Theorem 3 with $f_0 = v(x_0) + w(u_0)$, $g_0 = 0$.

A direct calculation gives us the Clacke derivatives

$$D_xP(x_0, u_0; 0)\overline{x} = \{\overline{x}\}, \quad (43)$$
$$D_xF(x_0, u_0; f_0)\overline{x} = \{v'(x_0)\overline{x}\}.$$

for each $\overline{x} \in R$. By (43), $(i_1)$ is easy to be checked with the regular section $p(., u_0) = x - x_0$. Assumptions $(ii)$ and $(iii')$ for $G$ are shown to be satified in a manner similar to that of the Example in [14]. As for $F$, to consider $(ii)$ let $\overline{x} \in R$ and $\varepsilon > 0$. We have to show that for all sufficiently small $\gamma > 0$, and all $x$ near to $\overline{x}$,

$$\frac{1}{\gamma}(F(x_0 + \gamma x, u_0) - f_0) - v'(x_0)\overline{x} \subset (-\epsilon, \varepsilon) - R_+.$$

This is true by (a) and (b) since the left-hand side is

$$\gamma x^2\mathcal{F}(u_0) + \frac{v(x_0 + \gamma x) - v(x_0)}{\gamma} - v'(x_0)\overline{x}.$$

Now examine the uniform lower semicontinuity of $F(., u)$ stated in $(iii')$. For given $\varepsilon > 0$, we have

$$F(x, u) - F(x_0, u) = (x - x_0)^2\mathcal{F}(u) + v(x) - v(x_0) \subset (-\varepsilon, \varepsilon)$$

whenever $x$ is near to $x_0$ since $\mathcal{F}(u)$ is bounded and $v(.)$ is continuous. So $F(., u)$ is $R_+$-u.l.s.c. with $F(x_0, u)$ at $x_0$. Similarly, $F(., u_0)$ is $-R_+$-u.l.s.c. with $f_0$ at $x_0$.

Now pass to $(iv")$. We choose $\overline{p}(x, u) = x - x_0$. Let $u_1, ..., u_s$ and $\delta$ be given. Put

$$\begin{aligned} M &= \max\{|u_i|,\ i = 1, ..., s\},\\ Q &= \max\{A, 1, max\{|f(u_i)|,\ i = 0, ..., s\}\},\\ \varepsilon &= \min\Big\{\frac{\delta}{4QM}, \frac{\delta}{4Q}, 1\Big\},\\ V &= (x_0 - \varepsilon, x_0 + \varepsilon),\\ v(x, \alpha) &= u_0 + \sum_{j=1}^{s}\alpha_j(u_j - u_0), \end{aligned}$$

$e = 1$ and $q = 1$. Then, for $x, x' \in V$ we take

$$\begin{aligned} f_x^{u_i} &= (x - x_0)^2 f(u_i) + v(x) + w(u_i)\\ g_x^{u_i} &= (x - x_0)^2 u_0. \end{aligned}$$

Next, for all $\alpha, \alpha' \in \varepsilon\Sigma^s$ take

$$f_x = (x - x_0)^2 f(v(x, \alpha)) + v(x) + w(v(x, \alpha)).$$

Further more, in our case $\overline{p}' = x - x'$ and $\overline{p}^{u_i}_{x_0} = 0$. Then (23) is checked by the following estimation

$$
\begin{aligned}
& f^{u_0}_x + \sum_{i=1}^{s} \alpha_i \Big( f^{u_i}_x - f^{u_0}_x \Big) - f_x + \delta\Big(|x - x_0| + \sum_{i=1}^{s} \alpha_i\Big) \\
&= (x - x_0)^2 \Big( f(u_0) - f(u_0 + \sum_{i=1}^{s} \alpha_i (u_i - u_0)) \Big) \\
&\quad + (x - x_0)^2 \sum_{i=1}^{s} \alpha_i (f(u_i) - f(u_0)) + \delta\Big(|x - x_0| + \sum_{i=1}^{s} \alpha_i\Big) \\
&\geq -(x - x_0)^2 2Q - (x - x_0)^2 2Q \sum_{i=1}^{s} \alpha_i + \delta\Big(|x - x_0| + \sum_{i=1}^{s} \alpha_i\Big) \\
&\geq -\frac{\delta}{2}|x - x_0| - \frac{\delta}{2} \sum_{i=1}^{s} \alpha_i + \delta\Big(|x - x_0| + \sum_{i=1}^{s} \alpha_i\Big) \geq 0,
\end{aligned}
$$

Similarly as the Example in [15], condition (24) is satisfied. In turn (25) is clear for $\overline{p}(x, u) = x - x_0$.

## 4. Final remarks

The necessary optimality conditions of the Fritz John type obtained in [4], [17], [18], [19] for multifunction optimization need a crucial assumption that the ordering cone in the product of all the image spaces has nonempty interior because the main tool of the proof is a standard separation theorem. In our consideration the ordering cone in $W$ is $\{0\}$ since the optimization problem involves equality constraints. This leads to numerous applications in control problems.

The emptyness (together with the parameter involved) requires a complicated technical machine to overcome. The core of this machine is the Lusternik theorem. We restrict ourshelves to the case where the multifunction $P$ in the inclusion constraint has suitable (single-valued) sections. We think that, in order to omit this restriction a generalization of multifunctions of the Lusternik theorem is needed and general results of [7] may be concerned.

Another further consideration should be applications of the results presented here to control problems involving differential inclusions. Note that the corresponding results for the single-valued case have been succesfully applied to control problems involving diffrential equations.

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