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BOUNDEDNESS OF SYMMETRICALLY γ-CONVEX FUNCTIONS

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Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

Abstract. A function $f : D \to \mathbb{R}$ is said to be symmetrically γ-convex w.r.t. the roughness degree $\gamma > 0$ if the Jensen inequality

$$f(x_\lambda) \le (1-\lambda)f(x_0) + \lambda f(x_1), \quad x_\lambda := (1-\lambda)x_0 + \lambda x_1$$

is fulfilled for all x_0, $x_1 \in D$ satisfying $\|x_0 - x_1\| \ge \gamma$ and for

$$\lambda = \frac{\gamma}{\|x_1 - x_0\|} \quad \text{and} \quad \lambda = 1 - \frac{\gamma}{\|x_1 - x_0\|}\,.$$

Such a function also has some analytical properties which are similar to those of convex functions. For instance, if it is bounded above on some sphere $\{x \in X : \|x - x^*\| = \gamma\} \subset D$ then it is bounded on the ball $\overline{\mathcal{U}}_\gamma(x^*) := \{x \in X : \|x - x^*\| \le \gamma\}$ and bounded below on each bounded subset of D. If the domain D is so large that its interior contains some ball $\overline{\mathcal{U}}_\gamma(x^*)$, and if the symmetrically γ-convex function considered is locally bounded above at some interior point of D, then it is locally bounded in the interior of D.

1. Introduction

Let D be a nonempty convex subset of some normed space X. A function $f : D \to \mathbb{R}$ is said to be *convex* if the Jensen inequality

$$f(x_\lambda) \le (1-\lambda)f(x_0) + \lambda f(x_1), \quad x_\lambda := (1-\lambda)x_0 + \lambda x_1 \tag{1.1}$$

is fulfilled

$$\text{for all } \; x_0, x_1 \in D \; \text{ and } \; \text{for all } \; \lambda \in [0,1]. \tag{1.2}$$

One of the most interesting aspects of convex functions is that the algebraic condition (1.1)-(1.2) implies many nice analytical properties. For instance, if a convex function is locally bounded above at some interior point of D then it is locally Lipschitzian in $\operatorname{int} D$, and if $X = \mathbb{R}^n$ then it is differentiable almost everywhere in $\operatorname{int} D$ (see [9]).

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A natural question is if generalized convex functions still possess similar analytical properties. Our particular attention is paid to some kind of *rough convexities*, for which the Jensen inequality (1.1) is not required to be fulfilled for all $x_0, x_1 \in D$ like in (1.2) but only for all $x_0, x_1 \in D$ satisfying $\|x_0 - x_1\| \geq r$, where $r > 0$ is given and called as *roughness degree*. Such an investigation was already done by Hartwig [2] and Söllner [10] for ρ-*convex* functions which have to fulfill (1.1) for all $x_0, x_1 \in D$ satisfying $\|x_0 - x_1\| \geq r$ and for all $\lambda \in [0, 1]$ (or, with other words, for all $x_\lambda \in [x_0, x_1]$).

If (1.1) holds true for all $x_0, x_1 \in D$ satisfying $\|x_0 - x_1\| \geq r$ and for $x_\lambda \in [x_0, x_1]$ satisfying $\|x_\lambda - x_0\| \geq r/2$ and $\|x_\lambda - x_1\| \geq r/2$ then f is called δ-*convex*, as introduced by Hu, Klee and Larman in [3]. The boundedness and the continuity of such roughly convex functions were considered in [6].

In the following, let $\gamma > 0$ be a given roughness degree. For a pair of given points x_0 and x_1 in X, denote

$$
\begin{aligned}
x_0' &:= x_0 + \gamma \frac{x_1 - x_0}{\|x_1 - x_0\|} = \Big(1 - \frac{\gamma}{\|x_1 - x_0\|}\Big) x_0 + \frac{\gamma}{\|x_1 - x_0\|} x_1 \quad \text{and} \\
x_1' &:= x_1 - \gamma \frac{x_1 - x_0}{\|x_1 - x_0\|} = \frac{\gamma}{\|x_1 - x_0\|} x_0 + \Big(1 - \frac{\gamma}{\|x_1 - x_0\|}\Big) x_1.
\end{aligned} \tag{1.3}
$$

Due to (1.1), we have

$$
\begin{aligned}
x_0' = x_\lambda \quad &\text{for} \quad \lambda = \frac{\gamma}{\|x_1 - x_0\|} \quad \text{and} \\
x_1' = x_\lambda \quad &\text{for} \quad \lambda = 1 - \frac{\gamma}{\|x_1 - x_0\|}.
\end{aligned}
$$

As established in [4-5], $f : D \to \mathbb{R}$ is said to be γ-*convex* if

$$
f(x_0') + f(x_1') \leq f(x_0) + f(x_1) \quad \text{whenever} \quad \|x_0 - x_1\| \geq \gamma, \tag{1.4}
$$

which yields that (1.1) is satisfied at $x_\lambda = x_0'$ *or* at $x_\lambda = x_1'$. [7-8] showed that such roughly convex functions have some interesting analytical properties, but there exist γ-convex functions which are nowhere continuous and nowhere locally bounded.

To obtain analytical properties which are similar to those of convex functions, in [1] we defined a special class of γ-convex functions satisfying

$$
\begin{aligned}
& f(x_0') \leq \Big(1 - \frac{\gamma}{\|x_1 - x_0\|}\Big) f(x_0) + \frac{\gamma}{\|x_1 - x_0\|} f(x_1) \quad \text{and} \\
& f(x_1') \leq \frac{\gamma}{\|x_1 - x_0\|} f(x_0) + \Big(1 - \frac{\gamma}{\|x_1 - x_0\|}\Big) f(x_1) \\
& \text{whenever} \quad \|x_0 - x_1\| \geq \gamma.
\end{aligned} \tag{1.5}
$$

Obviously, (1.4) is fulfilled and (1.1) holds true at *both* $x_\lambda = x_0'$ *and* $x_\lambda = x_1'$. For that reason, such a function is said to be *symmetrically* γ-*convex*. It was shown in [1] that though symmetrically γ-convex functions only have to fulfill the Jensen inequality (1.1) at two points x_0' and x_1' (defined in (1.3)) in $[x_0, x_1]$ (satisfying

$\|x_0 - x_1\| \geq \gamma$), they already have some similar continuity properties as those of convex functions.

This paper continues the investigation in [1] and presents some sufficient conditions for the boundedness (in Section 2) and for the locally boundedness (in Section 3) of symmetrically γ-convex functions.

2. Sufficient conditions for boundedness

Let us denote

$$\begin{aligned} \mathcal{S}_r(x) &:= \{y \in X : \|y - x\| = r\}, \\ \mathcal{U}_r(x) &:= \{y \in X : \|y - x\| < r\}, \quad \text{and} \\ \overline{\mathcal{U}}_r(x) &:= \{y \in X : \|y - x\| \leq r\}. \end{aligned}$$

The following theorem states a sufficient condition for the boundedness of a symmetrically γ-convex function on some given subsets.

Theorem 2.1. *Suppose that $f : D \to \mathbb{R}$ is symmetrically γ-convex and bounded above on some sphere $\mathcal{S}_\gamma(x^*) \subset D$. Then*

(a) *f is bounded on the closed ball $\overline{\mathcal{U}}_\gamma(x^*)$ and*

$$\sup_{x \in \overline{\mathcal{U}}_\gamma(x^*)} f(x) = \sup_{x \in \mathcal{S}_\gamma(x^*)} f(x); \tag{2.1}$$

(b) *f is bounded below on each bounded subset of D.*

Proof. (a) Since f is bounded above on $\mathcal{S}_\gamma(x^*)$, we have

$$M := \sup_{x \in \mathcal{S}_\gamma(x^*)} f(x) < +\infty.$$

Assume that (2.1) is false, then there exists $x_0 \in \mathcal{U}_\gamma(x^*)$ such that

$$f(x_0) > M \geq f(x) \quad \text{for all} \quad x \in \mathcal{S}_\gamma(x^*). \tag{2.2}$$

If $\dim X \geq 2$ then there exist $x_1, x_2 \in \mathcal{S}_\gamma(x^*)$ such that $x_0 \in [x_1, x_2]$ and $\|x_0 - x_1\| = \gamma$. By the symmetrical γ-convexity of f, we get $f(x_0) \leq \max\{f(x_1), f(x_2)\}$, a contradiction. However, if $\dim X = 1$ the sphere $\mathcal{S}_\gamma(x^*)$ is disconnected, therefore such x_1 and x_2 do not exist. So we present here a proof which is valid in both cases.

For $x', x'' \in \mathcal{S}_\gamma(x^*)$ satisfying $x^* = (x' + x'')/2$, the symmetrical γ-convexity of f implies

$$f(x^*) \leq \frac{1}{2} f(x') + \frac{1}{2} f(x'') \leq M.$$

Hence $x_0 \neq x^*$ follows from (2.2). Let

$$x_1 := x_0 + \gamma \frac{x^* - x_0}{\|x^* - x_0\|}$$

then $\|x_1 - x_0\| = \gamma$ and

$$\|x_1 - x^*\| = \left\| \Big(\frac{\gamma}{\|x^* - x_0\|} - 1 \Big)(x^* - x_0) \right\| = \gamma - \|x^* - x_0\|,$$

i.e., $x_1 \in \mathcal{U}_\gamma(x^*) \setminus \{x^*\}$. Define

$$y := x^* - \frac{\gamma(x^* - x_0)}{\|x^* - x_0\|} \quad \text{and} \quad z := x^* + \frac{\gamma(x^* - x_0)}{\|x^* - x_0\|}$$

then $y, z \in \mathcal{S}_\gamma(x^*)$ and $x_0, x_1, x^* \in [y, z]$. Thus (2.2) implies $f(x_0) > f(y)$ and $f(x_0) > f(z)$. By the symmetrical γ-convexity of f, we get

$$\begin{aligned} f(x_0) &\le \frac{\gamma}{\gamma + \|x_0 - y\|} f(y) + \frac{\|x_0 - y\|}{\gamma + \|x_0 - y\|} f(x_1) \\ &< \frac{\gamma}{\gamma + \|x_0 - y\|} f(x_0) + \frac{\|x_0 - y\|}{\gamma + \|x_0 - y\|} f(x_1) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} f(x_1) &\le \frac{\gamma}{\gamma + \|x_1 - z\|} f(z) + \frac{\|x_1 - z\|}{\gamma + \|x_1 - z\|} f(x_0) \\ &< \frac{\gamma}{\gamma + \|x_1 - z\|} f(x_0) + \frac{\|x_1 - z\|}{\gamma + \|x_1 - z\|} f(x_0) \\ &= f(x_0). \end{aligned} \tag{2.4}$$

(2.3) implies $f(x_0) < f(x_1)$, which contradicts (2.4). Therefore, (2.1) must be true. Consequently, by using (b) which we are going to prove next, f is bounded on $\overline{\mathcal{U}}_\gamma(x^*)$.

(b) Assume that D' is some bounded subset of D and $x \in D' \setminus \{x^*\}$. Let

$$d := \sup_{x' \in D'} \|x^* - x'\| \quad \text{and} \quad y := x^* + \gamma \frac{x^* - x}{\|x^* - x\|},$$

then

$$d < +\infty \quad \text{and} \quad y \in S_\gamma(x^*).$$

Since f is symmetrically γ-convex and $x^* \in [x, y]$, we have

$$\begin{aligned} f(x^*) &\le \frac{\gamma}{\gamma + \|x^* - x\|} f(x) + \frac{\|x^* - x\|}{\gamma + \|x^* - x\|} f(y) \\ &\le \frac{\gamma}{\gamma + \|x^* - x\|} f(x) + \frac{\|x^* - x\|}{\gamma + \|x^* - x\|} M. \end{aligned}$$

Hence,

$$f(x) \ge \Big(1 + \frac{\|x^* - x\|}{\gamma}\Big) f(x^*) - \frac{\|x^* - x\|}{\gamma} M \ge -\Big(1 + \frac{d}{\gamma}\Big)|f(x^*)| - \frac{d}{\gamma} M.$$

Thus f is bounded below on D'. □

[1, Proposition 3.1] stated that if a symmetrically γ-convex function is bounded on some ball $\mathcal{U}_r(x^*) \subset D$ with $r > \gamma$ then it is locally Lipschitzian at x^*. Combining this and Theorem 2.1 we have

Corollary 2.1. *Suppose $f : D \subset X \to \mathbb{R}$ is symmetrically γ-convex. If f is bounded above on some ball $\mathcal{U}_r(x^*) \subset D$ with $r > \gamma$ then it is locally Lipschitzian at x^*.*

Let us consider the special case $X = \mathbb{R}$.

Corollary 2.2. *Suppose that $f : D \subset \mathbb{R} \to \mathbb{R}$ is symmetrically γ-convex.*

(a) *If $[a, b] \subset D$ and $b - a = 2\gamma$ then f is bounded on $[a, b]$ and*

$$f(x) \leq \max\{f(a), f(b)\} \quad \textit{for all} \ \ x \in [a, b].$$

(b) *If* diam $D > 2\gamma$ *then f is bounded on each closed interval of D.*

Proof. (a) Assume that $[a, b] \subset D$ and $b - a = 2\gamma$. Let $x^* = (a + b)/2$, then $\mathcal{S}_\gamma(x^*) = \{a, b\}$ and hence, f is bounded on $\mathcal{S}_\gamma(x^*)$. Theorem 2.1 yields that f is bounded on $\overline{\mathcal{U}}_\gamma(x^*) = [a, b]$ and

$$f(x) \leq \max\{f(a), f(b)\} \quad \text{for all} \quad x \in [a, b].$$

(b) Assume $[\alpha, \beta] \subset D$. If $\beta - \alpha \leq 2\gamma$ then since diam $D > 2\gamma$, there exists an interval $[a, b]$ such that $[\alpha, \beta] \subset [a, b] \subset D$ and $b - a = 2\gamma$. By (a), f is bounded on $[a, b]$ and so is f on $[\alpha, \beta]$. If $\beta - \alpha > 2\gamma$ we define

$$\alpha_i := \alpha + 2i\gamma \quad \text{for} \quad 0 \leq i \leq n := \left[\frac{\beta - \alpha}{2\gamma}\right] \quad \text{and} \quad \alpha_{n+1} := \beta$$

(where $[\sigma]$ denotes the integer part of the real number σ). Then $\alpha_{i+1} - \alpha_i \leq 2\gamma,\ 0 \leq i \leq n$. Since f is bounded on each interval $[\alpha_i, \alpha_{i+1}],\ 0 \leq i \leq n$, so is f on $[\alpha, \beta] = \cup_{i=0}^n [\alpha_i, \alpha_{i+1}]$. ☐

Note that the assertion (b) of the above corollary was stated in [1, Proposition 3.4] where the proof was more complicated.

The following examples show that the hypothesis of Corollary 2.2 cannot be weakened.

Example 2.1. Consider the function

$$f(x) := \begin{cases} 1/(1 + x) - \ln(1 + x) & \text{if} \quad x \in]-1, 0] \\ -\ln x & \text{if} \quad x \in]0, 1]. \end{cases}$$

We now show that f is symmetrically γ-convex for $\gamma = 1$. Assume $x_0, x_1 \in]-1, 1]$ and $x_1 - x_0 > 1$. Then

$$-1 < x_0 < x_1' := x_1 - 1 \leq 0 \quad \text{and} \quad x_1 > x_0' := x_0 + 1 > 0.$$

Therefore,

$$f(x_0) = \frac{1}{1+x_0} - \ln(1+x_0),\ \ f(x_0+1) = -\ln(1+x_0),$$
$$f(x_1) = -\ln x_1,\ \ f(x_1-1) = \frac{1}{x_1} - \ln x_1.$$

Since f is continuous on $[x_0+1, x_1] \subset]0,1]$ and differentiable on $]x_0+1, x_1[$, we have

$$\frac{f(x_1) - f(x_0+1)}{x_1 - x_0 - 1} = f'(\xi) = -\frac{1}{\xi} \quad \text{for some} \quad \xi \in]x_0+1, x_1[\ .$$

Consequently, it follows from

$$f(x_0+1) - f(x_0) = -\frac{1}{1+x_0} < -\frac{1}{\xi} \quad \text{for} \quad \xi > x_0+1$$

that

$$f(x_0+1) - f(x_0) < \frac{f(x_1) - f(x_0+1)}{x_1 - x_0 - 1},$$

which is equivalent to

$$f(x_0') = f(x_0+1) < \Big(1 - \frac{\gamma}{x_1 - x_0}\Big) f(x_0) + \frac{\gamma}{x_1 - x_0} f(x_1).$$

Analogously,

$$\frac{f(x_1-1) - f(x_0)}{x_1 - x_0 - 1} = f'(\theta) = -\frac{1}{(1+\theta)^2} - \frac{1}{1+\theta} \quad \text{for some} \quad \theta \in]x_0, x_1 - 1[$$

and

$$-\frac{1}{(1+\theta)^2} - \frac{1}{1+\theta} < -\frac{1}{x_1^2} - \frac{1}{x_1} < -\frac{1}{x_1} = f(x_1) - f(x_1-1) \text{ for } 0 < 1+\theta < x_1$$

imply

$$\frac{f(x_1-1) - f(x_0)}{x_1 - x_0 - 1} < f(x_1) - f(x_1-1),$$

which is equivalent to

$$f(x_1') = f(x_1-1) < \frac{\gamma}{x_1 - x_0} f(x_0) + \Big(1 - \frac{\gamma}{x_1 - x_0}\Big) f(x_1).$$

Hence, f is symmetrically γ-convex for $\gamma = 1$.

However, it is clear that diam$]-1,1] = 2 = 2\gamma$ but f is not bounded on $]-1,1]$. This example shows that the conclusion of Corollary 2.2 (a) does not hold any more if the closed interval [a,b] is replaced by $]a,b]$.

Example 2.2. Suppose $D = [a,b]$ and $0 < b-a < 2\gamma$. If $b-a \le \gamma$ then every function f on $[a,b]$ is symmetrically γ-convex. Hence we can say nothing about the boundedness of f. If $\gamma < b-a < 2\gamma$, consider a function f on $[a,b]$ satisfying

$$f(x) = 0 \quad \text{for} \quad x \in [a, b-\gamma] \cup [a+\gamma, b].$$

Suppose $x, y \in [a,b]$ and $x - y \geq \gamma$. Since

$$b - \gamma \geq x - \gamma \geq y \geq a \quad \text{and} \quad a + \gamma \leq y + \gamma \leq x \leq b,$$

we have

$$f(x) = f(x-\gamma) = f(y+\gamma) = f(y) = 0.$$

Thus f is symmetrically γ-convex on $[a,b]$. Since values of f on $]b-\gamma, a+\gamma[$ do not influence the symmetrical γ-convexity of f, f may be unbounded above and/or unbounded below on $[a,b]$. Hence, the assumption $\operatorname{diam} D > 2\gamma$ of Corollary 2.2 is really needed.

In addition to the above consideration, let us mention that if a convex function is bounded on an affine set then it is constant. This property remains true for symmetrically γ-convex functions.

Proposition 2.1. *If f is symmetrically γ-convex and bounded above on an affine set A then f is constant on A.*

Proof. It is worth noting that the assertion of this proposition is also valid for γ-convex functions but only if $\dim A \geq 2$ (see [7]). Moreover, Proposition 2.1 is also an easy corollary of [7, Theorem 3.5] and [1, Proposition 2.4]. However, we want to present here a direct proof.

Assume the contrary that there exist $x, y \in A$ such that $f(x) < f(y)$. Let

$$s := \frac{y-x}{\|y-x\|} \quad \text{and} \quad z := y + \gamma s.$$

Then

$$f(y) \leq \frac{\gamma}{\gamma + \|y-x\|} f(x) + \frac{\|y-x\|}{\gamma + \|y-x\|} f(z)$$

and $f(x) < f(y)$ imply $f(y) < f(z)$. Therefore,

$$f(z) \leq \Big(1 - \frac{\gamma}{t}\Big) f(y) + \frac{\gamma}{t} f(y+ts) \quad \text{for} \quad t > \gamma$$

yields

$$f(y+ts) \geq \frac{t}{\gamma}\left(f(z) - f(y)\right) + f(y) \to +\infty \quad \text{as} \quad t \to +\infty,$$

which contradicts the boundedness above of f on A. The proof is complete. □

3. Sufficient conditions for locally boundedness

As mentioned in Section 1, if a convex function is locally bounded above at some interior point of D then it is locally Lipschitzian in $\operatorname{int} D$, that means at least: it is locally bounded there. The next theorem states a similar property of symmetrically γ-convex functions defined on some convex domain which is so large such that it contains some ball of radius greater than γ.

Theorem 3.1. *Suppose that a symmetrically γ-convex function $f : D \to \mathbb{R}$ is locally bounded above at some $y \in \operatorname{int} D$. Then the following assertions hold true.*

(a) *f is locally bounded at each interior point x of D satisfying $\|x-y\|/\gamma \in \mathbb{N}$.*

(b) *If $\operatorname{int} D$ contains some ball $\overline{\mathcal{U}}_\gamma(x^*)$ then f is locally bounded in $\operatorname{int} D$.*

Proof. (a) Assume that $x \in \operatorname{int} D$, $\|x-y\| = \gamma$, and f is locally bounded above at $y \in \operatorname{int} D$. Then there exist two positive numbers ρ and K satisfying

$$\overline{\mathcal{U}}_\rho(x) \subset D, \quad \overline{\mathcal{U}}_\rho(y) \subset D, \text{ and } \sup_{y' \in \overline{\mathcal{U}}_\rho(y)} f(y') \le K.$$

[1, Lemma 3.1] shows that there are two points

$$x', x'' \in D, \quad x' := x - \rho\frac{x-y}{\|x-y\|}, \quad x'' := x + \rho\frac{x-y}{\|x-y\|}$$

and a ball $\overline{\mathcal{U}}_\sigma(x) \subset \overline{\mathcal{U}}_\rho(x)$ such that for each $z \in \overline{\mathcal{U}}_\sigma(x)$, both

$$z' := x' - \gamma\frac{z-x'}{\|z-x'\|} \quad \text{and} \quad z'' := z + \gamma\frac{z-x''}{\|z-x''\|}$$

are in $\overline{\mathcal{U}}_\rho(y)$. Without loss of generality we can assume

$$K \ge \max\{|f(x')|,\ |f(x'')|\}.$$

Since f is symmetrically γ-convex, it holds for each $z \in \overline{\mathcal{U}}_\sigma(x)$

$$\begin{aligned} f(z) &\le \Big(1 - \frac{\|z-x''\|}{\gamma + \|z-x''\|}\Big) f(x'') + \frac{\|z-x''\|}{\gamma+\|z-x''\|} f(z'') \\ &\le \max\{f(x''), f(z'')\} \le K \end{aligned}$$

and

$$-K \le f(x') \le (1-\mu)f(z) + \mu f(z') \le (1-\mu)f(z) + \mu K \text{ for } \mu = \frac{\|z-x'\|}{\gamma + \|z-x'\|}\,.$$

By

$$\|z-x'\| \le \|z-x\| + \|x-x'\| \le \sigma + \rho \le 2\rho$$

it follows that

$$f(z) \ge -\frac{1+\mu}{1-\mu}K = -\big(1 + \frac{2\|z-x'\|}{\gamma}\big)K \ge -\Big(1+\frac{4\rho}{\gamma}\Big)K$$

Thus f is locally bounded at x.

If $\|x-y\| = n\gamma$ for some $n \in \mathbb{N}$ then $x, y \in \operatorname{int} D$ implies

$$y_i := y + i\gamma\frac{x-y}{\|x-y\|} \in \operatorname{int} D \quad \text{for} \quad 1 \le i \le n$$

(see [11, Theorem 2.23]) and $x = y_n$. By the proof above, f is locally bounded at y_1 and hence, locally bounded at y_2, y_3,... Finally, f is locally bounded at $x = y_n$.
(b) If $\dim X = 1$ then the assertion follows immediately from Corollary 2.2. Assume now $\dim X \ge 2$. Since the sphere $\mathcal{S}_\gamma(x^*)$ is connected, there is a $y' \in \mathcal{S}_\gamma(x^*)$ such that $\|y-y'\|/\gamma \in \mathbb{N}$. By (a), f is locally bounded at y'. Thus f

is locally bounded at x^* and therefore, so is f at each $z \in \mathcal{S}_\gamma(x^*)$. Finally, if $x \in \operatorname{int} D$, there exists $z \in \mathcal{S}_\gamma(x^*)$ satisfying $\|x - z\|/\gamma \in \mathbb{N}$. Consequently, f is locally bounded at x. The proof is complete. □

It is easy to show that the assertion (a) in Theorem 3.1 is false for γ-convex functions, but a similar result was also stated in [1].

In a finite dimensional space X, a symmetrically γ-convex function $f : D \subset X \to \mathbb{R}$ is locally bounded in

$$\operatorname{int}_\gamma D := \{x \in D : \text{ there exists } r = r(x) > \gamma \text{ such that } \mathcal{U}_r(x) \subset D\}$$

(see [1]). Therefore, Theorem 3.1 yields that f is locally bounded in $\operatorname{int} D$ if $\operatorname{int}_\gamma D \neq 0$. In fact, we can obtain a stronger result, namely

Proposition 3.1. *If* $\dim X < +\infty$, $f : D \subset X \to \mathbb{R}$ *is symmetrically* γ*-convex and if* $\operatorname{int}_\gamma D \neq \emptyset$ *then* f *is bounded on each compact subset of* $\operatorname{int} D$.

Proof. Suppose that $K \subset \operatorname{int} D$. For each $x \in K$, there exists an open ball $\mathcal{U}(x)$ centered at x such that f is bounded on $\mathcal{U}(x)$. If K is compact then there exist finite balls $\mathcal{U}(x_i),\ i = 1, 2, ..., n$ which form a covering of K. Thus, for very $x \in K$,

$$|f(x)| \leq M := \max_{1 \leq i \leq n} \sup \{|f(y)| : y \in \mathcal{U}(x_i)\} < +\infty,$$

i.e., f is bounded on K. □

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