ACTA MATHEMATICA VIETNAMICA
Volume 26, Number 2, 2001, pp. 231-239

SOME REMARKS ON FIXED POINTS

DO HONG TAN AND HA DUC VUONG

Abstract. In this note we establish two independent results on fixed points. The first one is about the continuity of fixed points of limit-compact mappings introduced by Sadovskii in [7]. This result partially generalizes a result of Tan in [10] for collectively condensing mappings. The second result is a new fixed point theorem for the sum of a generalized contraction and a compact mapping, which improves a well-known result of Krasnoselskii [4]. A probabilistic version of this result is also presented here.

1. Preliminaries

The notions of condensing and limit-compact mappings were introduced by Sadovskii in [6, 7] and studied then by many authors. On the other hand, in [9, 10] Tan has proved the continuity of fixed points of singlevalued and multivalued collectively condensing mappings. In Section 2 we partially generalize a result in [10] for limit-compact mappings. For convenience to the readers, before stating the result, we recall some definitions that we shall use below.

Definition 1. Let X be a locally convex space, M a subset of X and φ the Kuratowski or Hausdorff measure of noncompactness on X. A mapping $T : M \to X$ (or 2^X) is called condensing [6] if for each bounded but not relatively compact subset A of M we have

$$\varphi(T(A)) < \varphi(A). \tag{1}$$

Let Λ be an arbitrary nonempty set and X, M, A, φ be as above. A mapping $T : \Lambda \times M \to X$ (or 2^X) is called collectively condensing [5] if instead of (1) we have

$$\varphi(T(\Lambda \times A)) < \varphi(A).$$

Clearly, if Λ consists of only one element then a collectively condensing mapping becomes condensing.

Definition 2. Let X, M, Λ be as in Definition 1 and $T : \Lambda \times M \to X$ (or 2^X) a mapping. We construct a transfinite sequence of subsets $\{M_\alpha\}$ of X as follows:

Received May 8, 2000.
This work was partially supported by the National Basic Research Program of Vietnam.

Put

$$\begin{aligned} M_0 &= \overline{\mathrm{co}}\, T(\Lambda \times M) & \\ M_\alpha &= \overline{\mathrm{co}}\, T(\Lambda \times (M \cap M_{\alpha-1})) & \text{if } \alpha - 1 \text{ exists,} \\ M_\alpha &= \bigcap_{\beta<\alpha} M_\beta & \text{if } \alpha - 1 \text{ does not exist,} \end{aligned}$$

where $\overline{\mathrm{co}}$ denotes the closure of the convex hull of a set.

It was shown in [7] that there always exists a transfinite number δ such that $M_\alpha = M_\delta$ for all $\alpha \geq \delta$. The set M_δ is called the limit range of T and denoted by $T^\infty(\Lambda \times M)$.

The mapping T is called limit-compact if the restriction of T on $\Lambda \times (M \cap T^\infty(\Lambda \times M))$ is a compact mapping, i.e. if the set $T(\Lambda \times (M \cap T^\infty(\Lambda \times M)))$ is relatively compact (in particular, if $T^\infty(\Lambda \times M) = \emptyset$).

It was also shown in [7] that each continuous collectively condensing mapping T is limit-compact if M is closed and Λ is a compact space because in this case we have that $T^\infty(\Lambda \times M)$ is compact. If in addition, the space X is complete then we also have $T^\infty(\Lambda \times M) \neq \emptyset$.

If Λ consists of only one element then the class of limit-compact mappings contains the class of condensing mappings, and the latter contains the class of compact mappings.

Definition 3. Let X, Y be two topological spaces and $T : X \to 2^Y$ a multivalued mapping. The domain and the graph of T are defined respectively as follows:

$$\begin{aligned} \mathrm{dom}\, T &= \{x \in X : Tx \neq \emptyset\}, \\ \mathrm{graph}\, T &= \{(x, y) \in X \times Y : x \in \mathrm{dom}\, T, y \in Tx\}. \end{aligned}$$

The mapping T is called upper semicontinuous (ore *usc*, for short) at $x_0 \in \mathrm{dom}\, T$ if for every open set G of Y containing Tx_0 there exists a neighborhood U of x_0 such that $T(U \cap \mathrm{dom}\, T) \subset G$. If T is *usc* at every point in $\mathrm{dom}\, T$, we say that T is *usc*. For singlevalued mappings the notion of upper semicontinuity coincides with that of continuity. The image of a compact set under an *usc* multivalued mapping with compact values remains compact.

The mapping T is called closed if its graph is closed in $X \times Y$. For details about multivalued mappings, see [1].

2. The continuity of fixed points of limit-compact mappings

Our first result can be stated as follows:

Theorem 1. *Let Λ be a topological space, X a locally convex space, M a subset of X, $T : M \to 2^X$ a closed limit-compact multivalued mapping. For each $\lambda \in \Lambda$ we set*

$$F(\lambda) = \{x \in M : x \in T(\lambda, x)\}.$$

Then the mapping $F : \Lambda \to 2^M$ *is usc on* $\operatorname{dom} F$.

Proof. Denoting $\operatorname{Fix}(T) = \bigcup_{\lambda\in\Lambda} F(\lambda)$, we shall prove that

$$\operatorname{Fix}(T) \subset T(\Lambda \times (M \cap T^\infty(\Lambda \times M))). \tag{2}$$

By definitions of $\operatorname{Fix}(T)$ and $F(\lambda)$ we have

$$x \in \operatorname{Fix}(T) \Leftrightarrow x \in \bigcup_{\lambda\in\Lambda} F(\lambda) \Leftrightarrow \exists\, \lambda \in \Lambda \text{ such that } x \in T(\lambda, x).$$

Hence $x \in \operatorname{Fix}(T)$ implies $x \in \overline{\operatorname{co}}\, T(\lambda \times M) = M_0$, so we get $\operatorname{Fix}(T) \subset M_0$. We shall prove (2) by induction on α.

Suppose that $\alpha - 1$ exists and $\operatorname{Fix}(T) \subset M_{\alpha-1}$. Take any $x \in \operatorname{Fix}(T)$, then $x \in M$ and $x \in M_{\alpha-1}$. Since $x \in T(\lambda, x)$ for some λ, we get

$$x \in \overline{\operatorname{co}}\, T(\Lambda \times (M \cap M_{\alpha-1})) = M_\alpha,$$

so $\operatorname{Fix}(T) \subset M_\alpha$.

Now suppose that $\alpha-1$ does not exist and $\operatorname{Fix}(T) \subset M_\beta$ for every $\beta < \alpha$. Then $\operatorname{Fix}(T) \subset \bigcap_{\beta<\alpha} M_\beta = M_\alpha$. By induction we get $\operatorname{Fix}(T) \subset M_\alpha$ for all α, hence $\operatorname{Fix}(T) \subset M_\delta = T^\infty(\Lambda \times M)$ (see Definition 2). Since we have also $\operatorname{Fix}(T) \subset M$, from this we get $\operatorname{Fix}(T) \subset M \cap T^\infty(\Lambda \times M)$. Take any $x \in \operatorname{Fix}(T)$ then there exists $\lambda \in \Lambda$ such that $x \in T(\lambda, x)$. This implies that

$$x \in T(\Lambda \times \operatorname{Fix}(T)) \subset T(\Lambda \times (M \cap T^\infty(\Lambda \times M))),$$

from which this we get (2).

From (2) it follows that $\operatorname{Fix}(T)$ is relatively compact by limit compactness of T. We now prove the upper semicontinuity of F. Suppose on the contrary that F is not *usc* at some point $\lambda_0 \in \operatorname{dom} F$. Then there exists an open set G containing $F(\lambda_0)$ such that for every neighborhood U of λ_0 there are $\lambda \in U \cap \operatorname{dom} F$ and $x \in F(\lambda) \setminus G$.

Ordering the family of all neighborhoods $\{U_\gamma\}$ of λ_0 by inclusion, we get two nets $\{x_\gamma\} \subset M$ and $\{\lambda_\gamma\} \subset \operatorname{dom} F$ such that $\lambda_\gamma \to \lambda_0$ and $x_\gamma \in F(\lambda_\gamma) \setminus G$ for each γ. This means that $x_\gamma \in T(\lambda_\gamma, x_\gamma)$ and $x_\gamma \notin G$.

Denoting $B = \{x_\gamma\}$ we have $B \subset \operatorname{Fix}(T)$, hence B is relatively compact. Then there exists a subset, denoted again by $\{x_\gamma\}$, which converges to some point $x_0 \in X$. By closedness of T we get $x_0 \in M$ and $x_0 \in T(\lambda_0, x_0)$, this implies $x_0 \in F(\lambda_0) \subset G$.

On the other hand, $x_0 \notin G$ because $x_\gamma \notin G$ for each γ. This contradiction proves the theorem. $\square$

Remark 1. The theorem partially improves a result in [10] when Λ is compact and M is closed because in this case each *usc* collectively condensing mapping is limit-compact.

3. A fixed point theorem of Krasnoselskii type

In this section we establish a new fixed point theorem for the sum of a generalized contraction and a compact mapping. First we state the following:

Definition 4. A mapping T of a metric space (M, d) into itself is called φ-contractive if there exists an upper semicontinuous from the right function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ for $t > 0$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

for every $x, y \in M$.

In the sequel we shall be concerned with such functions φ that satisfy one of (or both) the following conditions:

Condition A. If $t - \varphi(t) \to 0$ then $t \to 0$.

This means that the inverse of the function $\psi(t) = t - \varphi(t)$ exists in a neighborhood of 0 and is continuous at 0.

Condition B. $\lim\limits_{t \to 0} \dfrac{\varphi(t)}{t} = k < 1$.

This means that for every $k' > k$ there exists $\varepsilon > 0$ such that if $t \leq \varepsilon$ then $\varphi(t) \leq k't$. This implies that φ is continuous at 0.

In what follows we always take $k' = \dfrac{1+k}{2}$ and fix an ε corresponding to this k'.

Before stating a fixed point theorem we prove some lemmas.

Lemma 1. *Let $B = B(x_0, r)$ be an open ball in a complete metric space (M, d) and $T : B \to M$ a φ-contractive mapping with φ satisfying condition B.*

If $d(Tx_0, x_0) \leq (1 - k')\varepsilon$ with k', ε defined above and $\varepsilon < r$, then T has a fixed point in B.

Proof. For each $x \in \overline{B}(x_0, \varepsilon)$, the closure of $B(x_0, \varepsilon)$, we have

$$\begin{aligned} d(Tx, x_0) &\leq d(Tx, Tx_0) + d(Tx_0, x_0) \\ &\leq \varphi(d(x, x_0)) + (1 - k')\varepsilon \\ &\leq k'\varepsilon + (1 - k')\varepsilon = \varepsilon. \end{aligned}$$

Hence T maps $\overline{B}(x_0, \varepsilon)$ into itself. Since $\overline{B}(x_0, \varepsilon)$ is complete and T is φ-contractive on $\overline{B}(x_0, \varepsilon)$, by a result of Boyd and Wong in [2], T has a unique fixed point in $\overline{B}(x_0, \varepsilon) \subset B$. □

Lemma 2. (Invariance of domain for φ-contractive fields) *Let U be an open set in a Banach space $(X, \|\cdot\|)$, and $T : U \to X$ a φ-contractive mapping with φ satisfying condition B. Then the mapping $H = I - T : U \to H(U)$ is a homeomorphism, where I denotes the identity in X.*

Proof. First we prove that H is an open mapping. For this it suffices to show that for every ball $B(x_0, r) \subset U$ we have

$$B(Hx_0, (1-k')\varepsilon) \subset H(B(x_0, r))$$

with k', ε defined as in Lemma 1. Take any $y \in B(Hx_0, (1-k')\varepsilon)$ we must find an $x \in B(x_0, r)$ such that $Hx = y$.

Define a mapping $G : B(x_0, r) \to X$ by putting $Gz = y + Tz$ for $z \in B(x_0, r)$. Since T is φ-contractive, so is G. Moreover, we have

$$\|Gx_0 - x_0\| = \|y + Tx_0 - x_0\| = \|y - Hx_0\| \leq (1-k')\varepsilon.$$

By Lemma 1, G has a fixed point $x \in B(x_0, r)$. Then $x = Gx = y + Tx$, hence $y = Hx$ as claimed.

Further, for every $x, y \in U$ we have

$$\begin{aligned} \|Hx - Hy\| = \|x - Tx - y + Ty\| &\geq \|x - y\| - \|Tx - Ty\| \\ &\geq \|x - y\| - \varphi(\|x - y\|). \end{aligned}$$

Since $\varphi(t) = t$ only if $t = 0$, from this we see that H is injective. Being an open mapping, H is a homeomorphism between U and $H(U)$. The lemma is proved. □

In particular, for $U = X$ we obtain

Corollary 1. *If T is a φ-contractive mapping on a Banach space X then $I - T$ is a homeomorphism on X.*

Lemma 3. *Let X be a Banach space, $T : X \to X$ be a φ-contractive mapping with φ satisfying condition A, $S : X \to X$ be a continuous mapping. Then for each $y \in X$ the mapping $F_y x = Tx + Sy$ has a unique fixed point x_y which depends continuously on y.*

Proof. Since T is φ-contractive, so is F_y. Hence F_y has a unique fixed point x_y for each $y \in X$. Moreover, for every y, $y' \in X$ we have

$$\begin{aligned} \|x_y - x_{y'}\| &= \|Tx_y + Sy - Tx_{y'} - Sy'\| \\ &\leq \|Tx_y - Tx_{y'}\| + \|Sy - Sy'\| \\ &\leq \varphi(\|x_y - x_{y'}\|) + \|Sy - Sy'\|. \end{aligned}$$

Hence

$$\|x_y - x_{y'}\| - \varphi(\|x_y - x_{y'}\|) \leq \|Sy - Sy'\|.$$

By continuity of S and condition A of φ we get the continuity of x_y in y, this proves the lemma. □

Now we are able to state our second result of this note.

Theorem 2. *Let X be a Banach space, $T : X \to X$ be a φ-contractive mapping with φ satisfying conditions A and B, $S : X \to X$ be a compact mapping, C be a nonempty convex closed bounded subset of X such that $T(C) + S(C) \subset C$. Then $T + S$ has a fixed point in C.*

Proof. For each $y \in C$ we define a mapping $F_y : C \to C$ by putting $F_y x = Tx + Sy$. By Lemma 3, F_y has a unique fixed point x_y and the mapping $K : C \to C$ defined by $Ky = x_y$ is continuous. Moreover, since

$$x_y = F_y x_y = Tx_y + Sy,$$

by Corollary of Lemma 2 we have

$$Ky = x_y = (I - T)^{-1} Sy \subset (I - T)^{-1} S(C),$$

hence $K(C) \subset (I - T)^{-1} S(C)$. Since S is compact and $(I - T)^{-1}$ is continuous, K is a compact mapping in a convex closed bounded subset of X. By the well-known Schauder fixed point principle [8], K has a fixed point $y^* \in C$. Thus, we have

$$y^* = Ky^* = x_{y^*} = Tx_{y^*} + Sy^* = Ty^* + Sy^*$$

and the theorem is proved. □

Remark 2. Since $\varphi(t) = kt$ with $k < 1$ is continuous and satisfies conditions A and B, Theorem 2 improves a result of Krasnoselskii [4]. Moreover, since each Banach contraction is condensing, the mentioned Krasnoselskii theorem can be deduced from Sadovskii's theorem [6] for condensing mappings. But this does not work in our setting, because a φ-contractive mapping needs not be condensing.

Lemma 3 itself is also a new result on the continuity of fixed points.

Our next aim is to establish a probabilistic version of Theorem 2. For this we need to extend the theorem to locally convex spaces. So, let us consider a Hausdorff complete locally convex space (X, P) with a family $P = \{p_i : i \in I\}$ of seminorm. On X we consider a φ-contractive mapping T, i. e., for every $x, y \in X$ and $i \in I$ we have

$$p_i(Tx - Ty) \le \varphi(p_i(x - y)),$$

where φ is the function described in Definition 4.

For a fixed finite subset J of I, $x_0 \in X$ and $r > 0$ we set

$$B_J = B_J(x_0, r) = \bigcap_{i \in J} \big\{ x \in X : p_i(x - x_0) < r \big\}.$$

Since B_J belongs to the basis of neighborhoods of x_0, we can use it (instead of $B(x_0, r)$ in the proofs of Lemmas 1 and 2) to get analogous results for locally convex spaces using a modified version of Boyd-Wong's result for such spaces. A result similar to Lemma 3 can also be proved in the same way.

From the above observations, using Tychonoff's fixed point theorem [11] instead of Schauder's one, we can state the following result which will be used in the next section.

Remark 3. Theorem 2 can be extended to Hausdorff complete locally convex spaces.

4. Application to probabilistic Banach spaces

Let us first recall some definitions and facts on probabilistic Banach spaces.

Definition 5 [3]. A triplet $(X, \mathcal{F}, \min)$ is called a probabilistic normed space (PN-space, for short) if X is real vector space, $\mathcal{F} = \{F_x : x \in X\}$ a family of distribution functions $F_x : [0,1] \to R$ satisfying

$$F_x(0) = 0 \text{ for every } x \in X,$$
$$F_x(t) = 1 \text{ for every } t > 0 \text{ if and only if } x = 0,$$
$$F_{\alpha x}(t) = F_x\Big(\frac{t}{|\alpha|}\Big) \text{ for every } \alpha \in R \setminus \{0\} \text{ and } x \in X,$$
$$F_{x+y}(t+s) \geq \min\{F_x(t), F_y(s)\} \text{ for every } x, y \in X \text{ and } t, s \geq 0.$$

The topology in X is defined by a family of neighborhoods of 0 as follows

$$U(0; \varepsilon, \lambda) = \big\{x \in X : F_x(\varepsilon) > 1 - \lambda\big\} \quad \text{for } \varepsilon > 0,\ \lambda \in (0,1),$$

or equivalently, by the family of seminorms

$$p_\lambda(x) = \sup\big\{t \in R : F_x(t) \leq 1 - \lambda\big\} \quad \text{for } \lambda \in (0,1).$$

From this, by the left-continuity of F_x we get

$$F_x(p_\lambda(x)) \leq 1 - \lambda \tag{3}$$

and

$$t > p_\lambda(x) \quad \text{implies } F_x(t) > 1 - \lambda. \tag{4}$$

So each PN-space $(X, \mathcal{F}, \min)$ can be associated to a Hausdorff locally convex space $(X, p_\lambda : \lambda \in (0,1))$ with the same topology. In particular, a sequence $\{x_n\} \subset X$ converges to x if for each $\lambda \in (0,1)$, $p_\lambda(x_n - x) \to 0$ as $n \to \infty$, a sequence $\{x_n\}$ is a Cauchy sequence if for each λ, $p_\lambda(x_n - x_m) \to 0$ as $n, m \to \infty$. The space X is said to be complete if each Cauchy sequence converges to some point in X.

A complete PN-space is called a probabilistic Banach space.

Definition 6. A mapping T from a probabilistic Banach space X into itself is called a probabilistic φ-contractive mapping if there exists a strictly increasing continuous function $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\varphi(t) < t$ for $t > 0$ such that for every $x, y \in X$ we have

$$F_{Tx-Ty}(\varphi(t)) \geq F_{x-y}(t). \tag{5}$$

Proposition 1. *Each probabilistic φ-contractive mapping in* $(X, \mathcal{F}, \min)$ *is φ-contractive in the corresponding space* $(X, p_\lambda : \lambda \in (0,1))$

Proof. Suppose T is probabilistic φ-contractive and assume on the contrary that it is not φ-contractive, that is there exist $\lambda \in (0,1)$ and $x, y \in X$ such that

$$p_\lambda(Tx - Ty) > \varphi(p_\lambda(x - y)).$$

Since φ is strictly increasing and continuous, it is invertible and φ^{-1} is also strictly increasing. Then we have

$$\varphi^{-1}(p_\lambda(Tx - Ty)) > p_\lambda(x - y).$$

Denoting $t = \varphi^{-1}(p_\lambda(Tx - Ty))$ we have $t > p_\lambda(x - y)$. From (3) and (4) we get respectively

$$F_{Tx-Ty}(\varphi(t)) = F_{Tx-Ty}(p_\lambda(Tx - Ty)) \leq 1 - \lambda$$

and

$$F_{x-y}(t) > 1 - \lambda,$$

contradicting (5). Thus the proposition is proved. □

As a direct consequence of Remark 3 and the above proposition we obtain

Theorem 3. *Let X be a probabilistic Banach space, $T : X \to X$ be a probabilistic φ-contractive mapping with φ satisfying condition* A *and* B*, $S : X \to X$ be a compact mapping, C be a convex closed bounded subset of X such that $T(C) + S(C) \subset C$. Then $T + S$ has a fixed point in C.*

The readers are kindly asked to compare this result to a similar result due to Chang et al., [3, Theorem 2].

Remark 4. There are a lot of functions $\varphi(t)$ different from kt which satisfy Conditions A, B and conditions mentioned in Definitions 4 and 6.

For example,

$$\varphi(t) = \begin{cases} kt, & 0 \leq t \leq 1, \\ t - \dfrac{1-k}{t}, & 1 < t < \infty, \end{cases}$$

with $k < 1$.

References

[1] C. Berge, *Espaces Topologiques, Fonctions Multivoques*, Paris, 1959.

[2] D. Boyd and J. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 475-482.

[3] S. S. Chang, B. S. Lee, Y. J. Cho, Y. Q. Chen, S. M. Kang, and J. S. Jung, *Generalized contraction mapping principle and differential equations in probabilistic metric spaces*, Proc. Amer. Math. Soc. **124** (1996), N. 8, 2367-2376.

[4] M. A. Krasnoselskii, *Two remarks on the method of successive approximations* (in Russian), Uspekhi Mat. Nauk **10** (1955), N. 1, 123-127.

[5] Martin Fan Cheng, *On the continuity of fixed points of collectively condensing maps*, Proc. Amer. Math. Soc. **63** (1977), 74-76.

[6] B. N. Sadovskii, *On a fixed point principle* (in Russian), Funk. Anal. Priloz. **1** (1967), N. 2, 74-76.

[7] B. N. Sadovskii, *Limit-compact and condensing operators* (in Russian), Uspekhi Mat. Nauk **27** (1972), N. 1, 81-146.

[8] J. Schauder, *Der Fixpunktsatz in Funktionalraumen*, Studia Math. **2** (1930), 171-180.

[9] D. H. Tan, *On continuity of fixed points*, Bull. Polon. Acad. Sci. Math. **31** (1983), N. 5-8, 299-301.

[10] D. H. Tan, *On the continuity of fixed points of multivalued collectively condensing mappings*, Indian J. Pure Appl. Math. **15** (1984), N. 6, 631-632.
[11] A. Tychonoff, *Ein Fixpunktsatz*, Math. Ann. **111** (1935), 767-776.

Institute of Mathematics
P.O.Box 631, BoHo, Hanoi, Vietnam

Office of Education and Training
Hanam province, Vietnam