Volume 26, Number 2, 2001, pp. 219-230

ON SOME HEREDITARY PROPERTIES BETWEEN I AND IN(I)

HENRIK BRESINSKY AND LÊ TUÂN HOA

1. Introduction

Let $S = K[x_1, ..., x_n]$ be the polynomial ring in n variables over an infinite field K and let $I \subseteq S$ be a homogeneous polynomial ideal. Given an admissible term order on the terms of S, let $in_<(I)$ (or $in(I)$ if the term order is either immaterial or clear from the context) denote the ideal generated by the initial terms of elements in the ideal I. It is well known that certain properties of the ring S/I may be deduced from the ring $S/in(I)$ and conversely, sometimes with respect to certain specified term orders (see for instance Chapter 15 in [5]). The most fundamental result in this respect is perhaps Macaulay's Theorem linking the Hilbert function of a homogeneous ideal I to $in(I)$. This way certain questions about ideals in polynomial rings can be reduced to questions about monomial ideals.

In this paper we examine the properties of being Cohen-Macaulay (C.M.), generalized Cohen-Macaulay (generalized C.M.) and Buchsbaum in the above setting. It is well known that the C.M. and generalized C.M. properties for S/I are inherited from $S/in(I)$ (see, e.g., [7]). For the C.M. property it was shown by Bayer and Stillman in [2], that in generic coordinates the converse is also true for reverse lexicographic (rev. lex.) term orders. In the first part of this paper we show that, in a certain sense to be explained, the rev. lex. term order is the unique term order with this property. This once more points out the specific role of the rev. lex. term order as already noted in [2] and [5]. In contrast to the first part, we show in the second part, that there is no change of coordinates and no term order such that the generalized C.M. property of $S/in(I)$ follows hereditarily from S/I. In the last part we show that not even the Buchsbaum property of S/I follows hereditarily from $S/in(I)$.

2. The Cohen-Macaulay property

Throughout this paper let S be as specified in the introduction. When conve-

Received April 20, 2000.

1991 *Mathematics Subject Classification.* 13C05, 13A15.

Key words and phrases. Cohen-Macaulay, Buchsbaum, generalized Cohen-Macaulay, initial ideal.

The second author is partially supported by the National Basic Research Program (Vietnam).

nient we will denote the term $x_1^{a_1}...x_n^{a_n} \in S$, where $a = (a_1, ..., a_n)$ is an n-tuple of nonnegative integers, by x^a. An admissible term order $<$ on the terms of S is a linear order of the terms such that 1 is the unique minimal element, and $m_1 < m_2$ for terms $m_1,\ m_2$ implies $mm_1 < mm_2$ for any term m. We will consider only graded (admissible) term orders, i.e. $m_1 < m_2$ if $\deg(m_1) < \deg(m_2)$, where deg denotes the usual polynomial degree ($\deg(x_i) = 1,\ 1 \leq i \leq n$). As pointed out in the introduction the rev. lex. order plays a special role among graded term order. For the convennience of the reader we recall its definition here.

Definition 2.1. For the rev. lex. term order $x^a > x^b$ if either

(i) $\deg(x^a) > \deg(x^b)$ or

(ii) $\deg(x^a) = \deg(x^b)$ and the last nonzero entry of the vector $a - b$ is negative.

Note that by Definition 2.1 $x_1 > \cdots > x_n$ and Definition 2.1 must be modified if one changes the order of the variables.

For $0 \neq F \in S$, let $in(F)$ be the largest term resulting from a nonzero monomial term of F (a term is a monomial with coefficient 1, a nonzero monomial term may have any nonzero coefficient). For a homogeneous ideal $I \subseteq S$, let $in(I)$ denote the ideal generated by all $in(F),\ F \in I$. It is well known, that if $in(I)$ is perfect (i.e. $S/in(I)$ is a C.M. ring), then I is perfect (see, e.g., [4], Corollary 3.1, or [7], Satz 4.3). The converse to this statement (without restrictions on the term order) is known to be not true. The purpose of this section is to consider the following problem.

Problem 1. Characterize all term orders such that I perfect implies $in(I)$ is perfect.

First of all we show that the coordinates (variables) on which the term order is to be defined, cannot be chosen arbitrarily, but depend on the given ideal I.

Proposition 2.2. *There is no term order on the variables $x_1, ..., x_n$ ($n \geq 3$) such that $in(I)$ is a perfect ideal for all perfect ideals $I \subseteq S$.*

Proof. Let $<$ be any term order. W.l.o.g. we may assume $x_1 > x_2 > \cdots > x_n$. We consider the following example due to M. L. Green (see Example 2.22 in [8]): $I = (x_1^2, x_1x_2 + x_3^2)$. Note that we always have $x_1^2 > x_1x_2 > x_1x_3 > x_3^2$. Hence $in(I) = (x_1^2, x_1x_2, x_1x_3^2, x_3^4)$. Clearly $x_2, x_4, ..., x_n$ is a system of parameters (s.o.p.) and a regular sequence on S/I, but although $x_2, x_4, ..., x_n$ is a s.o.p. on $S/in(I)$, x_2 is not a regular element on $S/in(I)$. Thus S/I is C.M., but $S/in(I)$ is not. □

If the variables are chosen sufficiently generic modulo a given ideal I (see Definition 1.5 in [2]), then by [2], Lemma 2.3, the rev. lex. term order is a term order such that S/I being C.M. implies $S/in(I)$ is C.M. The next theorem shows that it is the unique term order with this property. This result generalizes Theorem 3.1(ii) in [4].

Theorem 2.3. *Let $<$ be a term order such that $x_1 > \cdots > x_n$ ($n \geq 3$). Assume that $in(I)$ is perfect for all perfect ideals $I \subseteq S$, $in(I)$ subject to the condition*

(*) $x_{n-d+1}, ..., x_n$ is a s.o.p. on $S/in(I)$, where $d = \dim(S/I)$.

Then $<$ is the rev. lex. term order.

Before embarking on the proof of Theorem 2.3, we note that a condition such as (*) is necessary by virtue of Proposition 2.2 and the example in its proof. The converse of Theorem 2.3, namely that (*) together with $<$ being the rev. lex. term order imply $in(I)$ perfect for all perfect I, follows by [2], Lemma 2.3 and Theorem 2.4.

Proof. Assume that $<$ is not the rev. lex. term order and that the variables (possibly after permuting subscripts) satisfy $x_1 > x_2 > \cdots > x_n$. Since $<$ is not the rev. lex. term order, there are two terms p and q of equal degree such that $p = x_{i_1}^{a_1}...x_{i_u}^{a_u}$, $q = x_{j_1}^{b_1}...x_{j_v}^{b_v}$, with all exponents greater than zero, $1 \leq i_1 < \cdots < i_u \leq n$, $1 \leq j_1 < \cdots < j_v \leq n$, $j_v < i_u$, but $p > q$. Let $U = \{i_1, ..., i_u\}$ and $V = \{j_1, ..., j_v\}$. W.l.o.g. we may assume $U \cap V = \emptyset$, since $mx_i > m'x_i$ if and only if $m > m'$. Hence g.c.d.$(p, q) = 1$. Also (by the arrangement of the variables with respect to $<$) $x_{i_u}^{\deg(p)} = x_{i_u}^{\deg(q)} < x_{j_v}^{\deg(q)} < q$, thus $u \geq 2$, $v \geq 1$ (hence $n \geq 3$). For notational convenience let $r = i_u$, $a = a_u$ and $p' = x_{i_1}^{a_1}...x_{i_{u-1}}^{a_{u-1}}$. Let $I = (x_1^{c_1}, ..., x_{r-1}^{c_{r-1}}, x_{j_1}p', ..., x_{j_v}p', p + q = p'x_r^a + q)$, where

$$c_i = \begin{cases} 1 & \text{if } i \notin U \cup V, \\ a_i + 1 & \text{if } i \in U, \\ b_i + 1 & \text{if } i \in V, \ 1 \leq i \leq r-1. \end{cases}$$

We have the following properties of I:

(i) $in(p + q) = p'x_r^a = p$.

(ii)

$$\text{g.c.d.}(p, x_i^{c_i}) = \begin{cases} 1 & \text{if } i \notin U, \\ x_i^{a_i} & \text{if } i \in U, \ 1 \leq i \leq r-1. \end{cases}$$

(iii) g.c.d.$(x_jp', p) = p'$ for $j \in V$ and $x_jq = x_j^{c_j}q_j$ for some term q_j if $j \in V$.

By (i)-(iii), applying Buchberger's algorithm to the listed generating set for I, results in

$$G(I) = \{x_1^{c_1}, ..., x_{r-1}^{c_{r-1}}, x_{j_1}p', ..., x_{j_v}p', p'x_r^a + q, x_{i_1}q, ..., x_{i_{u-1}}q\}$$

as a Gröbner basis for I. Therefore

$$in(I) = (x_1^{c_1}, ..., x_{r-1}^{c_{r-1}}, x_{j_1}p', ..., x_{j_v}p', x_r^a p', x_{i_1}q, ..., x_{i_{u-1}}q) \subseteq (x_1, ..., x_{r-1}).$$

Hence $x_r, ..., x_u$ is a s.o.p. for $S/in(I)$, i.e. $in(I)$ satisfies (*) of Theorem 2.3. Since $x_r^{a-1}p' \in in(I) : x_r \subseteq (in(I), x_{r+1}, ..., x_n) : x_r$, but $x_r^{a-1}p' \notin (in(I), x_{r+1}, ..., x_n)$, $in(I)$ is not perfect. It remains to show that I is a perfect ideal.

Since $in((I, x_r, ..., x_n)) \supseteq (in(I), x_r, ..., x_n)$, $x_r, ..., x_n$ is also a s.o.p. for S/I. For all $j \in V$ we have $j < r$ and also $i < r$ if $i \in U \setminus \{r\}$. From this one obtains that $x_{r+1}, ..., x_n$ is a regular sequence on S/I. Let $S' := K[x_1, ..., x_r]$ and $I' = I \cap S'$. If $<'$ is the rev. lex. term order on S' (with x_r as smallest linear term), we then obtain $in_{<'}(I') = (x_1^{c_1}, ..., x_{r-1}^{c_{r-1}}, x_{j_1}p', ..., x_{j_v}p', q)$. Clearly x_r is a regular element on $S'/in_{<'}(I')$, i.e. $S'/in_{<'}(I')$ is a C.M. ring. Therefore S'/I' is also a C.M. ring, from which it follows that I is perfect. □

3. The generalized Cohen-Macaulay property

Let $d = \dim(S/I)$ (the Krull dimension) and $\mathfrak{m} := (x_1, ..., x_n)$. S/I is called a *generalized C.M. ring* if one of the following equivalent conditions is satisfied (see [10], [11]):

(i) All local cohomology modules $H^i_{\mathfrak{m}}(S/I),\ i < d$, are of finite length.

(ii) There is a nonnegative integer k such that $\mathfrak{m}^k H^i_{\mathfrak{m}}(S/I) = 0$ for all $i,\ 0 \leq i < d$.

(iii) There is a positive integer t such that all s.o.p. $x_1, ..., x_d$ in $\mathfrak{m}^t$ are an $\mathfrak{m}^t$-weak sequence, i.e. $\mathfrak{m}^t((x_1, ..., x_{i-1}) : x_i) \subseteq (x_1, ..., x_{i-1}),\ 1 \leq i \leq d$.

In these cases we also say I is a *generalized C.M. ideal*.

S/I is said to be a *Buchsbaum ring* (and I a *Buchsbaum ideal*), if t can be taken to be 1 in (iii) and S/I is called a *quasi-Buchsbaum ring* if $k = 1$ in (ii). By Corollary 1 in [7], if $S/in(I)$ is a generalized C.M. ring with respect to some term order, then S/I is also a generalized C.M. ring. This section concerns itself with the following problem (see also [4] pp. 157).

Problem 2. Characterize the term orders such that if S/I is a generalized C.M. ring, then $S/in(I)$ is a generalized C.M. ring.

The following result provides a negative answer to Problem 2. For a given term order let $gin(I)$ denote the initial ideal in generic coordinates in Galigo's sense (see [6] or Section 15.9 in [5]). We then have:

Proposition 3.1. *Let $<$ be a term order, $I \subseteq S$ a homogeneous ideal of positive dimension. If $S/gin(I)$ is a generalized C.M. ring, then $H^i_{\mathfrak{m}}(S/I) = 0$ for all $i \neq 0, d$ (and also, by the previous, S/I is a generalized C.M. ring). The converse is true if $<$ is the rev. lex. term order.*

In order to prove Proposition 3.1, we need the following useful property of generalized C.M. rings.

Lemma 3.2. (for a proof see Lemma I.2.2 in [10]) *If S/I is a generalized C.M.ring, then I is unmixed up to an $\mathfrak{m}$-primary component.*

Proof of Proposition 3.1. W.l.o.g. assume $x_1 > x_2 > \cdots > x_n$ and $x_1, ..., x_n$ are generic coordinates in Galigo's sense, i.e. $gin(I) = in_<(I) = in(I)$.

$\Rightarrow$. Assume that S/I is a generalized C.M. ring. Since $in(I)$ is a Borel fixed ideal, any associated prime of $in(I)$ has the form $(x_1, ..., x_i)$ for some $i \leq n$(see Corollary 15.25 in [5]). By Lemma 3.2 it follows that the saturation $\widetilde{in(I)}$ of $in(I)$ has the unique associated prime $(x_1, ..., x_{n-d})$. This means that $x_{n-d+1}, ..., x_n$ is a regular sequence on $S/\widetilde{in(I)}$. Hence $H^i_{\mathfrak{m}}(S/in(I)) = H^i_{\mathfrak{m}}(S/\widetilde{in(I)}) = 0$ for $i \neq 0, d$. By Satz 4.3 in [7], it follows that $H^i_{\mathfrak{m}}(S/I) = 0$ for $i \neq 0, d$.

$\Leftarrow$. By Propositions 15.12 and 15.24 in [5], we have $in(I : x_n^\infty) = in(I) : x_n^\infty = \widetilde{in(I)}$. Since S/I is a generalized C.M. ring and x_n is a parameter element of S/I, $I : x_n^\infty = \tilde{I}$. By assumption $S/\tilde{I}$ is a C.M. ring. Hence by Lemma 2.3 in [2], $S/in(\tilde{I}) = S/\widetilde{in(I)}$ is also a C.M. ring. Therefore $S/in(I)$ is a generalized C.M. ring □

If the coordinates are not generic, the last proposition is false. As a trivial example one can take a monomial ideal $I \subseteq S$ such that S/I is a generalized C.M. ring, $0 < \operatorname{depth}(S/I) < d$, and $<$ is an arbitrary term order on the variables $x_1, ..., x_n$.

As a natural modification of Problem 2 one might consider the following:

Problem 2'. Assume that $I \subseteq S$ is a generalized C.M. ideal. Does there exist a change of variables and a term order (in the new variables) such that $S/in(I)$ is a generalized C.M. ring?

The answer again unfortunately is negative as the next theorem shows.

Theorem 3.3. *There exists a generalized C.M. ideal I such that with respect to any coordinates and with respect to any term order $S/in(I)$ is not a generalized C.M. ring.*

To prove this theorem we will give two examples. In the first example I is a Buchsbaum ideal, in the second S/I is a domain.

Example 3.4 (see Example 5.3 in [1]). Let

$$I = (x_1^2, x_1x_2, x_2^2, x_1x_3 - x_2x_4) \subseteq K[x_1, ..., x_4].$$

Then I is a Buchsbaum ideal and I satisfies the statement in Theorem 3.3.

Proof. In [1] it is shown that I is a Buchsbaum ideal with $H^0_{\mathfrak{m}}(S/I) = 0$ and $H^1_{\mathfrak{m}}(S/I) \cong K$. We have $\deg(I) := e(S/I) = 2$. Suppose there is a change of variables into x, y, u, v and a term order in these new variables such that $in(I) \subseteq S = K[x, y, u, v]$ is a generalized C.M. ideal. By Lemma 3.2, $in(I) = J \cap Q$, where J is an unmixed monomial ideal of dimension 2 and Q is an $\mathfrak{m}$-primary ideal. Then $\deg(J) = 2$ and $H^1_{\mathfrak{m}}(S/J) = H^1_{\mathfrak{m}}(S/in(I))$. From $H^1_{\mathfrak{m}}(S/in(I)) = 0$ it would follow by Satz 3.4 in [7], that $H^1_{\mathfrak{m}}(S/I) = 0$, which is not possible. Thus J is not a perfect ideal. W.l.o.g. we assume $J = (x, y) \cap (u, v) = (xu, xv, yu, yv)$. Since $\dim_K[in(I)]_2 = \dim_K[I]_2 = 4 = \dim_K[J]_2$, $[in(I)]_2 = [J]_2$ follows. In particular the term $in(x_1^2)$, $x_1^2 \in K[x, y, u, v]$, must be a product of two distinct variables from x, y, u, v. Assume $x_1 = a_1 z_1 + \cdots + a_i z_i \in K[x, y, u, v]$, where $a_j \in K \setminus \{0\}$, $1 \le j \le i$ and $z_1 > \cdots > z_i$ are distinct variables from $\{x, y, u, v\}$. But then $in(x_1^2) = z_1^2$, a contradiction. Thus $S/in(I)$ is not a generalized C.M. ring. □

Example 3.5. Let $I = \mathfrak{p}$ be the defining ideal of the monomial curve (s^5, s^4t, st^4, t^5) in $S = K[x_1, ..., x_4]$. Then $\mathfrak{p}$ satisfies the statement of Theorem 3.3.

Proof. We have $\deg(\mathfrak{p}) = 5$, the Hilbert polynomial $P_{S/\mathfrak{p}}(n) = 5n + 1$ and $S/\mathfrak{p}$ is a generalized C.M. domain with $H^1_{\mathfrak{m}}(S/\mathfrak{p}) \neq 0$ (see, e.g., [10], pp. 171). If $S/in(\mathfrak{p})$ is a generalized C.M. ring for some term order and change of variables, say $\{x_1, ..., x_4\}$ is mapped into $\{x, y, u, v\}$, then $in(\mathfrak{p}) = J \cap Q$, where $J \subseteq K[x, y, u, v]$ is an unmixed monomial ideal of dimension 2 and Q is an $\mathfrak{m}$-primary ideal. We list the following properties of J:

(i) $\deg(J) = 5 = \deg(\mathfrak{p})$.

(ii) $P_{S/J}(n) = 5n + 1 = P_{S/\mathfrak{p}}(n)$. In paricular, J has genus 0.

(iii) J contains at least one monomial of degree 2, since $in(\mathfrak{p}) \subseteq J$ and

$$\mathfrak{p} = (x_1x_4 - x_2x_3, x_2^4 - x_1^3x_3, x_2^3x_4 - x_1^2x_3^2, x_2^2x_4^2 - x_1x_3^3, x_2x_4^3 - x_3^4).$$

(iv) J is not a perfect ideal (see the proof in Example 3.4).

Note that an associtaed prime ideal of J has the form (z, w) with $\{z, w\} \subset \{x, y, u, v\}$. If $\mathfrak{q}$ is the corresponding (z, w)-primary component of J, then $\mathfrak{q} = (z^a, z^b w^c, w^d) = (z^b, w^d) \cap (z^a, w^c)$ for some integers $a > 0$, $d > 0$ and $0 \le b < a$, $0 \le c < d$. Hence $J = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap ... \cap \mathfrak{q}_t$ such that:

(a) Each $\mathfrak{q}_i$ is of the form (z^{a_i}, w^{b_i}), $\{z, w\} \subset \{x, y, u, v\}$.

(b) For all $\{z, w\}$ there are at most two $\mathfrak{q}_i$ with equal radical (z, w).

(c) No $\mathfrak{q}_i$, $1 \le i \le t$, can be omitted.

(d) $a_1 = \max\{a_1, b_1\} =: a \ge \max\{a_2, b_2\} \ge \cdots \ge \max\{a_t, b_t\}$.

Therefore, w.l.o.g., we may assume $\mathfrak{q}_1 = (x^a, y^b)$, $a \ge b \ge 1$. Note that $\deg(J) = \sum_{\mathfrak{p}} \text{mult}_J(\mathfrak{p})$, where $\mathfrak{p}$ runs over all associated prime ideals of J and $\text{mult}_J(\mathfrak{p}) = \ell(S_{\mathfrak{p}}/JS_{\mathfrak{p}})$ (ℓ = length).

Using (a)-(d), we can list all unmixed monomial ideals of dimension 2 and degree 5 in S. Up to a possible permutation of the variables x, y, u, v, there are exactly 59 such ideals, which are listed in the Appendix. Employing the computer system "Macaulay" of [3], we obtain that there are only two ideals satisfying (i)-(iv), namely $J_1 = (x^3, y) \cap (u^2, v)$ and $J_2 = (x^2, y^2) \cap (u, v)$. For J equal to J_1 or J_2, $\sqrt{J} = (x, y) \cap (u, v)$. By [9] the initial complex $\Delta(in(L))$, for an ideal $L \subseteq K[y_1, ..., y_n]$, is defined as $\Delta(in(L)) = \{\{y_{i_1}, ..., y_{i_t}\};\ K[y_{i_1}, ..., y_{i_t}] \cap in(L) = (0)\}$. Since J is the unmixed part of $in(\mathfrak{p})$, $\Delta(in(\mathfrak{p})) = \Delta(\sqrt{J}) = \{\{x\}, \{y\}, \{u\}, \{v\}, \{x, y\}, \{u, v\}\}$. This complex is not strongly connected, which contradicts Theorem 1 in [9]. Thus $S/in(\mathfrak{p})$ is not a generalized C.M. ring. $\square$

Remark 3.6. Let $\mathfrak{p} \subseteq K[x_1, ..., x_4]$ be the defining prime ideal of the so called Macaulay curve (s^4, s^3t, st^3, t^4). If $S/in(\mathfrak{p})$ was a generalized C.M. ring for some change of variables $\{x_1, ..., x_4\}$ into $\{x, y, u, v\}$ and some term order, then, as in the proof of Example 3.5, $in(\mathfrak{p}) = (x^2u, x^2v, xyv, yu)$. We believe that this also is unlikely to be the case. For instance if $\{x_1, ..., x_4\} = \{x, y, u, v\}$, then xyv is not a part of any reduced and normalized Gröbner basis as an inspection of $\mathfrak{p} = (x_1x_4 - x_2x_3, x_2^3 - x_1^2x_3, x_2^2x_4 - x_1x_3^2, x_2x_4^2 - x_3^3)$ will show.

4. The Buchsbaum property

In the previous section we did see that the generalized C.M. property is not hereditary from I to $in(I)$, but is hereditary from $in(I)$ to I. There are examples which show that, with respect to certain term orders, even the Buchsbaum property is not hereditary from $in(I)$ to I (see [4], Example 3.1 and pp. 217 in [7]). The following proposition renders a more complete description.

Proposition 4.1. *There is an ideal $I \subseteq K[x, y, z] = S$ such that for all term orders (on the variables x, y, z), $S/in(I)$ is a Buchsbaum ring, but S/I is not a quasi-Buchsbaum ring.*

Proof. Let $I = (x(x^2 + xy + y^2), z(x^2 + xy + y^2), y^3, x^2y) \subseteq S$. Then I is of dimension 1. For any term order $<_1$ such that $y <_1 x$, by a straightforward Gröbner basis calculation, $in_{<_1}(I) = (x^3, x^2y, y^3, xy^2z, x^2z)$. Clearly z^2 is a parameter element of $S/in_{<_1}(I)$ in $\mathfrak{m}^2$ and $in_{<_1}(I) : z^2 = (in_{<_1}(I), xy^2, x^2)$. Since $(x, y, z)(xy^2, x^2) \subseteq in_{<_1}(I)$, $S/in_{<_1}(I)$ is a Buchsbaum ring (see Propositions I.2.1 and I.2.12 in [10]). For any term order $<_2$ such that $x <_2 y$, a Gröbner basis calculation gives $in_{<_2}(I) = (x^4, y^3, xy^2, x^2y, x^3z, y^2z)$. As before $S/in_{<_2}(I)$ is a Buchsbaum ring. Hence $S/in(I)$ is a Buchsbaum ring for an arbitrary term order. For S/I, z is a parameter element and $x^2+xy+y^2 \in I : z$. But $y(x^2+xy+y^2) \notin I$, since if this were the case, then $xy^2 = y(x^2+xy+y^2)-x^2y-y^3 \in I$, which contradicts $xy^2 \notin in_{<_1}(I)$. Thus $(x, y, z)(I : z) \nsubseteq I$ and S/I is not a quasi-Buchsbaum ring by Proposition I.2.1 in [10]. $\square$

By Proposition I.1.9 in [10], a S/I-basis in $\mathfrak{m} = (x_1, ..., x_n) \subseteq K[x_1, ..., x_n]$ exists. Therefore, by the following result, Proposition 4.1 may no longer be true if one is allowed to change coordinates.

Proposition 4.2. *Assume that $I \subseteq S = K[x_1, ..., x_n]$ is a d-dimensional ideal such that any subset of d variables forms a s.o.p. of S/I (the definition of an S/I-basis in $\mathfrak{m}$). Assume further that for any subset $\{x_{i_1}, ..., x_{i_d}\}$ of d variables there is a rev. lex. term order such that:*

(i) $x_i > x_j$ *if* $i \notin \{i_1, ..., i_d\}$ *and* $j \in \{i_1, ..., i_d\}$.

(ii) $S/in(I)$ *is a Buchsbaum ring.*

Then S/I is a Buchsbaum ring.

Proof. Let $\{i_1, ..., i_d\} \subseteq \{1, ..., n\}$ be a set of d indices. Assume that there is a rev. lex. term order satisfying the hypothesis of Proposition 4.2. W.l.o.g. assume $x_{i_1} > \cdots > x_{i_d}$. By Lemma 2.2 in [2] $in(I, x_{i_1}, ..., x_{i_d}) = (in(I), x_{i_1}, ..., x_{i_d})$, hence $x_{i_1}, ..., x_{i_d}$ is also a s.o.p. of $S/in(I)$. Since $in(I, x_{i_1}^2, ..., x_{i_d}^2) \supseteq (in(I), x_{i_1}^2, ..., x_{i_d}^2)$, we have $\ell(S/(in(I), x_{i_1}^2, ..., x_{i_d}^2)) - \ell(S/(in(I), x_{i_1}, ..., x_{i_d})) \geq \ell(S/(I, x_{i_1}^2, ..., x_{i_d}^2)) - \ell(S/(I, x_{i_1}, ..., x_{i_d}))$. By Lemma 2.2 in [11], we obtain:

$$\begin{aligned} &\ell(S/(I, x_{i_1}^2, ..., x_{i_d}^2)) - \ell(S/(I, x_{i_1}, ..., x_{i_d})) \geq \\ &e(x_{i_1}^2, ..., x_{i_d}^2; S/I) - e(x_{i_1}, ..., x_{i_d}; S/I) = (d^2 - 1)e(S/I). \end{aligned} \tag{**}$$

(Note that $(x_{i_1}, ..., x_{i_d})$ is a minimal reduction of $\mathfrak{m}S/I$ and $e(x_{i_1}, ..., x_{i_d}; S/I) = e(S/I)$.) By assumption $S/in(I)$ is a Buchsbaum ring, therefore

$$\begin{aligned} &\ell(S/(in(I), x_{i_1}^2, ..., x_{i_d}^2)) - \ell(S/(in(I), x_{i_1}, ..., x_{i_d})) = \\ &e(x_{i_1}^2, ..., x_{i_d}^2; S/in(I)) - e(x_{i_1}, ..., x_{i_d}; S/in(I)) = (d^2 - 1)e(S/in(I)). \end{aligned}$$

Since $e(S/I) = e(S/in(I))$, we must have equality in (**), i.e. $x_{i_1}, ..., x_{i_d}$ is a standard s.o.p. of S/I in the sense of [11]. Therefore by Proposition 3.1 in [11], S/I is a Buchsbaum ring. ☐

Appendix for Example 3.5

Unmixed monomial ideals $J \subseteq K[x, y, u, v]$ of dimension 2 and degree 5 (up to equivalence of a permutation of variables), J=$\mathfrak{q}_1 \cap ... \cap \mathfrak{q}_t$, $\mathfrak{q}_i \subseteq \mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, $1 \leq i \leq t$.

Case I: $\mathfrak{p}_i \neq \mathfrak{p}_j$, $i \neq j$, $\mathfrak{q}_1 = (x^a, y^b) \subseteq (x, y) = \mathfrak{p}_1$.

$a = 5 \Rightarrow b = 1,\ t = 1.$

1.	$J = (x^5, y)$	C.M., excluded by (iv) in the proof of Ex. 3.5

$a = 4 \Rightarrow b = 1,\ t = 2.$

2.	$J = (x^4, y) \cap (x, u) = (x^4, yx, yu)$	genus 3, excluded by (ii) in the proof of Ex. 3.5
3.	$J = (x^4, y) \cap (y, u) = (x^4u, y)$	C.M.
4.	$J = (x^4, y) \cap (u, v) = (x^4u, x^4v, yu, yv)$	genus 2

$a = 3 \Rightarrow b = 1 \Rightarrow 2 \le t \le 3.$

	$t = 2$	
5.	$J = (x^3, y) \cap (x^2, u) = (x^3, yx^2, yu)$	genus 2
6.	$J = (x^3, y) \cap (x, u^2) = (x^3, yx, yu^2)$	genus 2
7.	$J = (x^3, y) \cap (y^2, u) = (y^2, x^3u, yu)$	genus 3
8.	$J = (x^3, y) \cap (y, u^2) = (y, x^3u^2)$	C.M.
9.	$J = (x^3, y) \cap (u^2, v) = (x^3u^2, x^3v, yu^2, yv)$	**genus 0**
	$t = 3$	
10.	$J = (x^3, y) \cap (x, u) \cap (x, v) = (x^3, y) \cap (x, uv)$ $= (x^3, yx, yuv)$	genus 2
11.	$J = (x^3, y) \cap (x, u) \cap (y, u) = (x^3, y) \cap (xy, u)$ $= (x^3u, xy, yu)$	genus 3
12.	$J = (x^3, y) \cap (x, u) \cap (y, v) = (x^3v, y) \cap (x, u)$ $= (x^3v, xy, yu)$	genus 3
13.	$J = (x^3, y) \cap (x, u) \cap (u, v) = (x^3, y) \cap (xv, u)$ $= (x^3v, x^3u, xyv, yu)$	genus 1
14.	$J = (x^3, y) \cap (y, u) \cap (y, v) = (x^3uv, y)$	C.M.
15.	$J = (x^3, y) \cap (y, u) \cap (u, v) = (x^3u, y) \cap (u, v)$ $= (x^3u, yu, yv)$	genus 3
	$a = 2,\ b = 2 \Rightarrow t = 2.$	
16.	$J = (x^2, y^2) \cap (x, u) = (x^2, y^2x, y^2u)$	genus 2
17.	$J = (x^2, y^2) \cap (u, v) = (x^2u, x^2v, y^2u, y^2v)$	**genus 0**
	$a = 2,\ b = 1 \Rightarrow 3 \le t \le 4,\ \mathfrak{q}_1 = (x^2, y).$ $\mathrm{mult}_J(\mathfrak{p}_2) = 2 \Leftrightarrow t = 3.$	
18.	$J = (x^2, y) \cap (x^2, u) \cap (x, v) = (x^2, yu) \cap (x, v)$ $= (x^2, xyu, yuv)$	genus 2
19.	$J = (x^2, y) \cap (x^2, u) \cap (y, u) = (x^2, yu) \cap (y, u)$ $= (x^2u, x^2y, yu)$	genus 2
20.	$J = (x^2, y) \cap (x^2, u) \cap (y, v) = (x^2, yu) \cap (y, v)$ $= (x^2y, x^2v, yu)$	genus 2
21.	$J = (x^2, y) \cap (x^2, u) \cap (u, v) = (x^2, yu) \cap (u, v)$ $= (x^2u, x^2v, yu)$	genus 2
22.	$J = (x^2, y) \cap (x, u^2) \cap (x, v) = (x^2, y) \cap (x, u^2v)$ $= (x^2, xy, yu^2v)$	genus 3
23.	$J = (x^2, y) \cap (x, u^2) \cap (y, u) = (x^2u, y) \cap (x, u^2)$ $= (x^2u, xy, yu^2)$	genus 2
24.	$J = (x^2, y) \cap (x, u^2) \cap (y, v) = (x^2v, y) \cap (x, u^2)$ $= (x^2v, yx, yu^2)$	genus 2
25.	$J = (x^2, y) \cap (x, u^2) \cap (u, v) = (x^2, xy, yu^2) \cap (u, v)$ $= (x^2u, x^2v, xyu, xyv, yu^2)$	no deg.2 element, exluded by (iii)

26.	$J = (x^2, y) \cap (y^2, u) \cap (x, u) = (x^2u, y^2, yu) \cap (x, u)$	
	$= (x^2u, y^2x, yu)$	genus 2
27.	$J = (x^2, y) \cap (y^2, u) \cap (x, v) = (x^2u, y^2, yu) \cap (x, v)$	no deg.
	$= (x^2u, xy^2, xyu, y^2v, yuv)$	2 element
28.	$J = (x^2, y) \cap (y^2, u) \cap (y, v) = (x^2u, y^2, yu) \cap (y, v)$	
	$= (x^2uv, y^2, yu)$	genus 3
29.	$J = (x^2, y) \cap (y^2, u) \cap (u, v) = (x^2u, y^2, yu) \cap (u, v)$	
	$= (x^2u, yu, y^2v)$	genus 2
30.	$J = (x^2, y) \cap (y, u^2) \cap (x, u) = (x^2u^2, y) \cap (x, u)$	
	$= (xy, x^2u^2, yu)$	genus 3
31.	$J = (x^2, y) \cap (y, u^2) \cap (x, v) = (x^2u^2, y) \cap (x, v)$	
	$= (xy, x^2u^2, yv)$	genus 3
32.	$J = (x^2, y) \cap (y, u^2) \cap (y, v) = (x^2u^2v, y)$	C.M.
33.	$J = (x^2, y) \cap (y, u^2) \cap (u, v) = (x^2u^2, y) \cap (u, v)$	
	$= (x^2u^2, yu, yv)$	genus 3
34.	$J = (x^2, y) \cap (u^2, v) \cap (x, u) = (x^2u^2, x^2v, yu^2, yv) \cap (x, u)$	no deg. 2
	$= (x^2u^2, x^2v, yu^2, xyv, yuv)$	element
35.	$J = (x^2, y) \cap (u^2, v) \cap (x, v) = (x^2u^2, x^2v, yu^2, yv) \cap (x, v)$	
	$= (x^2u^2, x^2v, xyu^2, yv)$	genus 1
36.	$J = (x^2, y) \cap (u^2, v) \cap (y, u) = (x^2u, y) \cap (u^2, v)$	
	$= (x^2u^2, x^2uv, yu^2, yv)$	genus 1
37.	$J = (x^2, y) \cap (u^2, v) \cap (y, v) = (x^2v, y) \cap (u^2, v)$	
	$= (x^2v, yu^2, yv)$	genus 2

$t = 4 \Rightarrow \mathfrak{q}_2 = \mathfrak{p}_2,\ \mathfrak{q}_3 = \mathfrak{p}_3,\ \mathfrak{q}_4 = \mathfrak{p}_4.$

38.	$J = (x^2, y) \cap (x, u) \cap (x, v) \cap (y, u) = (x^2u, y) \cap (x, uv)$	
	$= (xy, x^2u, yuv)$	genus 2
39.	$J = (x^2, y) \cap (x, u) \cap (x, v) \cap (u, v)$	no deg.
	$= (x^2, xy, yu) \cap (xu, v) = (x^2u, x^2v, xyu, xyv, yuv)$	2 element
40.	$J = (x^2, y) \cap (x, u) \cap (y, u) \cap (y, v)$	
	$= (x^2uv, y) \cap (x, u) = (x^2uv, xy, yu)$	genus 3
41.	$J = (x^2, y) \cap (x, u) \cap (y, u) \cap (u, v)$	
	$= (x^2, y) \cap (xyv, u) = (xyv, x^2u, yu)$	genus 2
42.	$J = (x^2, y) \cap (x, u) \cap (y, v) \cap (u, v)$	
	$= (x^2v, y) \cap (xv, u) = (x^2v, xyv, yu)$	genus 2
43.	$J = (x^2, y) \cap (y, u) \cap (y, v) \cap (u, v)$	
	$= (x^2uv, y) \cap (u, v) = (x^2uv, yu, yv)$	genus 3

$a = 1 \Rightarrow b = 1$ and $t = 5$, i.e. intersections of (x, y) with four from (x, u), (x, v), (y, u), (y, v), (u, v).

44.	$J = (x, y) \cap (x, u) \cap (x, v) \cap (y, u) \cap (y, v)$ $= (x, yuv) \cap (y, uv) = (xy, xuv, yuv)$	genus 2
45.	$J = (x, y) \cap (x, u) \cap (x, v) \cap (y, u) \cap (u, v)$ $= (x, yuv) \cap (yv, u) = (xyv, xu, yuv)$	genus 2
46.	$J = (x, y) \cap (x, u) \cap (y, u) \cap (y, v) \cap (u, v)$ $= (xv, y) \cap (xyv, u) = (xyv, xuv, yu)$	genus 2

Case II: $\mathfrak{p}_1 = \mathfrak{p}_2 = (x, y)$. Then $\text{mult}_J(\mathfrak{p}_1) \geq 3$.

	$\text{mult}_J(\mathfrak{p}_1) = 5 \Rightarrow t = 2$.	
47.	$J = (x^4, y) \cap (x, y^2) = (x^4, y^2, xy)$	genus 3
48.	$J = (x^3, y) \cap (x^2, y^2) = (x^3, yx^2, y^2)$	genus 2
49.	$J = (x^3, y) \cap (x, y^3) = (x^3, xy, y^3)$	genus 2
	$\text{mult}_J(\mathfrak{p}_1) = 4 \Rightarrow \mathfrak{q}_1 \cap \mathfrak{q}_2 = (x^3, y) \cap (x, y^2)$ and $t = 3$.	
50.	$J = (x^3, y) \cap (x, y^2) \cap (x, u) = (x^3, y^2, xy) \cap (x, u)$ $= (x^3, xy, y^2u)$	genus 2
51.	$J = (x^3, y) \cap (x, y^2) \cap (y, u) = (x^3, y^2, xy) \cap (y, u)$ $= (x^3u, xy, y^2)$	genus 3
52.	$J = (x^3, y) \cap (x, y^2) \cap (u, v) = (x^3, y^2, xy) \cap (u, v)$ $= (x^3u, x^3v, y^2u, y^2v, xyu, xyv)$	no deg. 2 element
	$\text{mult}_J(\mathfrak{p}_1) = 3 \Rightarrow \mathfrak{q}_1 \cap \mathfrak{q}_2 = (x^2, y) \cap (x, y^2)$.	
53.	$J = (x^2, y) \cap (x, y^2) \cap (x^2, u) = (x^2, yu) \cap (x, y^2)$ $= (x^2, xyu, y^2u)$	genus 2
54.	$J = (x^2, y) \cap (x, y^2) \cap (x, u^2) = (x^2, y) \cap (x, y^2u^2)$ $= (x^2, xy, y^2u^2)$	genus 3
55.	$J = (x^2, y) \cap (x, y^2) \cap (u^2, v) = (x^2, xy, y^2) \cap (u^2, v)$ $= (x^2u^2, x^2v, xyu^2, xyv, y^2u^2, y^2v)$	no deg. 2 element
56.	$J = (x^2, y) \cap (x, y^2) \cap (x, u) \cap (x, v)$ $= (x^2, y) \cap (x, y^2uv) = (x^2, xy, y^2uv)$	genus 3
57.	$J = (x^2, y) \cap (x, y^2) \cap (x, u) \cap (y, u)$ $= (x^2u, y) \cap (x, y^2u) = (x^2u, xy, y^2u)$	genus 2
58.	$J = (x^2, y) \cap (x, y^2) \cap (x, u) \cap (y, v)$ $= (x^2v, y) \cap (x, y^2u) = (x^2v, xy, y^2u)$	genus 2
59.	$J = (x^2, y) \cap (x, y^2) \cap (x, u) \cap (u, v)$ $= (x^2, xy, y^2) \cap (u, xv) = (x^2u, x^2v, xyu, xyv, y^2u)$	no deg. 2 element

References

[1] M. Amasaki, *Basis sequences of homogeneous ideals in polynomial rings*, J. Algebra **190** (1997), 329-360.
[2] D. Bayer and M. Stillman, *A criterion for detecting m-regularity*, Invent. Math. **87** (1987), 1-11.
[3] D. Bayer and M. Stillman, *Macaulay: A system for computation in Algebraic Geometry and Commutative Algebra*.
[4] H. Bresinsky and W. Vogel, *Computational aspects for Cohen-Macaulay and Buchsbaum ideals*, Ann. Uni. Ferrara - Sez. VII, Sc. Mat. **39** (1993), 143-159.
[5] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry* (Springer-Verlag, Berlin-New York, 1995).
[6] A. Galligo, *A propos du théorème de préparation de Weierstrass*, Functions des Plusieurs Variables Complexes, Lecture Notes in Mathematics 409, Springer-Verlag, Berlin-New York, 1974, 543-579.
[7] H. J. Gräbe, *Moduln über Streckungsringen*, Results in Math. **15** (1989), 202-220.
[8] M. L. Green, *Generic intial ideals*, In "Six lectures on Commutative Algebra" (Eds: Elias, Giral, Miro-Roig and Zarzuela), pp. 119-186, Progr. Math. **166**, Birkhäuser, Basel, 1998.
[9] M. Kalkbrener and B. Sturmfels, *Initial complexes of prime ideals*, Adv. in Math. **116** (1995), 365-376.
[10] J. Stückrad and W. Vogel, *Buchsbaum Rings and Applications*, Springer-Verlag, Berlin, 1986.
[11] N. V. Trung, *Towards a theory of generalized Cohen-Macaulay modules*, Nagoya Math. J. **102** (1986), 1-49.

Department of Mathematics
University of Maine
Orono, Maine 04469-5752, USA.

E-mail address: Henrik@maine.maine.edu

Institute of Mathematics
Box 631, Bò Hô, Hanoi, Vietnam

E-mail address: lthoa@thevinh.ncst.ac.vn