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GENERALIZED QUASICONVEXITY VIA PROPERLY CHARACTERISTIC FUNCTIONS ASSOCIATED TO BINARY RELATIONS

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Abstract. As shown by Dinh The Luc in his well-known monograph on vector optimization, the cone-quasiconvex vector-valued functions can be characterized in terms of scalar quasiconvexity by means of the smallest strictly monotonic functions. The aim of this paper is to show that similar characterizations can be given for the general class of (Γ, Ω)-quasiconvex functions, this time by means of the so-called properly characteristic functions associated to the binary relation Ω.

1. Introduction

Among various notions of generalized quasiconvexity which have been applied in vector optimization (see [3], [5] or [1] and references therein), the concept of cone-quasiconvexity introduced by Dinh The Luc in [4] for vector-valued functions is of special interest because it can be characterized in terms of convex level sets. Recall that a function $f : X \to E_2$ defined on a nonempty convex subset X of a vector space E_1 which takes values in a vector space E_2, partially ordered by a convex cone C, is called C-quasiconvex on X if for any points $x^1, x^2 \in X$ and $y \in E_2$ one has

$$y \in \big[f(x^1) + C\big] \cap \big[f(x^2) + C\big] \Longrightarrow f(tx^1 + (1-t)x^2) \in y - C, \quad \forall\, t \in [0,1].$$

In other words, the function f is C-quasiconvex on X if for any point $y \in E_2$ the level set $L_f(y) = \{x \in X \mid f(x) \in y - C\}$ is convex. This property shows that, at least in the particular case when $E_2 = \mathbb{R}^n$ is the Euclidean space partially ordered by the standard positive cone $C = \mathbb{R}^n_+$, there is a direct relationship between the cone-quasiconvexity and the scalar quasiconvexity. In fact, a function $f = (f_1, \dots, f_n) : X \to \mathbb{R}^n$ is $\mathbb{R}^n_+$-quasiconvex on a convex set X if and only if its scalar components $f_1, \dots, f_n$ are quasiconvex on X in the usual sense. Besides this component-wise setting it was shown in [5] that if E_2 is a topological vector space, partially ordered by a closed convex cone C which has nonempty interior, then a function $f : X \to E_2$ is C-quasiconvex on a convex set X if and only if, for any point $a \in E_2$, the composite function $h_{e,a} \circ f : X \to \mathbb{R}$ is quasiconvex on X

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in the usual sense, where $h_{e,a} : E_2 \to \mathbb{R}$ denotes the "smallest strictly monotonic function" defined by

$$h_{e,a}(x) = \min\{t \in \mathbb{R} \mid x \in a + te - C\},\ \forall\ x\ \in E_2 \tag{1.1}$$

for an arbitrary fixed point $e \in \operatorname{int} C$.

In this paper we shall show that a similar characterization in terms of scalar quasiconvexity can be given for the class of (Γ, Ω)-quasiconvex functions, which was introduced by us in [7] in order to describe in a unifying way those functions which possess the characteristic property to have convex level sets. The next section is devoted to recall this concept.

2. (Γ, Ω)-quasiconvex functions

Let E_1 be a nonempty set and let $\Gamma : E_1 \times E_1 \to 2^{E_1}$ be a set-valued map, which assigns to each pair of points from E_1 a subset of E_1. We say that a subset X of E_1 is:

(i) Γ-convex, if $\Gamma(x^1, x^2) \subset X,\ \forall\ x^1, x^2 \in X$;

(ii) Γ-convex with respect to a point $x^0 \in X$, if $\Gamma(x, x^0) \subset X,\ \forall\ x \in X$.

Remark that even if Γ is not symmetric, a nonempty set $X \subset E_1$ is Γ-convex if and only if it is Γ-convex with respect to all its points. In this paper, such Γ-convex sets will play the role of the domain for some generalized quasiconvex functions. As codomain of these functions, we shall consider a nonempty set E_2 endowed with a binary relation $\Omega \subset E_2 \times E_2$, which will be identified with the set-valued map $\Omega : E_2 \to 2^{E_2}$, given by

$$\Omega y = \{y' \in E_2 \mid (y, y') \in \Omega\},\ \forall\ y \in E_2.$$

As usual, for any $y \in E_2$, we denote $\Omega^{-}y = \{y' \in E_2 \mid y \in \Omega y'\}$ and $\Omega^{c}y = E_2 \setminus (\Omega y)$.

Definition 2.1. Let X be a nonempty subset of E_1, which is Γ-convex with respect to a point $x^0 \in X$. A function $f : X \to E_2$ is called (Γ, Ω)-quasiconvex at x^0 if

$$\forall\ x \in X,\quad f(x) \in \Omega f(x^0) \implies f(\Gamma(x, x^0)) \subset \Omega f(x^0).$$

Definition 2.2. Let X be a nonempty and Γ-convex subset of E_1. A function $f : X \to E_2$ is called (Γ, Ω)-quasiconvex on X if

$$\forall\ x^1, x^2 \in X,\ \forall y \in E_2,\ f(\{x^1, x^2\}) \subset \Omega y \implies f(\Gamma(x^1, x^2)) \subset \Omega y.$$

Remark 2.1. If Ω is reflexive and $f : X \to E_2$ is (Γ, Ω)-quasiconvex on a nonempty and Γ-convex set $X \subset E_1$ then f is (Γ, Ω)-quasiconvex at every point of X. As shown by us in [8], the converse is true whenever Ω is transitive and complete and Γ is symmetric, each of these assumptions being essential.

Remark 2.2. Let $E_2 = \mathbb{R}$ be endowed with the usual order relation $\Omega ='' \geq ''$ defined for all $y \in \mathbb{R}$ by $\Omega y =]-\infty, y]$ and let $f : X \to \mathbb{R}$ be defined on a nonempty Γ-convex set $X \subset E_1$. The following assertions hold:

(i) f is (Γ, Ω)-quasiconvex at a point $x^0 \in X$, if and only if

$$\forall\, x \in X,\ f(x) \leq f(x^0) \implies \forall\, x' \in \Gamma(x, x^0),\ f(x') \leq f(x^0);$$

(ii) f is (Γ, Ω)-quasiconvex on X, if and only if

$$\forall\, x^1, x^2 \in X, \forall\, x \in \Gamma(x^1, x^2) \implies f(x) \leq \max\{f(x^1), f(x^2)\}.$$

Remark 2.3. When E_1 and E_2 are vector spaces and C is a convex cone in E_2, then for Γ and Ω defined by

$$\Gamma(x^1, x^2) = \mathrm{co}\{x^1, x^2\},\ \forall\, x^1, x^2 \in E_1 \quad \text{and} \quad \Omega y = y - C,\ \forall\, y \in E_2,$$

the (Γ, Ω)-quasiconvexity coincides with the C-quasiconvexity in the sense of Dinh The Luc.

The propositions below show that the (Γ, Ω)-quasiconvexity is a natural extension of the cone-quasiconvexity since it can also be characterized in terms of some appropriate generalized level sets, defined for any $y \in E_2$ by

$$L_f(y) = \{x \in X \mid f(x) \in \Omega y\}.$$

Proposition 2.1. *Let $X \subset E_1$ be Γ-convex with respect to $x^0 \in X$ and let $f : X \to E_2$. If $f(x^0) \in \Omega f(x^0)$, then the function f is (Γ, Ω)-quasiconvex at x^0 if and only if the set $L_f(f(x^0))$ is Γ-convex with respect to x^0.*

Proposition 2.2. *Let $f : X \to E_2$ be defined on a nonempty and Γ-convex set $X \subset E_1$. Then the function f is (Γ, Ω)-quasiconvex on X if and only if for any point $y \in E_2$ the set $L_f(y)$ is Γ-convex.*

Some other characterizations of the (Γ, Ω)-quasiconvexity in terms of polarities in the sense of Dolecki and Malivert [2] or in terms of level sets can be found in [8] or [7]. In what follows we will focus on the relationship between the (Γ, Ω)-quasiconvexity and the scalar quasiconvexity.

3. Properly characteristic functions

As in the previous section, E_1 and E_2 denote two nonempty sets endowed with the set-valued maps $\Gamma : E_1 \times E_1 \to 2^{E_1}$ and $\Omega : E_2 \to 2^{E_2}$, the latter one representing a binary relation on E_2. The definition below plays a key role in the sequel.

Definition 3.1. Let λ be a real number. A function $g : E_2 \times E_2 \to \mathbb{R}$ is called

(i) λ-characteristic for the binary relation Ω, if for any $y^1, y^2 \in E_2$, one has

$$g(y^1, y^2) \leq \lambda \Longleftrightarrow y^1 \in \Omega y^2;$$

(ii) properly λ-characteristic if, in addition, there exists $y^0 \in E_2$ such that

$$g(y^1, y^0) \le g(y^2, y^0) \le \lambda < g(y^3, y^0),$$

whenever $g(y^1, y) \le g(y^2, y) < g(y^3, y)$ for some $y, y^1, y^2, y^3 \in E_2$.

Example 3.1. Consider the Heaviside-type function $h : E_2 \times E_2 \to \mathbb{R}$ defined by

$$h(y^1, y^2) = \begin{cases} 0 & \text{if} \quad y^1 \in \Omega y^2, \\ 1 & \text{if} \quad y^1 \in \Omega^c y^2. \end{cases}$$

It is easy to see that h is 0-characteristic for Ω. Moreover, h is properly 0-characteristic since for all $y, y^1, y^2, y^3 \in E_2$ such that

$$h(y^1, y) \le h(y^2, y) < h(y^3, y)$$

we actually have $h(y^1, y) = h(y^2, y) = 0$ and $h(y^3, y) = 1$. Hence $y^0 = y$ satisfies the property in demand.

Example 3.2. The "signum"-type function $s : E_2 \times E_2 \to \mathbb{R}$, defined by

$$s(y^1, y^2) = \begin{cases} -1 & \text{if} \quad y^1 \in [\Omega \cap (\Omega^-)^c] y^2 \\ 0 & \text{if} \quad y^1 \in (\Omega \cap \Omega^-) y^2 \\ 1 & \text{if} \quad y^1 \in \Omega^c y^2 \end{cases}$$

is 0-characteristic for Ω, but it is not properly characteristic in general. For instance, if $E_2 = \mathbb{R}^2$ and $\Omega y = y - \mathbb{R}^2_+$, $\forall\, y \in \mathbb{R}^2$ then by taking $y^1 = (1, 0)$, $y^2 = (0, 1)$, $y^3 = (1, 1)$ and $y = y^3$ we have $s(y^1, y) = s(y^2, y) = -1 < s(y^3, y) = 0$, but the condition

$$s(y^1, y^0) \le s(y^2, y^0) \le 0 < s(y^3, y^0)$$

cannot be satisfied because it is equivalent with

$$y^0 \in (y^1 + \mathbb{R}^2_+) \cap (y^1 + \mathbb{R}^2_+) \setminus (y^3 + \mathbb{R}^2_+) = \emptyset.$$

As shown in the following theorem, a special class of properly characteristic functions can be constructed in partially ordered topological vector spaces by means of the smallest strictly monotonic functions.

Theorem 3.1. *Let E_2 be a topological vector space and let $\Omega y = y - C$ for any $y \in E_2$, where $C \subset E_2$ is a closed convex cone, which has nonempty interior. Then, the function $g : E_2 \times E_2 \to \mathbb{R}$ defined by*

$$g(y^1, y^2) = h_{e, y^2}(y^1),\ \forall\, y^1, y^2 \in E_2 \tag{3.1}$$

is properly 0-characteristic for Ω.

Proof. Let us firstly remark that according to (1.1) we have, for any $y^1, y^2 \in E_2$,

$$g(y^1, y^2) = \min\{t \in \mathbb{R} \mid y^2 - y^1 + te \in C\} = \min\{t \in \mathbb{R} \mid y^1 \in \Omega(y^2 + te)\}.$$

Now, for proving that g is 0-characteristic for Ω, it is easy to see that if $y^1, y^2 \in E_2$ are such that $y^1 \in \Omega y^2$ then $g(y^1, y^2) \le 0$. Conversely, if $g(y^1, y^2) \le 0$ then taking into account that C is a cone, we have $-g(y^1, y^2)e \in C$ and therefore $y^2 - y^1 \in y^2 - y^1 + g(y^1, y^2)e + C$. On the other hand, by the definition of

$g(y^1, y^2)$ we have $y^2 - y^1 + g(y^1, y^2)e \in C$. Then, by the convexity of the cone C we infer $y^2 - y^1 \in C$, that is $y^1 \in \Omega y^2$.

In order to prove that g is properly characteristic, let $y, y^1, y^2, y^3 \in E_2$ be such that

$$g(y^1, y) \le g(y^2, y) < g(y^3, y). \tag{3.2}$$

By considering $y^0 = y + g(y^2, y)e$ and by taking into account that

$$\begin{aligned} g(y^i, y^0) &= \min\{t \in \mathbb{R} \mid y - y^i + [t + g(y^2, y)]e \in C\} \\ &= g(y^i, y) - g(y^2, y),\ \forall\, i \in \{1, 2, 3\} \end{aligned}$$

we can deduce, by substracting $g(y^2, y)$ in (3.2), that

$$g(y^1, y^0) \le g(y^2, y^0) \le 0 < g(y^3, y^0).$$

□

4. Scalar characterizations of the (Γ, Ω)-quasiconvexity

The aim of this section is to show that the (Γ, Ω)-quasiconvexity can be characterized in terms of scalar quasiconvexity by using certain characteristic functions. The following theorems extend some similar results, which have been obtained by Dinh The Luc in [6] by using the particular function g defined by (3.1) in a vectorial framework.

Theorem 4.1. *Let $X \subset E_1$ be Γ-convex with respect to a point $x^0 \in X$ and let $f : X \to E_2$. If the function $g : E_2 \times E_2 \to \mathbb{R}$ is λ_0-characteristic for Ω, where $\lambda_0 = g(f(x^o), f(x^0))$, then the following assertions are equivalent:*

(i) *f is (Γ, Ω)-quasiconvex at x^0;*

(ii) *$g(f(\cdot), f(x^0)) : X \to \mathbb{R}$ is $(\Gamma, \ge)$-quasiconvex at x^0.*

Proof. By Definition 2.1, the assertion (i) is equivalent to:

$$\forall x \in X,\ f(x) \in \Omega f(x^0) \Longrightarrow \forall x' \in \Gamma(x, x^0),\ f(x') \in \Omega f(x^0).$$

Since g is λ_0-characteristic for Ω, this condition can be rewritten as

$$\forall x \in X,\ g(f(x), f(x^0)) \le \lambda_0 \Longrightarrow \forall x' \in \Gamma(x, x^0),\ g(f(x'), f(x^0)) \le \lambda_0.$$

By virtue of Remark 2.2 and taking into account that $\lambda_0 = g(f(x^o), f(x^0))$, the above condition means exactly (ii). □

Lemma 4.1. *Let $f : X \to E_2$ be a function defined on a nonempty and Γ-convex set $X \subset E_1$ and let $g : E_2 \times E_2 \to \mathbb{R}$ be λ-characteristic for Ω, with respect to some $\lambda \in \mathbb{R}$. If the function $g(f(\cdot), y) : X \to \mathbb{R}$ is $(\Gamma, \ge)$-quasiconvex on X, for every point $y \in E_2$, then f is (Γ, Ω)-quasiconvex on X.*

Proof. Suppose on the contrary that f is not (Γ, Ω)-quasiconvex on X. Then, according to Definition 2.2, there exist $x^1, x^2 \in X,\ y \in E_2$ and $x^0 \in \Gamma(x^1, x^2)$

such that $f(x^1),\ f(x^2) \in \Omega y$ and $f(x^0) \in \Omega^c y$. The function g is λ-characteristic for Ω and according to Definition 3.1 i) we obtain

$$g(f(x^1), y) \leq \lambda,\ g(f(x^2), y) \leq \lambda \quad \text{and} \quad g(f(x^0), y) > \lambda. \tag{4.1}$$

On the other hand, since $g(f(\cdot), y)$ is $(\Gamma, \geq)$-quasiconvex on X, we have

$$g(f(x), y) \leq \max\{g(f(x^1), y), g(f(x^2), y)\}\ \forall x \in \Gamma(x^1, x^2)$$

by virtue of Remark 2.2. In particular, for $x = x^0$ we obtain a contradiction with (4.1). □

Theorem 4.2. *Let $f : X \to E_2$ be a function defined on a nonempty and Γ-convex set $X \subset E_1$ and let $g : E_2 \times E_2 \to \mathbb{R}$ be a properly λ-characteristic function for Ω, with respect to some $\lambda \in \mathbb{R}$. The following assertions are equivalent:*

(i) f *is* (Γ, Ω)*-quasiconvex on* X*;*

(ii) $g(f(\cdot), y) : X \to \mathbb{R}$ *is* $(\Gamma, \geq)$*-quasiconvex on* X*, for any point* $y \in E_2$*.*

Proof. In view of Lemma 4.1 we just need to prove the implication (i) $\Rightarrow$ (ii).

For this aim, suppose on the contrary that there exists $y \in E_2$ such that $g(f(\cdot), y)$ is not $(\Gamma, \geq)$-quasiconvex on X. Then, in view of Remark 2.2, there exist $x^1, x^2 \in X$ and $x^3 \in \Gamma(x^1, x^2)$ such that

$$g(f(x^3), y) > \max\{g(f(x^1), y), g(f(x^2), y)\}.$$

Without loss of generality, we can suppose that $g(f(x^1), y) \leq g(f(x^2), y)$. The function g is properly λ-characteristic for Ω. According to Definition 3.1 (ii) applied for $y^1 = f(x^1)$, $y^2 = f(x^2)$ and $y^3 = f(x^3)$, we can find a point $y^0 \in E_2$ such that

$$g(f(x^1), y^0) \leq g(f(x^2), y^0) \leq \lambda < g(f(x^3), y^0).$$

According to Definition 3.1 (i), we deduce that $f(x^1), f(x^2) \in \Omega y^0$ and $f(x^3) \in \Omega^c y^0$. Finally, since $x^3 \in \Gamma(x^1, x^2)$, we infer

$$f(\{x^1, x^2\}) \subset \Omega y^0 \quad \text{and} \quad f(\Gamma(x^1, x^2)) \not\subset \Omega y^0,$$

contradicting the assumption (i). □

5. Conclusions

We have shown that the (Γ, Ω)-quasiconvex functions can be characterized in terms of scalar quasiconvexity by means of the properly characteristic functions associated to the binary relation Ω. Beyond the classical framework of quasiconvex vector optimization, where Γ is the convex hull operator and Ω is the partial order induced by a convex cone, our results may be useful for studying the structure of the efficient solutions set of more general optimization problems involving (Γ, Ω)-quasiconvex objective functions. Such problems arise, for instance, when the decision maker needs to consider a preference relation Ω which is defined in the image space by a supplementary utility function, this one being directly related to a properly characteristic function associated to Ω. On the other hand, by considering the concept of Γ-convexity, our study may be applied for a large class of optimization problems involving non-convex feasible sets, whenever some generalized convexity concepts are considered, such as: midpoint- or rational-convexity, polygonal-convexity, metric-convexity, or other specific convexity notions which appear for instance in transportation problems on graphs.

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