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SOME CONDITIONS FOR NONEMPTINESS OF γ-SUBDIFFERENTIALS OF γ-CONVEX FUNCTIONS

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Abstract. γ-subdifferential is a concept which can be used for global optimization. If x_* is a global minimizer of an arbitrary function f then $0 \in \partial_\gamma f(x_*)$, where $\partial_\gamma f(x_*)$ is the γ-subdifferential of f at x_*. In particular, $\partial_\gamma f(x_*) \neq \emptyset$ at a global minimizer x_*. In this paper we investigate the nonemptiness and the monotonicity of γ-subdifferentials of γ-convex functions. Some sufficient conditions are stated for the nonemptiness of the γ-subdifferential of a symmetrically γ-convex function at a point. It is proved that for a symmetrically γ-convex function, the Gâteaux derivative (when it exists) at a point belongs to the γ-subdifferential at that point. A relation between the γ-subdifferential and the Clarke generalized gradient of a symmetrically γ-convex function is also presented.

1. Introduction

Global optimization is a very active field of mathematical programming. To find an extremum of a given function f, a popular method is to seek all the local extrema and then compare them. If f is differentiable then a necessary condition for local extremality is $f'(x_*) = 0$. Since, however, many functions are not differentiable, new tools generalizing the concept of differential have been introduced. For convex functions, subdifferentials are often used where derivatives do not exist. If f is convex, a necessary and sufficient condition for x_* to be a local (and also a global) minimizer is $0 \in \partial f(x_*)$, where $\partial f(x_*)$ denotes the subdifferential of f at x_*. For locally Lipschitzian functions, in order to to seek a local extremum, one may solve the inclusion $0 \in \partial f(x)$ where $\partial f(x)$ is now the Clarke generalized gradient of f at x.

In [5–6], H. X. Phu introduced the notion of γ-subdifferential $\partial_\gamma f$ of an arbitrary function f and proved that $0 \in \partial_\gamma f(x_*)$ is a necessary condition for a global minimizer. For a γ-convex function, this condition implies that if it has a global minimum, there is a global minimizer near x_*. Some basic properties of $\partial_\gamma f(x)$ of an arbitrary function f were investigated in [5–6] (in [5], γ is not a constant, it is a continuous and positive function such that $x + \gamma(x)$ is strictly increasing).

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In this paper we first consider the monotonicity of γ-subdifferentials of γ-convex functions and prove that in multidimensional pre-Hilbert spaces, γ-subdifferential of each additive function that is not linear is empty at every point (see Section 2). Then, in Section 3, we restrict ourselves to symmetrically γ-convex functions and state some conditions under which the γ-subdifferential of a symmetrically γ-convex function is nonempty. We establish a relation between Gâteaux derivatives and γ-subdifferentials of symmetrically γ-convex functions. It is also shown that under some assumptions, the Clarke generalized gradient of a symmetrically γ-convex function at a point is a subset of its γ-subdifferential at that point. From this result it follows that for a symmetrically γ-convex function on a finite dimensional space, its γ-subdifferential at certain points is always nonempty.

2. Definitions and some properties

Let X be a real normed space and γ be a fixed positive number. Consider a function f whose effective domain is D, i.e. such that $D = \{x|\ f(x) < +\infty\}$ (in this paper we assume that $f(x) > -\infty$ for all $x \in X$). For the convenience of the reader we recall from [6] the definition of the γ-subdifferential of the function f and its properties.

First let us introduce some notations. For an $r > 0$, set

$$\begin{aligned} &S_r(x) = \{y \in X : \|y - x\| = r\}, \quad S = S_1(0),\\ &\mathcal{U}_r(x) = \{y \in X : \|y - x\| \le r\},\\ &\mathrm{int}_r D = \{x \in D : \exists r' = r'(x) > r,\ \mathcal{U}_{r'}(x) \subset D\}. \end{aligned}$$

Definition 2.1. The γ-subdifferential of f at $x \in D$, denoted by $\partial_\gamma f(x)$, is the subset of the dual space X^* given by

$$\begin{aligned} \{\xi \in X^* :\ &\text{for } s \in S, \text{ there exists } \lambda \in [0,\gamma] \text{ such that}\\ &\gamma\langle \xi, s\rangle \le f(x + \lambda s) - f(x - (\gamma - \lambda)s)\}. \end{aligned}$$

In other words,

$$\begin{aligned} \partial_\gamma f(x) = \{\xi \in X^* :\ &\text{for } s \in S, \text{ there exist } x' \text{ and } x'' \text{ such that}\\ &x' - x'' = \gamma s,\ x \in [x', x''],\ \langle \xi, x' - x''\rangle \le f(x') - f(x'')\}. \end{aligned}$$

Here, $[x', x'']$ denotes the closed line segment with endpoints x' and x''.

Proposition 2.1. [6, Theorem 2.1] *Let*

$$\Delta_{f,\gamma}(x, s) := \Big\{ \frac{f(x') - f(x'')}{\gamma} : x \in [x', x''], x' - x'' = \gamma s \Big\},\ x \in D,\ s \in S.$$

Then

$$\partial_\gamma f(x) = \{\xi \in X^* :\ \langle \xi, s\rangle \in \text{ conv } \Delta_{f,\gamma}(x, s),\ s \in S\}.$$

If f is continuous on $\mathcal{U}_\gamma(x)$ then

$$\partial_\gamma f(x) = \{\xi \in X^* :\ \langle \xi, s\rangle \in \Delta_{f,\gamma}(x, s),\ s \in S\}.$$

Corollary 2.1. [6, Proposition 2.1] *Let* $X = \mathbb{R}$ *and*

$$\mathcal{M}_{f,\gamma}(x) := \Big\{ \frac{f(x'+\gamma) - f(x')}{\gamma} : \ x \in [x', x'+\gamma] \subset \mathbb{R}, \ x' \in D \Big\}, \ x \in D.$$

Then $\partial_\gamma f(x) = \mathrm{conv}\mathcal{M}_{f,\gamma}(x)$. *If* f *is continuous on* $[x-\gamma, x+\gamma] \subset D \subset \mathbb{R}$ *then* $\partial_\gamma f(x) = \mathcal{M}_{f,\gamma}(x)$.

Corollary 2.2. [6, Corollary 2.1] *Let* $x \in D$ *and* $S_D(x) := \{s \in S : x + \gamma s \in D\}$. *Then,*

$$\partial_\gamma f(x) = \{\xi \in X^* : \ \langle \xi, s\rangle \in \ \mathrm{conv}\Delta_{f,\gamma}(x,s), \ s \in S_D(x)\}. \tag{2.1}$$

Corollary 2.1 shows that the γ-subdifferential of a function on the real line is convex at each point. This is also true for functions on a normed space.

Proposition 2.2. [6, Propositions 2.2–2.3] *Suppose* $f : D \subset X \to \mathbb{R}$. *Then*

(a) $\partial_\gamma f(x)$ *is convex for all* $x \in D$ *and*

(b) $\partial_\gamma f(x)$ *is compact if* $\dim X < \infty$ *and* f *is continuous on* $\mathcal{U}_\gamma(x)$.

γ-subdifferential can be used for global optimization. More precisely, we have

Proposition 2.3. *Let* $f : D \to \mathbb{R}$.

(i) *If* $f(x^*) \le f(x)$ *for* $x \in \mathcal{U}_\gamma(x^*) \cap D$ *then* $0 \in \partial_\gamma f(x^*)$.

(ii) *If* $f(x^*) \ge f(x)$ *for* $x \in \mathcal{U}_\gamma(x^*) \cap D$ *then* $0 \in \partial_\gamma f(x^*)$.

This proposition gives a necessary condition for global optimization, see [6, Theorems 4.1–4.2]. For any γ-convex function f, if it has a global minimum then the inclusion $0 \in \partial_\gamma f(x^*)$ will be a sufficient condition for a global minimizer near x^* as the next proposition shows.

From now on, we assume that D is a nonempty convex subset of X.

Definition 2.2. A function $f : D \to \mathbb{R}$ is said to be

(i) γ-convex if $x_0, x_1 \in D$, $\|x_1 - x_0\| \ge \gamma$ imply $f(x_0') + f(x_1') \le f(x_0) + f(x_1)$;

(ii) symmetrically γ-convex if $x_0, x_1 \in D$, $|x_1 - x_0\| \ge \gamma$ imply

$$f(x_0') \le \Big(1 - \frac{\gamma}{\|x_1 - x_0\|}\Big) f(x_0) + \frac{\gamma}{\|x_1 - x_0\|} f(x_1),$$
$$f(x_1') \le \frac{\gamma}{\|x_1 - x_0\|} f(x_0) + \Big(1 - \frac{\gamma}{\|x_1 - x_0\|}\Big) f(x_1),$$

where $x_0' := x_0 + \gamma \dfrac{x_1 - x_0}{\|x_1 - x_0\|}, \ \ x_1' := x_1 - \gamma \dfrac{x_1 - x_0}{\|x_1 - x_0\|}.$

Obviously, a symmetrically γ-convex function is also γ-convex.

The following may be useful for optimization of γ-convex functions.

Proposition 2.4. [6, Theorems 4.3–4.4] *Suppose that* $f : D \subset X \to \mathbb{R}$ *is* γ-*convex and* $x^* \in D$.

(i) *If* $f(x_*) \le f(x)$ *for all* $x \in \mathcal{U}_\gamma(x_*) \cap D$ *then* x_* *is a global minimizer.*

(ii) *If* $0 \in \partial_\gamma f(x^*)$ *then for each* $x_0 \in D \setminus \mathcal{U}_\gamma(x^*)$ *there exists an* $x_k \in \mathcal{U}_\gamma(x^*) \cap D$ *with* $f(x_k) \le f(x_0)$.

(iii) *If* $0 \in \partial_\gamma f(x^*)$ *and if* $f(x_*) \le f(x)$ *for all* $x \in \mathcal{U}_\gamma(x^*) \cap D$ *then* x_* *is a global minimizer.*

We now present another formula which describes γ-subdifferentials of γ-convex functions.

Proposition 2.5. *Suppose that* $f : D \subset X \to \mathbb{R}$ *is* γ*-convex and* $x \in D$. *Then*

$$\partial_\gamma f(x) = \Big\{\xi \in X^* : \ \langle \xi, s\rangle \le \frac{f(x+\gamma s) - f(x)}{\gamma}, \ s \in S_D(x)\Big\}. \tag{2.2}$$

Proof. Let $\mathcal{A}$ be the right hand side of (2.2). If $\xi \in \mathcal{A}$ then

$$\gamma\langle \xi, s\rangle \le f(x+\gamma s) - f(x) \quad \text{for all} \quad s \in S$$

because $f(x+\gamma s) - f(x) = \infty$ whenever $s \notin S_D(x)$. Choosing $x' = x + \gamma s$ and $x'' = x$, we have $\gamma\langle \xi, s\rangle \le f(x') - f(x'')$. Hence $\xi \in \partial_\gamma f(x)$, so that $\mathcal{A} \subset \partial_\gamma f(x)$. Conversely, suppose $\xi \in \partial_\gamma f(x)$. For $s \in S_D(x)$, there are $x', x'' \in X$ satisfying

$$x' - x'' = \gamma s, \ x \in [x', x''] \quad \text{and} \quad \gamma\langle \xi, s\rangle \le f(x') - f(x'').$$

Since f is γ-convex and $x, x' \in [x'', x+\gamma s]$,

$$f(x') - f(x'') \le f(x+\gamma s) - f(x).$$

Hence

$$\gamma\langle \xi, s\rangle \le f(x+\gamma s) - f(x) \quad \text{for all} \quad s \in S_D(x).$$

Thus $\xi \in \mathcal{A}$ and $\partial_\gamma f(x) \subset \mathcal{A}$. Consequently, $\mathcal{A} = \partial_\gamma f(x)$. □

Corollary 2.3. *Suppose* $f : D \subset \mathbb{R} \to \mathbb{R}$ *is* γ*-convex. Let*

$$D_\gamma^+ = \{x \in D : \ x + \gamma \in D\},$$
$$D_\gamma^- = \{x \in D : \ x - \gamma \in D\}.$$

(a) $\partial_\gamma f(x) = \mathbb{R}$ *if* $x \in D \setminus (D_\gamma^+ \cup D_\gamma^-)$, *and* $\partial_\gamma f(x) =]-\infty, (f(x+\gamma) - f(x))/\gamma]$ *if* $x \in D_\gamma^+ \setminus D_\gamma^-$.

(b) $\partial_\gamma f(x) = [(f(x) - f(x-\gamma))/\gamma, (f(x+\gamma) - f(x))/\gamma]$ *if* $x \in D_\gamma^+ \cap D_\gamma^-$.

(c) $\partial_\gamma f(x) = [(f(x) - f(x-\gamma))/\gamma, +\infty[$ *if* $x \in D_\gamma^- \setminus D_\gamma^+$.

Proof. (a) Suppose $x \in D \setminus D_\gamma^-$. If $x \notin D_\gamma^+$ then $S_D(x) = \emptyset$ and (2.2) yields $\partial_\gamma f(x) = \mathbb{R}$. If $x \in D_\gamma^+$ then $S_D(x) = \{1\}$ and applying (2.2) once more, we get

$$\partial_\gamma f(x) = \Big\{\xi \in \mathbb{R} : \xi \le \frac{f(x+\gamma) - f(x)}{\gamma}\Big\} = \Big] -\infty, \ \frac{f(x+\gamma) - f(x)}{\gamma}\Big].$$

(b) If $x \in D_\gamma^- \cap D_\gamma^+$ then $S_D(x) = \{-1, 1\}$. Hence $\xi \in \partial_\gamma f(x)$ iff

$$\xi \le \frac{f(x+\gamma) - f(x)}{\gamma} \quad \text{and} \quad -\xi \le \frac{f(x-\gamma) - f(x)}{\gamma}.$$

Thus the assertion (b) holds. Similar arguments apply to the case (c). □

It follows from Corollary 2.3 that for any γ-function $f : D \subset \mathbb{R} \to \mathbb{R}$, $\partial_\gamma f(x)$ is always nonempty and has a simple structure at every $x \in D$. There arises a question: Is it true that for every γ-convex function $f : D \subset X \to \mathbb{R}$, $\partial_\gamma f(x)$ is nonempty at least for every x satisfying $\mathcal{U}_\gamma(x) \subset D$? The answer is negative as Proposition 2.6 below shows.

We recall that a function f on a normed space X is said to be additive if $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. It is easy to verify that an additive function is γ-convex for arbitrary positive number γ. If $X \neq \{0\}$ then there exists an additive function f on X such that f is not linear. Indeed, there is an additive function $H : \mathbb{R} \to \mathbb{R}$ that is not linear, see [2, p. 6]. Since $X \neq \{0\}$, by the Hahn-Banach theorem, there is a continuous linear functional $\varphi : X \to \mathbb{R}$ such that $\varphi \neq 0$, i.e., $\varphi(X) = \mathbb{R}$. Let $f = H \circ \varphi$ then f is additive. If f is linear then, for any $\alpha, \beta \in \mathbb{R}$, there is an $x \in X$ such that $\varphi(x) = \beta$ and we have

$$H(\alpha\beta) = H(\alpha\varphi(x)) = H(\varphi(\alpha x)) = f(\alpha x) = \alpha f(x) = \alpha H(\varphi(x)) = \alpha H(\beta),$$

i.e., H is linear, a contradiction. Thus f is not linear.

Proposition 2.6. *Suppose that X is a pre-Hilbert space and* $\dim X \geq 2$. *Then there exists a γ-convex function $f : X \to \mathbb{R}$ such that $\partial_\gamma f(x) = \emptyset$ for all $x \in X$.*

Proof. Let f be an arbitrary additive function on X such that f is not linear. Then for all $x \in X$ and all rational number r, $f(rx) = rf(x)$. Suppose, contrary to our claim, that $\partial_\gamma f(x_0) \neq \emptyset$ for some $x_0 \in X$. Choose any $\xi \in \partial_\gamma f(x_0)$. For each $s \in S$ there exists $\lambda \in [0, \gamma]$ such that

$$\gamma\langle \xi, s\rangle \leq f(x + \lambda s) - f(x - (\gamma - \lambda)s) = f(\gamma s).$$

Replacing s by $-s$ we get

$$-\gamma\langle \xi, s\rangle \leq f(\gamma(-s)) = -f(\gamma s)$$

i.e., $\gamma\langle \xi, s\rangle \geq f(\gamma s)$. Hence

$$\gamma\langle \xi, s\rangle = f(\gamma s) \quad \text{for all} \quad s \in S. \tag{2.3}$$

Suppose now that $s \in S$ and $r \in \mathbb{R}$, $0 \leq |r| < 1$. Choose $t \in \mathbb{R}$ such that $r^2 + t^2 = 1$. Since $\dim X \geq 2$, there exists $s' \in S$ satisfying $(s|s') = 0$ (where $(\cdot|\cdot)$ denotes the inner product of X). Let $y = rs + ts'$ and $z = rs - ts'$ then $y, z \in S$ and $2rs = y + z$. Applying (2.3) we have

$$2f(r\gamma s) = f(2r\gamma s) = f(\gamma y + \gamma z) = f(\gamma y) + f(\gamma z) = \gamma\langle \xi, y\rangle + \gamma\langle \xi, z\rangle$$

and

$$\gamma\langle \xi, y\rangle + \gamma\langle \xi, z\rangle = \gamma\langle \xi, y + z\rangle = \gamma\langle \xi, 2rs\rangle = 2r\gamma\langle \xi, s\rangle = 2rf(\gamma s).$$

Thus,

$$f(r\gamma s) = rf(\gamma s) \quad \text{for all} \quad s \in S \quad \text{and} \quad 0 \leq |r| < 1. \tag{2.4}$$

Now for each $x \in X$, $x \neq 0$ and $\alpha \in \mathbb{R}$, we choose a natural number n satisfying $\alpha\|x\|/n\gamma < 1$ and $\|x\|/n\gamma < 1$. Then (2.4) yields

$$\frac{1}{n} f(\alpha x) = f\big(\frac{\alpha}{n} x\big) = f\Big(\frac{\alpha\|x\|}{n\gamma}\gamma\frac{x}{\|x\|}\Big) = \frac{\alpha\|x\|}{n\gamma} f\big(\gamma\frac{x}{\|x\|}\big)$$

and

$$\frac{\alpha\|x\|}{n\gamma} f\big(\gamma\frac{x}{\|x\|}\big) = \alpha f\Big(\frac{\|x\|}{n\gamma}\gamma\frac{x}{\|x\|}\Big) = \alpha f\big(\frac{1}{n}x\big) = \frac{\alpha}{n} f(x).$$

Consequently, $f(\alpha x) = \alpha f(x)$ for all $x \in X$ and $\alpha \in \mathbb{R}$, i.e., f is linear. This contradiction completes the proof. □

Remark. Proposition (2.6) still holds if it is only assumed that X is a normed space with $\dim X \geq 2$. However, the proof is more complicated.

We conclude this section with a property of monotonicity of γ-subdifferentials of γ-convex functions. We know that subdifferentials of convex functions are monotone. γ-subdifferentials of γ-convex functions have a similar property.

Proposition 2.7. *Suppose that $f : D \to \mathbb{R}$ is γ-convex. Then*

$$x, y \in D,\ \xi \in \partial_\gamma f(x),\ \eta \in \partial_\gamma f(y),\ \|x - y\| \geq \gamma \Longrightarrow \langle \xi - \eta, x - y\rangle \geq 0. \tag{2.5}$$

Proof. Let $s = \dfrac{x-y}{\|x-y\|}$. For $\xi \in \partial_\gamma f(x)$, $\eta \in \partial_\gamma f(y)$, there exist $\lambda, \mu \in [0, \gamma]$ such that

$$\gamma\langle \xi, -s\rangle \leq f(x + \lambda(-s)) - f(x - (\gamma - \lambda)(-s))$$

and

$$\gamma\langle \eta, s\rangle \leq f(y + \mu s) - f(y - (\gamma - \mu)s).$$

Hence

$$\gamma\langle \xi, s\rangle \geq f(x + (\gamma - \lambda)s) - f(x - \lambda s)$$

and

$$-\gamma\langle \eta, s\rangle \geq f(y - (\gamma - \mu)s) - f(y + \mu s).$$

Therefore

$$\gamma\langle \xi - \eta, s\rangle \geq f(x + (\gamma - \lambda)s) + f(y - (\gamma - \mu)s) - [f(x - \lambda s) + f(y + \mu s)] \geq 0.$$

The last inequality holds because

$$\|x + (\gamma - \lambda)s - (y - (\gamma - \mu)s)\| = \|x - y\| + 2\gamma - (\lambda + \mu) \geq \|x - y\| \geq \gamma.$$

□

Remark. If $D \subset \mathbb{R}$ then (2.5) is sufficient for the γ-convexity. In fact, by [7, Theorem 2.3], a function $f : D \subset \mathbb{R} \to \mathbb{R}$ is γ-convex iff

$$\xi \leq \eta \text{ for all } \xi \in \partial_\gamma f(x),\ \eta \in \partial_\gamma f(x + \gamma) \text{ and } \{x, x + \gamma\} \subset D.$$

3. γ-SUBDIFFERENTIALS OF SYMMETRICALLY γ-CONVEX FUNCTIONS

In this section, we consider γ-subdifferentials of symmetrically γ-convex functions. It is possible that the intersection of $\partial_\gamma f(x)$ of a γ-convex function f and its Clarke generalized gradient $\partial f(x)$ at x is empty. For instance, $f(x) = \cos x$ is γ-convex for $\gamma = 2\pi$ (see [6, Example 2.3]) and $\partial_\gamma f(x) = \{0\}$ while $\partial f(x) = \{-\sin x\}$ for all $x \in \mathbb{R}$. However, the situation is slightly different for symmetrically γ-convex functions.

Proposition 3.1. *Suppose that $f : D \to \mathbb{R}$ is symmetrically γ-convex and $x_0 \in \operatorname{int} D$. If f is Gâteaux differentiable at x_0 then $Df(x_0) \in \partial_\gamma f(x_0)$.*

Proof. Suppose $s \in S_D(x_0)$ and $t > 0$ such that $x_0 - ts \in D$. Symmetrical γ-convexity of f implies

$$f(x_0) \le \frac{\gamma}{\gamma + t} f(x_0 - ts) + \frac{t}{\gamma + t} f(x_0 + \gamma s).$$

Hence,

$$\frac{f(x_0 - ts) - f(x_0)}{t} \ge \frac{f(x_0) - f(x_0 + \gamma s)}{\gamma}\,.$$

Letting $t \downarrow 0$ we get

$$\langle Df(x_0), -s \rangle \ge \frac{f(x_0) - f(x_0 + \gamma s)}{\gamma},$$

i.e.,

$$\langle Df(x_0), s \rangle \le \frac{f(x_0 + \gamma s) - f(x_0)}{\gamma}, \quad s \in S_D(x_0).$$

That $Df(x_0) \in \partial_\gamma f(x_0)$ follows from Proposition 2.5. □

Corollary 3.1. *If $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is symmetrically γ-convex then the set of all $x \in \operatorname{int}_\gamma D$ such that $\partial_\gamma f(x) = \emptyset$ is a set of Lebesgue measure 0.*

Proof. This follows from the preceding proposition and the fact that f is differentiable almost everywhere in $\operatorname{int}_\gamma D$, see [3, Corollary 3.7]. □

Combining Propositions 2.4 and 3.1 we obtain

Corollary 3.2. *Suppose that $f : D \to \mathbb{R}$ is symmetrically γ-convex. If $f'(x_0) = 0$ and if $f(x_*) \le f(x)$ for some $x_* \in \mathcal{U}_\gamma(x_0) \cap D$ and for all $x \in \mathcal{U}_\gamma(x_0) \cap D$, then x_* is a global minimizer of f.*

We now state a relation between the γ-subdifferential and the Clarke generalized gradient of a symmetrically γ-convex function.

Proposition 3.2. *Suppose that $f : D \to \mathbb{R}$ is symmetrically γ-convex and $x_0 \in \operatorname{int}_\gamma D$. If f is locally Lipschitzian at x_0 and continuous at each point of $S_\gamma(x_0)$ then $\partial f(x_0) \subset \partial_\gamma f(x_0)$, where $\partial f(x_0)$ is the Clarke generalized gradient of f at x_0. In particular, $\partial_\gamma f(x_0)$ is nonempty.*

Proof. Suppose $s \in S$. The generalized directional derivative of f at x_0 in the direction s, denoted as $f^\circ(x_0; s)$, is defined by

$$f^\circ(x_0; s) = \lim_{\varepsilon \downarrow 0} \sup_{\|x - x_0\| < \varepsilon} \sup_{0 < t < \varepsilon} \frac{f(x + ts) - f(x)}{t} \tag{3.1}$$

(see [1, p. 36]). Since f is locally Lipschitzian at x_0, there exist a positive number K and a ball $\mathcal{U}_r(x_0)$ such that

$$|f(x) - f(x')| \le K\|x - x'\| \quad \text{for all} \quad x, x' \in \mathcal{U}_r(x_0).$$

Let ε satisfy $0 < \varepsilon < r$ and $\mathcal{U}_{\gamma+2\varepsilon}(x_0) \subset D$. For

$$x \in X, \ \|x - x_0\| < \varepsilon, \ s \in S \quad \text{and} \quad 0 < t < \varepsilon,$$

we have $x + (t + \gamma)s \in \mathcal{U}_{\gamma+2\varepsilon}(x_0)$. Hence

$$f(x + ts) \le \frac{\gamma}{\gamma + t} f(x) + \frac{t}{\gamma + t} f(x + (t + \gamma)s).$$

Consequently,

$$\frac{f(x + ts) - f(x)}{t} \le \frac{f(x + (t + \gamma)s) - f(x)}{\gamma + t}. \tag{3.2}$$

On the other hand, since $\varepsilon < r$,

$$\begin{aligned}
&\big|[f(x + (t + \gamma)s) - f(x)] - [f(x_0 + (t + \gamma)s) - f(x_0)]\big| \\
\le &\big|f(x + (t + \gamma)s) - f(x_0 + (t + \gamma)s)\big| \ + \ \big|f(x) - f(x_0)\big| \\
\le &\big|f(x + (t + \gamma)s) - f(x_0 + (t + \gamma)s)\big| \ + \ K\|x - x_0\| \\
\le &\big|f(x + (t + \gamma)s) - f(x_0 + (t + \gamma)s)\big| + K\epsilon.
\end{aligned}$$

Thus, (3.2) yields

$$\sup_{0 < t < \varepsilon} \frac{f(x + ts) - f(x)}{t} \le \tag{3.3}$$

$$\sup_{0 < t < \varepsilon} \Big[\frac{f(x_0 + (t + \gamma)s) - f(x_0) + |f(x + (t + \gamma)s) - f(x_0 + (t + \gamma)s)|}{t + \gamma} + \frac{K\varepsilon}{t + \gamma}\Big].$$

Since f is continuous at $x_0 + \gamma s$, (3.1) and (3.3) imply

$$f^\circ(x_0; s) \le \frac{f(x_0 + \gamma s) - f(x_0)}{\gamma}.$$

If $\xi \in \partial f(x_0) := \{\varphi \in X^* : \langle \varphi, v \rangle \le f^\circ(x_0; v) \text{ for all } v \in X\}$ (see [1, p. 27]) and if $s \in S$ then $\langle \xi, s \rangle \le f^\circ(x_0; s)$. Hence

$$\langle \xi, s \rangle \le \frac{f(x_0 + \gamma s) - f(x_0)}{\gamma} \quad \text{for all } s \in S.$$

Thus by (2.2), $\xi \in \partial_\gamma f(x_0)$, i.e., $\partial f(x_0) \subset \partial_\gamma f(x_0)$. Finally, $\partial f(x_0)$ is nonempty (see [1, p. 27]) and so is $\partial_\gamma f(x_0)$. □

Corollary 3.3. *If $f : D \to \mathbb{R}$ is symmetrically γ-convex and locally Lipschitzian at an $x_0 \in \operatorname{int}_{2\gamma} D$ then $\emptyset \ne \partial f(x_0) \subset \partial_\gamma f(x_0)$.*

Proof. Since $S_\gamma(x_0) \subset \operatorname{int}_\gamma D$, the proof follows from Proposition 3.3 and [3, Corollary 3.6]. $\square$

An immediate result of the preceding corollary and [4, Corollary 2.1] is the following.

Corollary 3.4. *If $f : D \to \mathbb{R}$ is symmetrically γ-convex and bounded from above on a ball $\mathcal{U}_r(x_0) \subset D$ for some $r > 2\gamma$ then $\emptyset \neq \partial f(x_0) \subset \partial_\gamma f(x_0)$.*

Corollary 3.5. *If $\dim X < \infty$ and $f : D \subset X \to \mathbb{R}$ is symmetrically γ-convex then $\partial_\gamma f(x_0)$ is nonempty, convex and compact whenever $x_0 \in \operatorname{int}_{2\gamma} D$.*

Proof. Applying Corollary 3.3 and [3, Theorem 3.1] again, we get $\emptyset \neq \partial f(x_0) \subset \partial_\gamma f(x_0)$. The convexity and the compactness of $\partial_\gamma f(x_0)$ follow from [3, Theorem 3.1] and Proposition 2.2. $\square$

4. Concluding remarks

In this paper we have presented some sufficient conditions for nonemptiness of the γ-subdifferential of a symmetrically γ-convex function. Some open questions are the following:

Is there a γ-convex function f defined on all of the space X that is continuous and $\partial_\gamma f(x) = \emptyset$ at some point $x \in X$?

Under which condition, a set-valued map $T : X \to 2^{X^*}$ will be the γ-subdifferential of a γ-convex function?

These and other questions will be subjects of further investigation.

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