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LYAPUNOV STABILITY OF NONLINEAR TIME-VARYING DIFFERENTIAL EQUATIONS

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Abstract. The paper studies asymptotic stability of nolinear time-varying differential equations by Lyapunov's direct method. Sufficient conditions for asymptotic stability are given in terms of nondifferentiable Lyapunov-like functions. An application to stabilizability of a class of nonlinear control systems with feedback controls is also given.

1. Introduction

Consider a nonlinear time-varying differential equation of the form

$$\dot{x}(t) = f(t, x(t)), \quad t \geq 0 \tag{1}$$

It is well known that there are two major approaches to the Lyapunov stability analysis of system (1): the first linearization method and the second direct method. Stability of system (1) can be investigated via the first linearization method, but in general and the most powerful technique is the second direct method. For this method one usually assumes the existence of, so called Lyapunov function, a positive definite function with negative derivative along the trajectory of the system. In the last decade the Lyapunov direct second method has been a fruitful technique in stability analysis of nonlinear differential equations and has gained increasing significance in the development of qualitative theory of dynamical systems [5, 6, 9, 18]. There are a number of books and papers available expouding the extensions and generalizations of Lyapunov-like functions, see, e.g., [2, 7, 8, 14, 16, 19] and references therein. It is recognized that the Lyapunov-like functions serve as a main tool to reduce a given complicated system into a relatively simpler system and provide useful applications to control systems [1, 3, 10, 11, 15, 17].

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Unlike the previous papers [4, 12, 13], where some stability results are given by non-Lyapunov function approach, in this paper we investigate the asymptotic stability of a class of nonlinear differential equations in terms of nonsmooth Lyapunov-like functions. In this general setup, the class of systems considered is allowed to be time-varying and we relax the boundedness condition on the system. The Lyapunov-like functions proposed in the paper need not be differentialble and not be even continuous. Based on the stability results obtained for system (1), as an application, we derive sufficient conditions for stabilizability of nonlinear control systems by nonlinear feedback controls.

The paper is organized as follows. In Section 2 we give main notations and definitions of Lyapunov-like functions needed later. Section 3 presents main theorems on sufficient conditions for asymptotic stability with the proposed Lyapunov-like functions. An application to stabilizability of nonlinear control systems is also given. The conclusion is drawn in Section 4.

2. Notations and definitions

We shall employ the following notations and definitions thoroughout this paper: $X = R^n$ denotes the n-dimensional Euclidean space with the corresponding norm $\|.\|$; B_ε denotes the open unit ball with radius ε centered at zero, R denotes the real line; R^+ denotes the set of non-negative real numbers; Z^+ denotes the set of non-negative integers.

Consider the following nonlinear time-varying differential equation with the initial condition:

$$\begin{cases} \dot{x}(t) & = f(t, x(t)), \quad t \geq t_0 \in R^+, \\ x(t_0) & = x_0, \end{cases} \tag{2}$$

where the states $x(t)$ take values in X, $f(t, x) : R^+ \times X \to X$ is a given nonlinear function and $f(t, 0) = 0$ for all $t \in R^+$. We shall assume that conditions are imposed on the system (2) such that the existence of its solutions is guaranteed.

Definition 2.1. The zero solution of (2) is said to be stable if for every $\varepsilon > 0$, $t_0 \in R^+$, there exists a number $\delta > 0$ (depending upon ε and δ) such that for any solution $x(t)$ of (2) with $\|x_0\| < \delta$ implies $\|x(t)\| < \varepsilon$, for all $t \geq t_0$.

Definition 2.2. The zero solution of (2) is said to be asymptotically stable if it is stable and there is a number $\delta > 0$ such that any solution $x(t)$ with $\|x_0\| < \delta$ satisfies $\lim\limits_{t \to \infty} \|x(t)\| = 0$.

In the above definitions, if the number $\delta > 0$ is independent of t_0, then the zero solution of the system is said to be uniformly (asymptotically) stable.

Definition 2.3. A function $f(t,x)$: $R^+ \times X \to X$ is Lipschitz in x uniformly with respect to (u.w.r.t.) $t \in R^+$ if there is a number $L > 0$ such that

$$\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|, \quad \forall (t,x_1,x_2) \in R^+ \times X \times X.$$

Let $V(t,x)$: $R^+ \times W \to R$ be a given function, where $W \subseteq X$ is some open neighborhood of the origin. We define

$$D_f^+ V(t,x) = \lim \sup_{h \to 0^+} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h},$$
$$D_- V(t,x) = \lim_{y \to 0} \inf_{h \to 0^+} \Delta_{h,y} V(t,x),$$

where

$$\Delta_{h,y} V(t,x) = \frac{V(t+h, x+hf(t,x)+hy) - V(t,x)}{h}.$$

Definition 2.4. [8, 19] A positive definite function $V(t,x)$: $R^+ \times W \to R$ is a weak Lyapunov function of system (1) if it is continuous in $(t,x) \in R^+ \times W$ and Lipschitz in $x \in W$ (u.w.r.t. $t \in R^+$) and there is a non-decreasing continuous function $\gamma(.) : R^+ \to R^+ \setminus \{0\}$ such that

$$D_f^+ V(t,x) \leq -\gamma(\|x\|) < 0, \quad \forall t \in R^+, x \in W \setminus \{0\}. \tag{3}$$

Definition 2.5. A function $V(t,x)$: $R^+ \times W \to R$ is a Lyapunov-like function of system (1) if it satisfies the following conditions:
(i) There exist a non-decreasing function $a(t)$: $R^+ \to R^+ \setminus \{0\}$, a non-increasing function $b(t)$: $R^+ \to R^+ \setminus \{0\}$, and numbers $a > 0$, $b > 0$ such that

$$a(t)\|x\|^a \leq V(t,x) \leq b(t)\|x\|^b, \quad \forall (t,x) \in R^+ \times W. \tag{4}$$

(ii) There are non-decreasing functions $\gamma(.)$: $R^+ \to R^+ \setminus \{0\}$, $c(.)$: $R^+ \to R^+ \setminus \{0\}$, such that

$$D_- V(t,x) \leq -c(t)\gamma(V(t,x)), \quad \forall t \in R^+, x \in W \setminus \{0\}. \tag{5}$$

Definition 2.6. A function $V(t,x)$: $R^+ \times W \to X$ is a generalized Lyapunov-like function of system (1) if it satisfies the following conditions:
(i) There exist functions $a(t,h)$, $b(t,h)$: $R^+ \times R^+ \to R^+ \setminus \{0\}$, $a(t,0) = b(t,0) = 0$ which are continuously strictly increasing in $h \in R^+$, $a(t,h)$ is non-decreasing in t, $b(t,h)$ is non-increasing in t, such that

$$a(t,\|x\|) \leq V(t,x) \leq b(t,\|x\|), \quad \forall (t,x) \in R^+ \times W. \tag{6}$$

(ii) For every $T > 0$, there are a sequence of positive numbers $\{t_n\}$ going to zero and a function $\gamma(t,h)$: $R^+ \times R^+ \to R^+ \setminus \{0\}$, which is integrable, non-decreasing in (t,h), such that

$$\lim_{n\to\infty, y\to 0} \Delta_{t_n,y} V(t,x) \leq -\int_t^{t+T} \gamma(s,\|x\|)ds, \quad \forall t \in R^+, x \in W \setminus \{0\}. \tag{7}$$

Remark 2.1. If the function $V(t,x)$ is continuous in $t \in R^+$ and Lipschitzian in x (uniformly in $t \in R^+$) and satisfies condition (3), then the Lyapunov function $V(t,x(t))$ is, as in [19], non-increasing in t. However, the Lyapunov-like function in terms of Definitions 2.5, 2.6 is not necessarily continuous in t, Lipschitzian in x and then it is not, in general, non-increasing in t, since the functions $a(t)$, $b(t)$ are not assumed to be continuous in t. The condition (5) or (7) means that the Lyapunov-like function $V(t,x(t))$ is non-increasing along the solution of the system on a sequence.

3. Stability results

In this section we give the stability conditions using the Lyapunov-like functions. We start with the following theorem given in [8] which gives a sufficient condition for the asymptotic stability of system (1) with the weak Lyapunov function in terms of Definition 2.4.

Theorem 3.1. [8] *Assume that*

$$\|f(t,x)\| \leq M, \quad \forall (t,x) \in R^+ \times W.$$

If the system (1) *admits a weak Lyapunov function, then the zero solution is uniformly asymptotically stable.*

In the sequel we need the following lemma.

Lemma 3.1. *Let $G(t,x), V(t,x)$: $R^+ \times W \to R$ be given functions, where $V(t,x)$ is continuous in $x \in W \subset X$ and satisfies the condition*

$$D_-V(t,x) \le G(t,x), \quad \forall (t,x) \in R^+ \times W,$$

then for every solution $x(t)$ of the system (1):

$$\lim\inf_{h\to 0^+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h} \le G(t, x(t)), \quad \text{u.w.r.t. } t \in R^+.$$

Proof. We now assume to the contrary that for every sequence $\{t_n\}$ going to 0^+ there exists a solution $\bar{x}(t)$ of system (1) such that

$$\lim_{n\to\infty} \frac{V(T+t_n, \bar{x}(T+t_n)) - V(T, \bar{x}(T))}{t_n} > G(T, \bar{x}(T)),$$

for some $T > 0$. Then, there exist an integer $N_1 > 0$, a positive number ε_0 small enough such that for all $n > N_1$,

$$\frac{V(T+t_n, \bar{x}(T+t_n)) - V(T, \bar{x}(T))}{t_n} > G(T, \bar{x}(T)) + \varepsilon_0.$$

Noting $\bar{x}(T+t_n) = \bar{x}(T) + t_n f(T, \bar{x}(T)) + o(t_n)$, for some small function $o(h)$: $R^+ \to R^+$: $\lim_{h\to 0} o(h)/h = 0$, we have

(*)
$$\frac{V(T+t_n, \bar{x}(T) + t_n f(T, \bar{x}(T)) + o(t_n)) - V(T, \bar{x}(T))}{t_n} > G(T, \bar{x}(T)) + \varepsilon_0.$$

On the other hand, by the assumption, there exist a sequence $\{t_n^1\}$ going to 0^+, a number $N_2 > 0$, and for $t = T$, $x = \bar{x}(T) = \bar{x}$, $\varepsilon_0 > 0$, there is a number $\delta > 0$, such that

$$(**) \qquad \frac{V(t+t_n^1, x + t_n^1 f(t,x) + t_n^1 y) - V(t,x)}{t_n^1} \le G(t,x) + \varepsilon_0,$$

for all $y \in B_\delta$, $n > N_2$. Let us set in the inequality (*): $t_n = t_n^1$. Then, taking a number $N > \max\{N_1, N_2\}$ and $n > N$ large enough so that $y = \dfrac{o(t_n^1)}{t_n^1} \in B_\delta$, from (*) it follows that

$$\frac{V(T+t_n^1, \bar{x} + t_n^1 f(T, \bar{x}) + t_n^1 y) - V(T, \bar{x})}{t_n^1} > G(T, \bar{x}) + \varepsilon_0,$$

which contradicts the condition (**). □

Remark 3.1. If we assume that $V(t,x)$ is Lipschitz in $x \in R^n$ uniformly w.r.t. $t \in R^+$, then Lemma 3.1 holds true. Indeed, we have

$$\begin{aligned} &V(t+h, x(t+h)) - V(t, x(t)) \\ &= V(t+h, x(t+h)) - V(t+h, x + hf(t,x) + hy) \\ &\quad + V(t+h, x+hf(t,x)+hy) - V(t,x) \\ &\le L\|x(t+h) - x(t) - hf(t,x(t)) - hy\| \\ &\quad + V(t+h, x+hf(t,x)+hy) - V(t,x), \end{aligned}$$

and then

$$\begin{aligned} &\frac{V(t+h, x(t+h)) - V(t, x(t))}{h} \\ &\le L\Big\{\frac{\|x(t+h) - x(t)\|}{h} - f(t, x(t))\Big\} + L\|y\| \\ &\quad + \frac{V(t+h, x+hf(t,x)+hy) - V(t,x)}{h}, \end{aligned}$$

where L is the Lipschitz constant of $V(t,.)$. Then, we have

$$\lim\inf_{h\to 0^+} \frac{V(t+h, x(t+h)) - V(t,x)}{h} \le D_- V(t,x),$$

as desired.

In the following theorem we relax the boundedness condition on the system and give a sufficient condition for asymptotic stability of system (1) with a Lyapunov-like function $V(t,x)$ which is nondifferentiable in t and x in terms of Definition 2.5.

Theorem 3.2. *Assume that*

$$\|f(t,x)\| \le M(t), \quad \forall (t,x) \in R^+ \times W, \tag{8}$$

where $M(t) : R^+ \to R^+$ *is an integrable function satisfying the condition*

$$\lim_{h\to 0} \int_t^{t+h} M(s)ds = 0, \quad u.w.r.t.\ t \in R^+. \tag{9}$$

If the system (1) *admits a Lyapunov-like function then the zero solution is asymptotically stable.*

Proof. a) Stability: Let $\delta_1 > 0$ be chosen so that $B_{\delta_1} \subset W$. From the condition (5) and Lemma 3.1 it follows that there is a sequence $\{t_n > 0\} \to 0$ such that for all solution $x(t)$ of the system, $t \in R^+$

$$\lim_{n\to\infty} \frac{V(t+t_n, x(t+t_n)) - V(t, x(t))}{t_n} \le -c(t)\gamma(V(t, x(t)) \le 0. \tag{10}$$

Let us take $\varepsilon > 0$ an arbitrary number satisfying $\varepsilon < \delta_1$ such that $B_\varepsilon \subset B_{\delta_1} \subset W$. Let $a(t)$, $b(t)$, $a > 0$, $b > 0$ be the functions and numbers given in the assumption (4). For any $t_0 \in R^+$ we set

$$\delta_2 = \Big[\frac{a(t_0)}{b(t_0)}\varepsilon^a\Big]^{1/b} > 0, \quad 0 < \delta_3 \le \min\{\delta_1, \delta_2, \varepsilon\}.$$

Let $x(t)$ be an arbitrary solution of system (1) with $\|x_0\| < \delta_3$. We shall prove that $\|x(t)\| < \varepsilon$ for all $t > t_0$. For this, by the condition (9), there is a positive number $N \in Z^+$ such that for all $n > N$, $t > t_0$,

$$\int_t^{t+t_n} M(s)ds < \min\{\delta_1 - \varepsilon, \delta_2 - \delta_3, \delta_1 - \delta_3\}. \tag{11}$$

For any fixed number $n_0 > N$ and setting $t_{n_0} = h$, from (10) it follows that

$$V(t+h, x(t+h)) - V(t, x(t)) \le 0, \quad \forall t \in R^+. \tag{12}$$

Consider the solution of system (1) evaluated at $t_0 + h$, and using (11), we have the estimate

$$\|x(t_0+h)\| \le \|x_0\| + \int_{t_0}^{t_0+h} M(s)ds \le \delta_3 + \int_{t_0}^{t_0+h} M(s)ds < \delta_1.$$

which gives $x(t_0 + h) \in B_{\delta_1}$. Using (4) and (12) we have

$$\begin{aligned} a(t_0)\|x(t_0+h)\|^a &\le V(t_0+h, x(t_0+h)) \le V(t_0, x_0) \\ &\le b(t_0)\|x_0\|^b < b(t_0)\delta_2^b = a(t_0)\varepsilon^a, \end{aligned}$$

which gives $\|x(t_0 + h)\| < \varepsilon$. We now consider the solution $x(t)$ evaluated at time $(t_0 + 2h)$. By the same way, we can show that $\|x(t_0 + 2h)\| < \delta_1$ and then applying (4) and (12) again, we obtain

$$\begin{aligned} a(t_0)\|x(t_0 + 2h)\|^a \le V(t_0 + 2h, x(t_0 + 2h)) \le V(t_0, x_0) \\ \le b(t_0)\delta_2^b = a(t_0)\varepsilon^a, \end{aligned}$$

which gives $\|x(t_0 + 2h)\| < \varepsilon$. Repeating the same arguments we obtain

$$\|x(t_0 + kh)\| < \varepsilon, \quad \forall k \in Z^+. \tag{13}$$

Let $t \ge t_0$ be an arbitrary number. For $n_0 > N$, $t_{n_0} = h$, there are numbers $k_0 \in Z^+ \setminus \{0\}$ and $\tau \in [0, h)$ such that $t - t_0 = k_0 h + \tau$. Consider now the solution $x(t)$ evaluated at $t_0 + k_0 h + \tau$. From (9), (13) it is easy to see that $x(t_0 + k_0 h + \tau) \in B_{\delta_1}$. Taking the conditions (4), (12) into account, we obtain

$$\begin{aligned} a(t_0)\|x(t)\|^a &\le V(t_0 + k_0 h + \tau, x(t_0 + k_0 h + \tau)) \\ &\le V(t_0 + (k_0 - 1)h + \tau, x(t_0 + (k_0 - 1)h + \tau)) \\ &\le \cdots \le V(t_0 + \tau) \le b(t_0 + \tau)\|x(t_0 + \tau)\|^b \\ &< b(t_0)\|x(t_0 + \tau)\|^b. \end{aligned}$$

On the other hand, estimating the solution $x(t)$ evaluated at $t+\tau$ by using (11), we have $\|x(t_0 + \tau)\| \le \delta_2$. Therefore, we obtain

$$a(t_0)\|x(t)\|^a < b(t_0)\delta_2^b = a(t_0)\varepsilon^a,$$

which means that $\|x(t)\| < \varepsilon$, as desired.

b) Asymptotic stability: We have to show that there is a number $\delta > 0$ such that, for every solution $x(t)$ of (1) with $\|x(t_0)\| < \delta$, for every $\varepsilon > 0$, there exists a number $N > 0$ such that $\|x(t)\| < \varepsilon$ for all $t > t_0 + N$. For this, we first note that the system is stable, then for $\delta_1 > 0$, where δ_1 is chosen so that $B_{\delta_1} \subset W$, we can find a number $\delta_2 > 0$ such that any solution $x(t)$ of the system with $\|x(t_0)\| < \delta_2$ it holds

$$\|x(t)\| < \delta_1, \quad \forall t > t_0.$$

Consider any solution $x(t)$ of (1) with $\|x_0\| < \delta = \min\{\delta_1, \delta_2\}$. We have $x(t) \in W$, for all $t > t_0$. Let $\varepsilon > 0$ be an arbitrarily given number. We define

$$\delta_3 = \Big[\frac{a(t_0)\varepsilon^a}{b(t_0}\Big]^{1/b}.$$

Let $\delta_4 \in (0, \delta_3)$. In view of (5), there are a sequence $\{t_n > 0\}$ going to zero and a number $N_1 > 0$ such that for all $n > N_1$, the condition (10) holds. Due to the condition (9), there is a number $N_2 > 0$ such that for all $n > N_2$,

$$\int_t^{t+t_n} M(s)ds < \delta_3 - \delta_4. \tag{14}$$

We shall show that for any fixed number $n_0 > N = \max\{N_1, N_2\}$ there is an integer $K > 0$ such that

$$\|x(t_0 + Kt_{n_0})\| < \delta_4. \tag{15}$$

Indeed, if (15) is not satisfied, then $\|x(t_0 + kt_{n_0})\| \geq \delta_4$ for all $k \in Z^+$. Since $x(t) \in B_{\delta_1}$, $t > t_0$, by the condition (10) we have

$$\begin{aligned}
&V(t_0 + (k+1)t_{n_0}, x(t_0 + (k+1)t_{n_0})) \\
&\leq V(t_0 + kt_{n_0}, x(t_0 + kt_{n_0})) \\
&\quad - t_{n_0} c(t_0 + kt_{n_0})\gamma(V(t_0 + kt_{n_0}, x(t_0 + kt_{n_0})) \\
&\leq V(t_0 + kt_{n_0}, x(t_0 + kt_{n_0})) - t_{n_0} c(t_0)\gamma(b(t_0)\delta_4^b) \\
&\leq \cdots \leq V(t_0, x_0) - (k+1)M,
\end{aligned}$$

where $M := t_{n_0} c(t_0)\gamma(b(t_0)\delta_4^b) > 0$. Therefore

$$0 \leq V(t_0, x_0) - (k+1)M \leq b(t_0)\|x_0\|^b - (k+1)M,$$

or equivalently,

$$(k+1)M \leq b(t_0)\delta^b, \quad \forall k \in Z^+,$$

which leads to a contradiction when $k \to \infty$. Thus (15) is proved. The proof is completed as follows. Let $t > t_0 + Kt_{n_0}$. There are an integer $k_0 > 0$ and number $\tau_0 \in [0, t_{n_0})$ such that $t - t_0 = k_0 t_{n_0} + \tau_0$. Since $t > t_0 + Kt_{n_0}$ we can claim that $k_0 > K$. We have

$$\begin{aligned}
a(t_0)\|x(t)\|^a &\leq a(t)\|x(t)\|^a \leq V(t, x(t)) \\
&\leq V(t_0 + (k_0 - 1)t_{n_0} + \tau_0, x(t_0 + (k_0 - 1)t_{n_0} + \tau_0)) \\
&\leq \cdots \leq V(t_0 + Kt_{n_0} + \tau_0, x(t_0 + Kt_{n_0} + \tau_0)).
\end{aligned} \tag{16}$$

Combining (14) and (15) gives

$$\begin{aligned}\|x(t_0+Kt_{n_0}+\tau_0)\| &\le \|x(\bar{t}_0)\| + \int\limits_{\bar{t}_0}^{\bar{t}_0+\tau_0} M(s)ds \\ &< \delta_4 + \int\limits_{\bar{t}_0}^{\bar{t}_0+\tau_0} M(s)ds < \delta_3,\end{aligned}$$

where $\bar{t}_0 := t_0 + Kt_{n_0}$. Hence, from (16) it follows that

$$\begin{aligned}a(t_0)\|x(t)\|^a &< V(t_0+Kt_{n_0}+\tau_0, x(t_0+Kt_{n_0}+\tau_0)) \\ &< b(t_0)\|x(t_0+Kt_{n_0}+\tau_0)\|^b < b(t_0)\delta_3^b = a(t_0)\varepsilon^a,\end{aligned}$$

which gives

$$\|x(t)\| < \varepsilon, \quad \forall t > t_0 + Kt_{n_0},$$

as desired. The proof is complete. □

Remark 3.2. If the functions $a(t)$, $b(t)$ are independent of t, i.e., constant, then Theorem 3.2 holds for the uniform asymptotic stability. Moreover, from the proof we see that the condition $D_-V(t,x) \le 0$ is secured for the stablility of system (1) and the condition (9) can be replaced by the non-increasing condition of $M(S)$.

Example 3.1. Consider the differential equation in R^1:

$$\dot{x} = -\big(e^{-t}+1\big)^{\frac{1}{2}}\gamma(x), \quad t \ge 0,$$

where $\gamma(.) : R^+ \to R$ is any given non-decreasing bounded in W function satisfying: $x\gamma(x) > 0$, $\gamma(0) = 0$, where $W = \{x \in R : |x| \le a, a > 0\}$. For $(t,x) \in R^+ \times W$ we consider $V(t,x) = |x|^3$. It is easy to verify that

$$D_-V(t,x) \le -\big(e^{-t}+1\big)^{\frac{1}{2}} 3x^2|\gamma(x)| \le -3x^2|\gamma(x)|$$

and hence the above system is then uniformly asymptotically stable. Note that we can take the function $\gamma(x) = x^n$, where n is an arbitrary odd positive integer.

Remark 3.3. It is noticed that instead of the boundedness condition (8) we can assume that $f(t,x)$ is a Lipschitz function in x uniformly w.r.t. $t \in R^+$. In this case, we will take a number $n > N_1$ such that

$$t_n < \min\Big\{\frac{1}{L}\ln\,\frac{\delta_1}{\delta_3}, \frac{1}{L}\ln\,\frac{\delta_2}{\delta_3}, \frac{1}{L}\ln\,\frac{\delta_1}{\varepsilon}\Big\},$$

where $L > 0$ is the Lipschitz constant of $f(t,x)$, and use the same arguments of the proof by applying the Gronwall's inequality to the estimate

$$\|x(t_0 + (k+1)t_n)\| \le \|x(t_0 + kt_n)\|e^{Lt_n}.$$

Theorem 3.3. *Assume the condition* (8)*. If system* (1) *admits a generalized Lyapunov-like function, then the zero solution is asymptotically stable.*

Proof. In view of Remark 3.2, if system (1) admits generalized Lyapunov-like function satisfying (7), then $D_-V(t,x) \le 0$, and then stability of the system is proved by the same arguments used in the proof of part a) of Theorem 3.2, where the number $\delta_2 > 0$ is a solution of the following equation

$$b(t_0, \delta_2) = a(t_0, \varepsilon),$$

which is always solvable due to the strict continuously increasing assumption on the function $b(t_0, .)$. It remains to prove the asymptotic stability of the system. For this, we first note that from the stability of (1) it follows that for $\delta_1 > 0$, where $\delta_1 > 0$ is chosen so that $B_{\delta_1} \subset W$, there is a number $\delta_2 > 0$ such that for every solution $x(t)$ of (1) with $\|x_0\| < \delta_2$ implies $\|x(t)\| < \delta_1$. Consider any solution $x(t)$ of (1) with $\|x_0\| < \delta = \min\{\delta_1, \delta_2\}$. Let $\varepsilon > 0$ be an arbitrary number such that $B_\varepsilon \subset B_{\delta_1}$, and for any $t_0 \in R^+$ we take $\delta_3 > 0$ satisfying the equation $b(t_0, \delta_3) = a(t_0, \varepsilon)$. Let $\delta_4 \in (0, \delta_3)$. Using the condition (7) and Lemma 3.1, for every $T > 0$, there exist a sequence $\{t_n > 0\}$ going to zero and a number $N_1 > 0$ such that for every $n > N_1$,

$$\frac{V(t+t_n, x(t+t_n)) - V(t, x(t))}{t_n} \le -\int_t^{t+T} \gamma(s, \|x(s)\|)ds, \tag{17}$$

for all $(t, x(t)) \in R^+ \times W \setminus \{0\}$. In view of condition (9), there exists an integer $N > N_1$ large enough such that $t_N < T$ and

$$\int_t^{t+t_N} M(s)ds < \min\{\delta_4, \delta_3 - \delta_4\}, \quad \forall t \in R. \tag{18}$$

We first prove that there is an integer $K > 0$ such that

$$\|x(t_0 + Kt_N)\| < \delta_4. \tag{19}$$

Indeed, if (19) is violated, we have $\|x(t_0 + kt_N)\| \geq \delta_4$ for all $k \in Z^+$. From (17) we have

$$\begin{aligned} &V(t_0 + (k+1)t_N, x(t_0 + (k+1)t_N)) \\ &\leq V(t_0 + kt_N, x(t_0 + kt_N)) - t_N \int_{\bar{t}_0}^{\bar{t}_0+T} \gamma(s, \|x(t_0 + kt_N)\|)ds \\ &\leq V(t_0 + kt_N, x(t_0 + kt_N)) - t_N \int_{\bar{t}_0}^{\bar{t}_0+t_N} \gamma(s, \|x(s)\|)ds, \end{aligned} \tag{20}$$

where $\bar{t}_0 := t_0 + kT$. On the other hand, for every $t \in [\bar{t}_0, \bar{t}_0 + t_N]$ we have

$$\|x(t)\| \geq \|x(\bar{t}_0)\| - \int_{\bar{t}_0}^{t} \|f(s, x(s)\| \geq \delta_4 - \int_{\bar{t}_0}^{\bar{t}_0+t_N} M(s)ds.$$

In view of (18) we have

$$\|x(t)\| \geq \delta_4 - \int_{\bar{t}_0}^{\bar{t}_0+t_N} M(s)ds = \eta > 0,$$

which gives

$$\int_{\bar{t}_0}^{\bar{t}_0+t_N} \gamma(s, \|x(s)\|ds \geq t_N\gamma(t_0, \eta). \tag{21}$$

Therefore, from (20), (21) it follows that

$$\begin{aligned} V(t_0 + (k+1)t_N, x(t_0 + (k+1)t_N)) \leq V(t_0 + kt_N, x(t_0 + kt_N)) \\ - \gamma(t_0, \eta)t_N^2. \end{aligned}$$

Repeating the same arguments we obtain

$$V(t_0 + (k+1)t_N, x(t_0 + (k+1)t_N)) \le V(t_0, x_0) - (k+1)M, \, \forall k \in Z^+,$$

where $M := \gamma(t_0, \eta)t_N^2 > 0$. Since the Lyapunov-like function $V(t,x)$ is non-negative, we have

$$0 \le V(t_0, x_0) - (k+1)M, \quad \forall k \in Z^+,$$

or equivalently

$$(k+1)M \le V(t_0, x_0) \le b(t_0, \|x_0\|) < b(t_0, \delta) < +\infty,$$

which is a contradiction since the above inequality holds for all $k \in Z^+$. Thus, the condition (19) is proved. The proof is completed as follows. For very $t \ge t_0 + Kt_N$ there are numbers $k_0 > K, \tau_0 \in [0, t_N)$ such that $t - t_0 = k_0 t_N + \tau_0$. Taking (19) into account we have

$$\begin{aligned} \|x(t_0 + Kt_N + \tau_0)\| &\le \|x(t_0 + Kt_N)\| + \int_{t_0+Kt_N}^{t_0+Kt_N+\tau_0} M(s)ds \\ &< \delta_4 + \int_{t_0+Kt_N}^{t_0+Kt_N+t_N} M(s)ds < \delta_3. \end{aligned}$$

Therefore

$$\begin{aligned} a(t_0, \|x(t)\|) &\le a(t, \|x(t)\|) \le V(t, x(t)) \\ &\le V(t_0 + Kt_N + \tau, x(t_0 + Kt_N + \tau_0)) \\ &\le b(t_0, \|x(t_0 + Kt_N + \tau_0)\|) < b(t_0, \delta_3) = a(t_0, \varepsilon), \end{aligned}$$

which gives $\|x(t)\| < \varepsilon$. The theorem is proved. □

Remark 3.4. It is worth to note that Theorem 3.3 remains true in the case when we replace (7) by the following conditions: There are a sequence of positive numbers $\{t_n\}$ going to zero and integrable functions $\gamma(h) : R^+ \to R^+, c(.) : R^+ \to R$, such that

$$\lim_{n\to\infty, y\to 0} \left\{ \Delta_{t_n,y} V(t,x) + \frac{1}{t_n} \int_t^{t+t_n} c(s)\gamma(\|x\|)ds \right\} \le 0, \tag{22}$$

for all $t \in R^+$, $x \in W \times \{0\}$, where $\gamma(h)$ is strictly increasing in $h \in R^+$, vanishing at zero and $c(s)$ satisfies

$$\lim_{n\to\infty} \frac{1}{t_n} \int_t^{t+t_n} c(s)ds \geq c > 0, \quad \forall t \in R^+, \tag{23}$$

for some positive number $c > 0$. Indeed, in this case, we take a number $N > 0$ large enough so that

$$\int_t^{t+t_N} M(s)ds \leq \min\{\delta_4, \delta_3 - \delta_4\}, \quad \frac{1}{t_N} \int_t^{t+t_N} c(s) \geq c.$$

The inequality (20) becomes

$$V(t_0 + (k+1)t_N, x(t_0 + (k+1)t_N)) \leq V(t_0 + kt_N, x(t_0 + kt_N)) - t_N \int_{\bar{t}_0}^{\bar{t}_0+t_N} c(s)\gamma(\|x(s)\|)ds,$$

and (21) is

$$\int_{\bar{t}_0}^{\bar{t}_0+t_N} c(s)\gamma(\|x(s)\|)ds \geq ct_N\gamma(\eta).$$

The proof of Theorem 3.3 is then completed similarly.

Example 3.2. We consider the asymptotic stability of a semilinear system the form

$$\begin{cases} \dot{x}(t) = Ax(t) + g(t, x(t)), & t \geq 0, \\ x(t_0) = x_0, & x(t) \in R^n. \end{cases} \tag{24}$$

Let us assume that the matrix A is asymptotically stable, i.e., the linear system $\dot{x}(t) = Ax(t)$ is asymptotically stable. Then, by a classical Lyapunov theorem (see, e.g. [17]), there exist positive definite symmetric matrices X, Y, which are solutions of the Lyapunov equation

$$XA + A'X = -Y.$$

We assume that the nonlinear function $g(t,x)$ satisfies the following growth condition

$$\|g(t,x)\| \le K(t)\|x\|, \quad \forall (t,x) \in R^+ \times W,$$

where $K(t) : R^+ \to R^+$ is a bounded and integable function, $W = \{x \in R^n : \|x\| \le 1\}$. Consider the Lyapunov function $V(t,x) = \langle Xx, x\rangle : W \to R^+$. The derivative along the trajectories $x(t)$ of system (24) is given by

$$\frac{d}{dt}V(t,x) = -\langle Yx(t), x(t)\rangle + \langle Xx(t), g(t,x(t))\rangle.$$

Thus, we can not apply the classical stability Lyapunov theorem since the derivative of $V(t,x)$ may take positive and negative values. Taking Remark 3.4 into account, we can show that if

$$\liminf_{h\to 0^+} \frac{1}{h}\int_t^{t+h} [\alpha - \|X\|K(s)]ds \ge c > 0, \quad \forall t \in R^+, \tag{25}$$

where $\alpha > 0$ is defined from the positive definite property of the matrix Y (since Y is positive definite, there is a positive number $\alpha > 0$ such that $\langle Yx(t), x(t)\rangle \ge \alpha\|x(t)\|^2$, $\forall t \ge 0$), then the zero solution of system (24) is uniformly asymptotically stable. Indeed, from the condition (25), it follows that there is a sequence $\{t_n\}$ going to zero such that

$$\lim_{n\to\infty} \frac{1}{t_n}\int_t^{t+t_n} [\alpha - \|X\|K(s)]ds \ge c > 0, \quad \forall t \in R^+.$$

For this sequence t_n, and for any $t \ge 0$, $x \in W$, $y \in R^n$, and let $x(t)$ be any solution of (24), $x(t) = x$, we have

$$\begin{aligned}
&V(t+t_n, x+t_nf(t,x)+t_ny) - V(t,x)) \\
&\le V(t+t_n, x(t+t_n)) - V(t, x(t)) \\
&\quad + L\|x(t+t_n) - x(t) - t_nf(t,x(t)) - t_ny\| \\
&= \int_t^{t+t_n} \dot V(s, x(s))ds + H(t_n, y),
\end{aligned}$$

where $H(t_n, y) := L\|x(t+t_n) - x(t) - t_n f(t, x(t)) - t_n y)\|$, and $\dfrac{H(t_n, y)}{t_n} \to 0$, when $t_n \to 0$, $y \to 0$. Consequently, we have

$$\begin{aligned}
\Delta_{t_n,y}V(t,x) &\leq \frac{1}{t_n}\int_t^{t+t_n}[-\alpha\|x(s)\|^2 + \langle Xx(s), g(s,x(s))\rangle]ds + \frac{H(t_n,y)}{t_n} \\
&\leq -\frac{1}{t_n}\int_t^{t+t_n}[\alpha - \|X\|K(s)]\|x(s)\|^2 ds + \frac{H(t_n,y)}{t_n} \\
&= -\frac{1}{t_n}\int_t^{t+t_n} c(s)\gamma(\|x(s)\|)ds + \frac{H(t_n,y)}{t_n},
\end{aligned}$$

and hence

$$\lim_{n\to\infty, y\to 0}\Big\{\Delta_{t_n,y}V(t,x) + \frac{1}{t_n}\int_t^{t+t_n} c(s)\gamma(\|x\|)ds\Big\} \leq \lim_{n\to\infty}\frac{H(t_n,y)}{t_n} = 0,$$

where $c(s) = [\alpha - \|X\|K(s)], \gamma(h) = h^2$ and the conditions (22), (23) are satisfied, since $\gamma(h) : R^+ \to R^+$ is strictly inscreasing function and $c(s)$ satisfies (23).

We conclude this section with an application to some stabilizability problem of a class of nonlinear control systems using feedback controls. Consider a nonlinear control system of the form

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \geq 0, \tag{26}$$

where the state $x(t) \in X$; the control $u(t)$ takes values in some m-dimensional space $U = R^m$; $f(t, x, u)$ is a given nonlinear function with $f(t, 0, 0) = 0$, $t \geq 0$. We recall that the system (26) is stabilizable by a feedback control $u(t) = g(x(t))$, where $g(x) : X \to U$, $g(0) = 0$, is a given function, if the zero solution of the following system without control

$$\dot{x}(t) = f(t, x(t), g(x(t)) := F(t, x(t)), \quad x(t_0) = x_0,\ t_0 \geq 0, \tag{27}$$

is asymptotically stable. Stabilization of system (26) has become, during the last decades, one of the most impotant problems in control theory. This problem has been investigated usually by using the stability results of corresponding systems without controls (27) [1, 7, 10, 14, 15]. Some sufficient conditions below for the stabilizability using Lyapunov functions were established for a class of nonlinear autonomous system

$$\dot{x}(t) = f(x(t), u(t)), \quad t \geq 0. \tag{28}$$

Theorem 3.4. [10] *Consider autonomous system* (28)*. If there are a function $g(x) : R^n \to U$, $g(0) = 0$, $g(x) \in C^1_{t,x}$ and a positive definite function $V(x) : R^n \to R^+$, $V(x) \in C^1_x$ and $V(x)$ is proper (i.e. $\lim_{\|x\| \to \infty} V(x) = +\infty$) such that*

$$\frac{\partial V}{\partial x_i} f^i(x, g(x)) < 0, \quad \forall i = 1, 2, \ldots, n, \ \forall x \neq 0.$$

Then the system is stabilizable by feedback control $u(t) = g(x(t))$.

Note that the proposed Lyapunov function in the above theorems is, by the assumptions, non-increasing in t. Based on the stability results obtained in previous section, we can derive the following sufficient condition for the stabilizability of control system (26) with a discontinuous Lyapunov-like function.

Theorem 3.5. *Assume that there exists a function $g(x) : X \to U$ such that the system* (27) *satisfies the condition* (8)*. Assume that there exist a function $V(t, x) : R^+ \times W \to R$, where $W \subset X$ is a open neighborhood of zero, and functions $a(t, h), b(t, h) : R^+ \times R^+ \to R^+ \setminus \{0\}$, which are continuously strictly increasing in h; $a(t, h)$ is non-decreasing in t, $b(t, h)$ is non-increasing in t such that*
(i)

$$a(t, \|x\|) \leq V(t, x) \leq b(t, \|x\|), \quad \forall (t, x) \in R^+ \times W \setminus \{0\}. \tag{29}$$

(ii) *For every $T > 0$ there are a sequence of positive numbers t_n going to zero, a positive function $\gamma(t, h) : R^+ \times R^+$, which is integrable and strictly increasing in (t, h) such that*

$$\lim_{n \to \infty, y \to 0} \Delta_{t_n, y} V(t, x) \leq - \int_t^{t+T} \gamma(s, \|x\|) ds, \quad \forall (t, x) \in R^+ \times W \setminus \{0\}.$$

Then the system (26) *is stabilizable by feedback control $u(t) = g(x(t))$.*

Remark 3.5. Note that the condition (29), where $a(t,h)$, $b(t,h)$ need not be continuous in t, provides enough information to prove the stabilizability of the system.

4. Conclusions

Asymptotic stability of nonlinear time-varying differential equations by Lyapunov direct method has been investigated. Nondifferentiable Lyapunov-like functions are proposed for obtaining sufficient stability conditions. The stability conditions obtained in the paper can be extended to any infinite-dimensional Banach space X. The problem of finding numerical Lyapunov-like functions is usually a difficult task and has remained under our further investigation. However, the obtained stability results are applied to some stabilization problems of nonlinear control systems by feedback controls, which can be considered as an addendum to some control results given in [10, 14, 15, 17].

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