ACTA MATHEMATICA VIETNAMICA
Volume 25, Number 1, 2000, pp. 19–32

THE PROPERTIES $(\underline{DN})$ AND (DN_φ) OF SPACES OF GERMS OF HOLOMORPHIC FUNCTIONS

NGUYEN VAN HAO AND BUI DAC TAC

Abstract. In this note we prove that the space $[\mathcal{H}(O_E)]'$ is asymptotically normable (resp. has property $(\underline{DN})$) if E is an asymptotically normable Frechet space (resp. E is a Frechet $(\underline{DN})$-space) having absolute basis. We also investigate these properties of the space $[\mathcal{H}(K)]'$ when K is a balanced compact subset of an asymptotically normable Frechet-Hilbertisable space and K is a polynomially convex conpact set in $\mathbf{C}^n$.

Introduction

Let E be a Frechet space and K a compact set in E. By $\mathcal{H}(K)$ we denote the space of germs of holomorphic functions on K equipped with the inductive limit topology

$$\mathcal{H}(K) = \liminf_{U \supset K} \mathcal{H}^\infty(U),$$

where U ranges on all neighborhoods of K in E and $\mathcal{H}^\infty(U)$ is the Banach space of bounded holomorphic functions on U. It is known [3] that $\mathcal{H}(K)$ is regular and hence $[\mathcal{H}(K)]'$ is a Frechet space.

The aim of the present note is to study the properties $(\underline{DN})$ and (DN_φ) of the space $[\mathcal{H}(K)]'$. These properties and many other were introduced and investigated by Vogt (see, for example [6, 7]).

Let E be a Frechet space with a fundamental system of seminorms $\{\|\cdot\|_k\}_{k=1}^\infty$ and φ a strictly increasing positive function on $(0,+\infty)$. We say that E has property $(\underline{DN})$ (resp.(DN_φ)) if (1) (resp. (2)) holds:

$$(1) \qquad \exists p \quad \forall q \quad \exists k,\ C,\ d>0 \quad \forall r>0:\ U_q^0 \subset Cr^d U_p^0 + \frac{1}{r} U_k^0,$$

Received April 13, 1998; in revised form December 31, 1998.
1991 Mathematics Subject Classification. 46A08, 46A11, 46A13.
Key words and phrases. Linear topological invariants, space of germs of holomorphic functions, asymptotically normable Frechet spaces.
This work was supported by the National Basic Research Program.

$$\exists p \quad \forall q \quad \exists k,\ C>0 \quad \forall r>0: \ U_q^0 \subset C\varphi(r)U_p^0 + \frac{1}{r}U_k^0, \tag{2}$$

where $U_k = \{x \in E \big| \, \|x\|_k \le 1\}$ and U_k^0 is the polar of U_k.

It is obvious that $(\underline{DN}) \Rightarrow (DN_{\exp})$. In [5] Terzioglu and Vogt have proved the following result

Theorem. *Let E be a Frechet space. The following are equivalent*

(i) *E has property (DN_φ)*

(ii) *E is an asymptotically normable space*

(iii) *There exist a Banach space B and a nuclear Frechet space $\lambda(A)$ having a continuous norm such that E is a subspace of $B \underset{\pi}{\hat{\otimes}} \lambda(A)$.*

Recall [5] that E is called asymptotically normable if there exists p such that for every $q \ge p$ there is a k for which the seminorms $\|\cdot\|_p$ and $\|\cdot\|_q$ define equivalent topologies on U_k. The first section of the present paper is devoted to the property $(\underline{DN})$ of the space $[\mathcal{H}(K)]'$. First we prove that $[\mathcal{H}(O_E)]'$ has property $(\underline{DN})$ for every Frechet $(\underline{DN})$-space having an absolute basis and next that $[\mathcal{H}(K)]'$ has property $(\underline{DN})$ for every polynomially convex compact set K in $\mathbf{C}^n$. In the second section, by applying the theorem of Terzioglu-Vogt we prove that if E is an asymptotically normable Frechet-Hilbertisable space and K is a balanced compact set in E then $[\mathcal{H}(K)]'$ is an asymptotically normable space.

1. The Property $(\underline{DN})$

Theorem 1.1. *Let E be a Frechet space having an absolute basis. If E has property $(\underline{DN})$ then $[\mathcal{H}(O_E)]'$ also has property $(\underline{DN})$.*

Proof. Put

$$\mathbf{M} = \Big\{ m = (m_1, m_2, \dots, m_n, 0, \dots) \big| \, m_j \in \mathbf{N} \Big\}.$$

For each $m \in \mathbf{M}$ and $z \in w$, the space of all complex number sequences, we denote

$$|m| = m_1 + m_2 + \cdots + m_n, \quad m! = m_1! m_2! \dots m_n!$$

and

$$z^m = z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}.$$

From [4] we have

$$(1) \qquad \Big(\sum_{j\geq 1} z_j\Big)^n = \sum_{m\in\mathbf{M}_n} \frac{n!}{m!} z^m \quad \text{for every} \quad z = \big(z_j\big)_{j\geq 1} \in \ell^1,$$

where

$$\mathbf{M}_n = \big\{m \in \mathbf{M} : |m| = n\big\}.$$

For $f \in \mathcal{H}(O_{\ell^1})$, choose $R > 0$ such that f is bounded and holomorphic on $B(0,R) = \{\mathbf{z} \in \ell^1 : \|\mathbf{z}\| \leq R\}$. The following result is an improvement of Ryan [4].

Let $\rho = \big(\rho_k\big)_{k\geq 1}$ be a sequence of positive numbers such that

$$\sum_{k\geq 1} \rho_k \leq \frac{R}{2}.$$

For each $m \in \mathbf{M}_n$, we define

$$(2) \qquad C_m = \frac{1}{(2\pi i)^n} \int\limits_{|t_1|=\rho_1} \cdots \int\limits_{|t_n|=\rho_n} \frac{f(t_1e_1 + \cdots + t_ne_n)}{t_1^{m_1+1}\dots t_n^{m_n+1}} dt_1 \dots dt_n,$$

where $\{e_n\}$ is the canonical basis of ℓ^1. The Cauchy integral formula in several variables implies that C_m does not depend on the choise of (ρ_k). Since f is of bounded type, we may define

$$A = \sup\Big\{\|f(z)\| \,\Big|\, \|z\| \leq R\Big\}.$$

By (2), it follows that $|C_m|\rho^m \leq A$ for every $m \in \mathbf{M}$.

By taking $\rho_k = \dfrac{R}{2}\sigma_k$, we have $|C_m|\sigma^m\Big(\dfrac{R}{2}\Big)^{|m|} \leq A$ for every $m \in \mathbf{M}$ and $\sigma = \big(\sigma_k\big)_{k\geq 1}$ with $\sum\limits_{k\geq 1} \sigma_k \leq 1$. Since $\sigma = \Big(\dfrac{m_1}{|m|}, \dots, \dfrac{m_n}{|m|}, 0, \dots\Big)$ satisfies the above condition, it follows that

$$(3) \qquad |C_m|\frac{m^m}{|m|^{|m|}}\Big(\frac{R}{2}\Big)^{|m|} \leq A \quad \text{for every} \quad m \in \mathbf{M}.$$

Combining (3) with the inequality

$$\frac{m!}{|m|!} \leq \frac{e^{|m|}m^m}{|m|^{|m|}}\ ,$$

we get

$$|C_m|\frac{m!}{|m|!}\Big(\frac{R}{2e}\Big)^{|m|} \le A. \tag{4}$$

Hence for every $0 < r < \dfrac{R}{2e}$, we obtain from (1) and (4) the following estimations

$$\begin{aligned}\sum_{m\in\mathbf{M}}|C_m z^m| &\le \sup_{m\in\mathbf{M}}\Big\{|C_m|\frac{m!}{|m|!}r^{|m|}\Big\}\sum_{m\in\mathbf{M}} r^{-|m|}\frac{|m|!}{m!}|z|^m\\ &\le A\sum_{n\ge 0} r^{-n}\sum_{m\in\mathbf{M}_n}\frac{n!}{m!}|z|^m\\ &\le A\sum_{n\ge 0} r^{-n}\|z\|^n\\ &\le A\cdot\frac{r}{r-\delta}\quad \text{for}\quad \|z\|<\delta<r.\end{aligned}$$

Thus, the series $\sum\limits_{m\in\mathbf{M}} C_m z^m$ converges in $\mathcal{H}^\infty(B(0,R))$ to f for $0 < r < \dfrac{R}{2e}$.

We now assume that E is a Frechet space with ($\underline{DN}$) and that E has an absolute basis $\{e_j\}_{j\ge 1}$. Choose a fundamental system of seminorms $\{\|\cdot\|_k\}$ of E such that

$$2\|\cdot\|_k \le \|\cdot\|_{k+1}\quad\text{for every}\quad k\ge 1.$$

Put $a_{j,k} = \|e_j\|_k$ for $j,k\ge 1$ and $a_k = (a_{1,k}, a_{2,k}, ...)$. We deduce that

$$E \cong \Big\{\xi = (\xi_j)_{j\ge 1}\,\big|\,\|\xi\|_k = \sum_{j\ge 1}|\xi_j|a_{j,k} < +\infty, \forall k\ge 1\Big\}$$

and

$$\exists p\quad \forall q\ \exists k, C, \varepsilon > 0 : a_{j,q}^{1+\varepsilon} \le C a_{j,k} a_{j,p}^{\varepsilon}\quad \forall j\ge 1. \tag{5}$$

Replacing $\|\cdot\|_k$ by $\dfrac{1}{C}\|\cdot\|_k$ we may assume that $C = 1$. By the above argument, we have

$$\mathcal{H}(O_E) \cong \lim\operatorname{ind} F_k,$$

where

$$F_k = \Big\{ \big(C_m\big)_{m\in\mathbf{M}} \,\big|\, \|(C_m)\|_k = \sup_{m\in\mathbf{M}} \frac{|C_m| m^m}{|m|^{|m|} a_k^m} < +\infty \Big\}.$$

To prove that $[\mathcal{H}(O_E)]'$ has property ($\underline{DN}$), it remains to check the following condition

$$\exists p \quad \forall q \quad \exists k, C, d > 0 \quad \forall r > 0 : W_q \subseteq C r^d W_p + \frac{1}{r} W_k. \tag{6}$$

For each $k \geq 1$ we put

$$W_k = \Big\{ (C_m)_{m\in\mathbf{M}} \,\big|\, \|(C_m)\|_k \leq 1 \Big\}.$$

Let p, q, k, $\varepsilon > 0$ be as in (5). Obviously (6) holds for $0 < r \leq 1$ and every $d > 0$. For each $(C_m) \in W_q$ and $r > 1$, we have

$$\begin{aligned}\sup_{|m|>\alpha} \Big\{ \frac{|C_m| m^m}{|m|^{|m|} a_k^m} \Big\} &\leq \sup_{|m|>\alpha} \Big\{ \frac{|C_m| m^m}{|m|^{|m|} a_q^m} \Big\} \cdot \sup_{|m|>\alpha} \Big(\frac{a_q}{a_k} \Big)^m \\ &\leq \Big(\frac{1}{2}\Big)^\alpha \leq \frac{1}{r}\end{aligned}$$

if $\alpha = \alpha(r) \geq \dfrac{\log r}{\log 2}$.

On the other hand, using (5) we have

$$\begin{aligned}&\sup \Big\{ \frac{|C_m| m^m}{|m|^{|m|} a_k^m} \,\big|\, |m| \leq \alpha, m \in \widetilde{\mathbf{M}}_\alpha \Big\} \\ &\leq \sup \Big\{ \frac{|C_m| m^m}{|m|^{|m|} a_q^m} \,\big|\, m \in \mathbf{M} \Big\} \cdot \sup \Big(\frac{a_p}{a_k} \Big)^{\frac{m\varepsilon}{1+\varepsilon}} \\ &\leq \frac{1}{r},\end{aligned}$$

where

$$\widetilde{\mathbf{M}}_\alpha = \Big\{ m \in \mathbf{M} \,\big|\, |m| \leq \alpha,\ m_1 \log \frac{a_{1,k}}{a_{1,p}} + \cdots + m_n \log \frac{a_{n,k}}{a_{n,p}} \geq \frac{1+\varepsilon}{\varepsilon} \log r \Big\}$$

and

$$\begin{aligned}
&\sup\Big\{\frac{|C_m|m^m}{|m|^{|m|}a_p^m}\,\Big|\,|m|\le\alpha, m\notin\widetilde{\mathbf{M}}_\alpha\Big\}\\
&\le\sup\Big\{\frac{|C_m|m^m}{|m|^{|m|}a_q^m}\,\Big|\,m\in\mathbf{M}\Big\}\cdot\sup\Big\{\Big(\frac{a_k}{a_p}\Big)^{\frac{m}{1+\varepsilon}}\,\Big|\,|m|\le\alpha, m\notin\widetilde{\mathbf{M}}_\alpha\Big\}\\
&\le\sup\Big\{\Big(\frac{a_k}{a_p}\Big)^{\frac{m}{1+\varepsilon}}\,\Big|\,m\notin\widetilde{\mathbf{M}}_\alpha\Big\}\\
&\le r^d,
\end{aligned}$$

if $d>\varepsilon$. Hence

$$\begin{aligned}
\big(C_m\big)_{m\in\mathbf{M}}&=\big(C_m\big)_{|m|\le\alpha,m\notin\widetilde{\mathbf{M}}_\alpha}+\big(C_m\big)_{|m|\le\alpha,m\in\widetilde{\mathbf{M}}_\alpha}+\big(C_m\big)_{|m|>\alpha}\\
&\in r^dW_p+\frac{2}{r}W_k.
\end{aligned}$$

Remark. Recall that if K is a compact set in $\mathbf{C}$, then

$$[\mathcal{H}(K)]'\cong\mathcal{H}(\overline{\mathbf{C}}\setminus K).$$

Consequently, if $\bar{C}\setminus K$ has only finitely many numbers of connected components (in particular, if K is a polynomially convex set) then $[\mathcal{H}(\overline{\mathbf{C}}\setminus K)]'$ has property $(\underline{DN})$.

In the following theorem, we formulate the property $(\underline{DN})$ of $[\mathcal{H}(K)]'$ in more general situations.

Theorem 1.2. *If K is a compact set in $\mathbf{C}^n$ such that $K=\hat{K}_U$, the holomorphically convex hull of K in U for some Stein neighborhood U of K, then $[\mathcal{H}(K)]'$ has property $(\underline{DN})$.*

Proof. By the hypothesis $K=\hat{K}_U$ for some Stein neighborhood U of K. Hence we can choose a Stein neighborhood basis $\Big\{U_k\Big\}_{k\ge1}$ of K, $U=U_1$, such that $\mathcal{H}(U_k)$ is dense in $\mathcal{H}(U_{k+1})$ for $k\ge1$. This yields

$$K=\hat{K}_{U_k}\quad\text{for}\quad k\ge1.$$

For each $k\ge1$, we put

$$W_k=\Big\{f\in\mathcal{H}(U_k)\,\Big|\,\int_{U_k}|f|^2d\lambda\le1\Big\}.$$

To prove that $[\mathcal{H}(K)]'$ has property ($\underline{DN}$), it suffices to show that

$$\forall q\ \exists k, d, C > 0 \quad \forall r > 0 : W_q \subseteq Cr^d W_2 + \frac{1}{r} W_k.$$

Given $q \geq 2$. Since $K = \hat{K}_{U_1}$ we can find a plurisubharmonic function ρ on U_1 such that

$$K \subset Z_- := \{\rho < 1\} \subseteq U_q.$$

Let $\sup\limits_K \rho < \delta_2 < \delta_1 < 1$ and $\alpha = \sup\limits_{U_2} \rho < +\infty$. We define, for each $L > 0$, the function τ_L by

$$\tau_L(t) = \begin{cases} \dfrac{L}{\delta_2} t - L & \text{for} \quad \delta_2 \leq t < \alpha, \\ 0 & \text{for} \quad t \leq \delta_2. \end{cases}$$

Since τ_L is a convex function, the function $\rho_L = \tau_L \circ \rho$ becomes a plurisubharmonic function. Choose $k \geq q$ such that $U_k \subseteq \{\rho < \delta_1\}$ and put

$$Z_+ = U_2 \setminus \mathrm{Cl}\{\rho < \delta_1\}, \quad Z = Z_- \cap Z_+.$$

For each $f \in W_q$, we have

$$\int\limits_Z |f|^2 e^{-\rho_L} d\lambda \leq \sup_Z e^{-\rho_L} = \sup_Z e^{-\frac{L}{\delta_2}\rho + L} = e^{L(1 - \frac{\delta_1}{\delta_2})}. \tag{1}$$

By Aytuna [1, Lemma 1], we can find $f_+ \in \mathcal{H}(Z_+)$ and $f_- \in \mathcal{H}(Z_-)$ such that $f_+ - f_- = f$ on Z and

$$\int\limits_{Z_+} |f_+|^2 e^{-\rho_L} d\lambda \leq C e^{L(1 - \frac{\delta_1}{\delta_2})}, \tag{2}$$

$$\int\limits_{Z_-} |f_-|^2 e^{-\rho_L} d\lambda \leq C e^{L(1 - \frac{\delta_1}{\delta_2})},$$

where C is a constant that depends neither upon f nor L. Since $\rho_L = 0$ on $Z_{\delta_2} = \{\rho < \delta_2\}$ we have

$$\int\limits_{Z_{\delta_2}} |f_-|^2 d\lambda = \int\limits_{Z_{\delta_2}} |f_-|^2 e^{-\rho_L} d\lambda \le C e^{L(1-\frac{\delta_1}{\delta_2})}, \tag{3}$$

$$\begin{aligned} \int\limits_{Z_+} |f_+|^2 d\lambda &\le \sup_{Z_+} e^{\rho_L} \int\limits_{Z_+} |f_+|^2 e^{-\rho_L} d\lambda \\ &\le C e^{L(1-\frac{\delta_1}{\delta_2})} \sup_{\delta_1 \le \rho \le \alpha} e^{\frac{L}{\delta_2}\rho - L} \\ &= C e^{\frac{L}{\delta_2}(\alpha-\delta_1)}, \end{aligned} \tag{4}$$

$$\int\limits_{Z_-\setminus Z_+} |f_-|^2 d\lambda \le \sup_{Z_-\setminus Z_+} e^{\rho_L} \int\limits_{Z_-\setminus Z_+} |f_-|^2 e^{-\rho_L} d\lambda \le C e^{\frac{L}{\delta_2}(1-\delta_1)}. \tag{5}$$

From (5) and the fact that $\int\limits_{Z_-\setminus Z_+} |f|^2 d\lambda \le 1$ we get

$$\begin{aligned} \int\limits_{Z_-\setminus Z_+} |f - f_-|^2 d\lambda &\le \Big[\Big\{\int\limits_{Z_-\setminus Z_+} |f|^2 d\lambda\Big\}^{1/2} + \Big\{\int\limits_{Z_-\setminus Z_+} |f_-|^2 d\lambda\Big\}^{1/2}\Big]^2 \\ &\le (1+\sqrt{C})^2 e^{\frac{L}{\delta_2}(1-\delta_1)} \\ &\le (1+\sqrt{C})^2 e^{\frac{L}{\delta_2}(\alpha-\delta_1)}. \end{aligned} \tag{6}$$

Since $f_+ - f_- = f$ on Z, the function g defined by

$$g(z) = \begin{cases} f_+ & \text{on} \quad Z_+, \\ f - f_- & \text{on} \quad Z_- \end{cases}$$

is holomorphic on U_2. We write

$$f = h + g \quad \text{where} \quad h = f - g.$$

With this notation the estimate (3) implies

$$\int\limits_{Z_{\delta_2}} |h|^2 d\lambda = \int\limits_{Z_{\delta_2}} |f_-|^2 d\lambda \le C e^{L(1-\frac{\delta_1}{\delta_2})}. \tag{7}$$

From (4) and (6), we have

$$\begin{aligned}\int\limits_{U_2} |g|^2 d\lambda &= \int\limits_{Z_+} |f_+|^2 d\lambda + \int\limits_{Z_-\setminus Z_+} |f - f_-|^2 d\lambda \\ &\le Ce^{\frac{L}{\delta_2}(\alpha-\delta_1)} + (1+\sqrt{C})^2 e^{\frac{L}{\delta_2}(\alpha-\delta_1)} \\ &= [(1+\sqrt{C})^2 + C] e^{\frac{L}{\delta_2}(\alpha-\delta_1)}.\end{aligned} \tag{8}$$

Thus

$$\forall q \quad \exists k,\ C>0 \quad \forall L: \quad W_q \subseteq Ce^{L(1-\frac{\delta_1}{\delta_2})} W_k + [(1+\sqrt{C})^2 + C] e^{\frac{L}{\delta_2}(\alpha-\delta_1)} W_2.$$

Putting $s = e^{-L(1-\frac{\delta_1}{\delta_2})}$ we have

$$W_q \subset \frac{C}{s} W_k + [(1+\sqrt{C})^2 + C] s^d W_2 \quad \text{with} \quad d = \frac{\alpha - \delta_1}{\delta_1 - \delta_2} \cdot$$

Consequently, $[\mathcal{H}(K)]'$ has property ($\underline{DN}$). □

Remarks.
1. Since $\mathcal{H}(D)$ has property ($\underline{DN}$) for every domain D in $\mathbf{C}^n$ (see [6]), it follows from the Grothendieck dual theorem for a compact set $K \subset \mathbf{C}$ that $[\mathcal{H}(K)]'$ has property ($\underline{DN}$) if and only if $[\mathcal{H}(K)]'$ has a continuous norm.
2. The proof of Theorem 1.2 is an improvement of the ($\overline{\Omega}$)-case of [1] to the ($\underline{DN}$)-case.

2. The Property (DN_φ)

In this section we will prove the following result.

Theorem 2.1. *If E is an asymptotically normable Frechet-Hilbertisable space and K a balanced compact set in E then $[\mathcal{H}(K)]'$ is asymptotically normable.*

We need the following lemma.

Lemma 2.2. *Let E be as in the theorem. Then there exist an index set I and an asymptotically normable nuclear Frechet space F such that E is a subspace of $\ell^2(I)\hat{\otimes}_\pi F$.*

Proof. By Lemma 5.4 of [7], there exists a nuclear Köthe space $\lambda(B)$ with a continuous norm such that $(E, \lambda(B)) \in (S_1^*)$. Since $\lambda(B)$ is also a

Schwartz space, we can construct an exact sequence (see [5, Proposition 3.2])

$$0 \to \lambda(B) \to \lambda(A) \to w \to 0.$$

In the case where E is a Hilbert space the lemma is trivial. If E is a proper Frechet space, then $(E, \lambda(B)) \in (S_1^*)_0$ by Lemma 3.3 of [8]. We choose a set I such that each Hilbert space E_k is isomorphic to a subspace of $\ell^2(I)$. We identify the tensor product $\ell^2(I)\hat{\underset{\pi}{\otimes}}\lambda(B)$ with the space of all $y = \{y_j\}_{j\geq 1}$ such that each $y_j \in \ell^2(I)$ and for every $k \in \mathbf{N}$ we have

$$\|y\|_k = \sup_j \|y_j\|_2 b_{j,k} < +\infty,$$

where $\|\cdot\|_2$ denotes the norm of $\ell^2(I)$. An obvious modification of Proposition 3.5 of [8], we have $(E, \ell^2(I)\hat{\underset{\pi}{\otimes}}\lambda(B)) \in (S_1)$. Taking tensor product with $\ell^2(I)$, we get an exact sequence

$$0 \to \ell^2(I)\hat{\underset{\pi}{\otimes}}\lambda(B) \to \ell^2(I)\hat{\underset{\pi}{\otimes}}\lambda(A) \to \big(\ell^2(I)\big)^{\mathbf{N}} \to 0.$$

Since $\mathrm{Ext}^1(E, \ell^2(I)\hat{\underset{\pi}{\otimes}}\lambda(B)) = 0$, the natural imbedding $T : E \to \big(\ell^2(I)\big)^{\mathbf{N}}$ can be lifted to a continuous linear map $\hat{T} : E \to \ell^2(I)\hat{\underset{\pi}{\otimes}}\lambda(A)$. Obviously, $\hat{T}$ is injective and has a closed range. $\square$

Proof of Theorem 2.1. Choose an index set I and an asymptotically normable nuclear Frechet space F such that E is a subspace of $\ell^2(I)\hat{\underset{\pi}{\otimes}}F$. Since $\ell^2(I)\hat{\underset{\pi}{\otimes}}F$ has a fundamental system of Hilbert seminorms, by applying the Taylor expansion of each element of $\mathcal{H}(K)$ at $O \in K$, it is easy to check that every bounded set in $\mathcal{H}(K)$ is the image of a bounded set in $\mathcal{H}(e(K))$ under the restriction map, where $e : E \hookrightarrow \ell^2(I)\hat{\underset{\pi}{\otimes}}F$ is the embedding defined by Lemma 2.2. This yields that $[\mathcal{H}(K)]'$ is a subspace of $[\mathcal{H}(e(K))]'$. Thus, it suffices to prove that $[\mathcal{H}(e(K)]'$ is asymptotically normable.

Let $\{\|\cdot\|_k\}_{k\geq 1}$ be a fundamental system of seminorms of F for which

$$2\|\cdot\|_k \leq \|\cdot\|_{k+1} \quad \text{for} \quad k \geq 1.$$

Since F is asymptotically normable,we have

$$\text{(AS)} \qquad \exists p \quad \forall q \quad \exists k : \|\cdot\|_p \sim \|\cdot\|_q \quad \text{on } U_k.$$

Here we write $\|\cdot\|_p \sim \|\cdot\|_q$ on U_k if the topologies on U_k generated by $\|\cdot\|_p$ and $\|\cdot\|_q$ are the same.

We check that for p, q and k as in (AS):

$$\pi_p^n \sim \pi_q^n \quad \text{on} \quad W_k^n \quad \text{for} \quad n \geq 1,$$

where W_k^n denotes the unit ball in $\big(\ell^2(I) \underbrace{\hat{\otimes}_\pi F\big) \hat{\otimes}_\pi \dots \hat{\otimes}_\pi \big(\ell^2(I)}_{n} \hat{\otimes}_\pi F\big)$ of the seminorm π_k^n induced by $\|\cdot\|_k$.

For simplicity we may consider only the case $n = 2$. Let $\{f_m\} \subset W_k^2$ with $\pi_p^2(f_m) \to 0$ as $m \to \infty$. Since

$$\begin{aligned}\big(\ell^2(I)\hat{\otimes}_\pi F\big)\hat{\otimes}_\pi\big(\ell^2(I)\hat{\otimes}_\pi F\big) &\cong F\hat{\otimes}_\pi\big(\ell^2(I)\hat{\otimes}_\pi\ell^2(I)\hat{\otimes}_\pi F\big)\\ &\cong \mathcal{L}\big(F', \ell^2(I)\hat{\otimes}_\pi\ell^2(I)\hat{\otimes}_\pi F\big),\end{aligned}$$

the sequence $\{f_m\}$ can be considered as a sequence

$$\{\hat{f}_m\} \subset \mathcal{L}\big(F', \ell^2(I)\hat{\otimes}_\pi\ell^2(I)\hat{\otimes}_\pi F\big)$$

for which

$$\sup\Big\{\big|(\omega\hat{\otimes}\hat{f}_m)(u)\big| \,\Big|\, u \in U_k^0,\ \omega \in \big(V \otimes V \otimes U_k\big)^0,\ m \geq 1\Big\} \leq 1$$

and

$$\pi_p^2(\hat{f}_m) = \sup\Big\{\big|\omega \circ \hat{f}_m(u)\big| \,\Big|\, u \in U_p^0, \omega \in \big(V \otimes V \otimes U_p\big)^0\Big\} \to 0$$

as $m \to \infty$, where V is the unit ball of $\ell^2(I)$.

Assume that $\pi_q^2(\hat{f}_m) \not\to 0$ as $m \to 0$, then for each $m \geq 1$ there exists $\omega_m \in (V \otimes V \otimes U_q)^0$ such that

$$\sup\Big\{|\omega_m \circ \hat{f}_m(u)| \,\big|\, u \in U_q^0\Big\} \geq \varepsilon \quad \text{for some } \varepsilon > 0.$$

This is impossible, because $\{\omega_m \circ \hat{f}_m\} \subset U_k^0$ and $\|\omega_m \circ \hat{f}_m\|_p \to 0$ as $m \to \infty$.

It remains to show that for p, q and k as in (AS) we have $\|\cdot\|_p^* \sim \|\cdot\|_q^*$ on W_k, where

$$\|\mu\|_k^* = \sup\Big\{|\mu(f)| \,\Big|\, f \in \mathcal{H}^\infty(e(K) + \operatorname{conv}(V \otimes V \otimes U_k)), \|f\|_k \le 1\Big\}$$

for $\mu \in [\mathcal{H}(e(K))]'$ and

$$W_k = \Big\{\mu \in [\mathcal{H}(e(K))]' \,\Big|\, \|\mu\|_k^* \le 1\Big\}.$$

Assume that $\{\mu_j\} \subset W_k$ with $\|\mu_j\|_p^* \to 0$ as $j \to \infty$. Choose $\delta_k > 1$ such that

$$e(K) + \operatorname{conv}(V \otimes V \otimes U_k) \subset \delta_k(e(K) + \operatorname{conv}(V \otimes V \otimes U_q)).$$

Writing each $f \in \mathcal{H}^\infty(e(K) + \operatorname{conv}(V \otimes V \otimes U_q))$ in the Taylor series form

$$f(\omega) = \sum_{n\ge 0} P_n f(\omega),$$
$$P_n f(\omega) = \frac{1}{(2\pi i)^n} \int\limits_{|\lambda|=1} \frac{f(\lambda\omega)}{\lambda^{n+1}} d\lambda,$$

we see that for each $\varepsilon > 0$ there exists N such that

$$\Big|\mu_j\Big(\sum_{n>N} P_n f\Big)\Big| < \varepsilon$$

for $j \ge 1$ and $f \in \mathcal{H}^\infty\big(e(K) + \operatorname{conv}(V \otimes V \otimes U_q)\big)$, $\|f\|_q \le 1$. We infer with j sufficiently large that

$$\Big|\mu_j\Big(\sum_{0<n\le N} P_n f\Big)\Big| < \varepsilon$$

for every $\|f\|_q \le 1$. Hence

$$|\mu_j(f)| < 2\varepsilon \quad \text{for } f \in \mathcal{H}^\infty\big(e(K) + \operatorname{conv}(V \otimes V \otimes U_q)\big), \quad \|f\|_q \le 1.$$

It follows that $\|\mu_j\|_q^* \to 0$ as $j \to \infty$. Thus, Theorem 2.1 is proved. $\square$

Theorem 2.3. *If E is an asymptotically normable Frechet space with an absolute basis, then $[\mathcal{H}(O_E)]'$ is asymptotically normable.*

Proof. By the notations of Theorem 1.1 we have to show that there is a strictly increasing positive function ψ on $(0,+\infty)$ such that

$$\exists p\ \forall q\ \exists k, C>0\ \forall r>0 : W_q \subset C\psi(r)W_p + \frac{1}{r}W_k. \tag{1}$$

Since E is asymptotically normable, there exists a strictly increasing positive function φ on $(0,+\infty)$ such that

$$\exists p\ \forall q\ \exists k, C>0\ \forall j\geq 1 : \varphi\Big(\frac{a_{j,q}}{a_{j,p}}\Big) \leq C\frac{a_{j,k}}{a_{j,q}}\ \cdot \tag{2}$$

Obviously, (1) holds for every ψ and $0<r\leq 1$. Let $\big(C_m\big)_{m\in\mathbf{M}} \in W_q$, $r>1$. We have, as in Theorem 1.1, that

$$\big(C_m\big)_{|m|\geq\alpha} \in \frac{1}{r}W_k \quad \text{with} \quad \alpha=\alpha(r)\geq\frac{\log r}{\log 2}.$$

On the other hand, using (2) we have

$$\begin{aligned}
&\sup\Big\{\frac{|C_m|m^m}{|m|^{|m|}a_k^m}\,\Big|\,|m|\leq\alpha,\ m\in\widetilde{\mathbf{M}}_\alpha\Big\}\\
&\leq \sup\Big\{\frac{|C_m|m^m}{|m|^{|m|}a_q^m}\,\Big|\,m\in\mathbf{M}\Big\}\cdot\sup\Big\{\Big(\frac{a_q}{a_k}\Big)^m\,\Big|\,|m|\leq\alpha,\ m\in\widetilde{\mathbf{M}}_\alpha\Big\}\\
&\leq C^\alpha\sup\Big\{\Bigg(\frac{1}{\varphi\big(\frac{a_q}{a_p}\big)}\Bigg)^m\,\Big|\,|m|\leq\alpha,\ m\in\widetilde{\mathbf{M}}_\alpha\Big\}\\
&\leq \frac{1}{r}
\end{aligned}$$

with

$$\widetilde{\mathbf{M}}_\alpha = \Big\{m\in\mathbf{N}\,\Big|\,-\alpha\log C+m_1\log\varphi\Big(\frac{a_{1,q}}{a_{1,p}}\Big)+\cdots+m_n\log\varphi\Big(\frac{a_{n,q}}{a_{n,p}}\Big)\geq\log r\Big\}$$

and

$$\begin{aligned}
&\sup\Big\{\frac{|C_m|m^m}{|m|^{|m|}a_p^m}\,\Big|\,|m|\leq\alpha,\ m\notin\widetilde{\mathbf{M}}_\alpha\Big\}\\
&\leq \sup\Big\{\frac{|C_m|m^m}{|m|^{|m|}a_q^m}\,\Big|\,m\in\mathbf{M}\Big\}\cdot\sup\Big\{\Big(\frac{a_q}{a_p}\Big)^m\,\Big|\,|m|\leq\alpha,\ m\notin\widetilde{\mathbf{M}}_\alpha\Big\}\\
&\leq \psi_{q,p}(r),
\end{aligned}$$

where
$$\psi_{q,p}(r) = \sup\Big\{\Big(\frac{a_q}{a_p}\Big)^m \,\big|\, |m| \le \alpha,\ m \notin \widetilde{\mathbf{M}}_\alpha\Big\} < +\infty.$$

We choose ψ such that $\lim\limits_{r\to\infty} \dfrac{\psi_{q,p}(r)}{\psi(r)} = 0$. Then

$$\forall q\ \exists k, C > 0\ \forall r > 0 : W_q \subset C\psi(r)W_p + \frac{2}{r}W_k.$$

Thus, $[\mathcal{H}(O_E)]'$ has property (DN_φ). $\square$

Acknowledgement

The authors would like to thank Prof. Nguyen Van Khue for helpful suggestions.

References

1. A. Aytuna, *On Stein manifolds M for which $\mathcal{O}(M)$ is isomorphic to $\mathcal{O}(\Delta^n)$ as Frechet spaces*, Manuscripta Math. **62** (1988), 297-315.
2. L. Hormander, *An Introduction to Complex Analysis in Several Variables*, North Holland, 1973.
3. J. Mujica, *Spaces of germs of holomorphic functions*, Studies in Analysis Advances in Mathematics Supplementary Studies **4** (1979), 1-41.
4. A. Ryan, *Holomorphic mappings on ℓ_1,* Trans. Amer. Math. Soc. **302** (1987), 797-811.
5. T. Terzioglu and D. Vogt, *On asymptotically normable Frechet spaces*, Preprint.
6. D. Vogt, *Charakterisierung der Unterräume eines nuklearen stabilen Potenzreihenräumes von endlichen Typ,* Studia Math. **71** (1982), 251-270.
7. D. Vogt, *Some results on continuous linear maps between Frechet spaces,* in: Bierstedt K. D., Fuchssteiner B. (eds), Functional analysis: surveys and recent results II, North-Holland Math. Studies **90** (1984), 349-381.
8. D. Vogt, *On the functor $Ext^1(E,F)$ for Frechet spaces*, Studia Math. **85** (1987), 163-197.

Department of Mathematics
Pedagogical Institute
Caugiay, Hanoi, Vietnam.