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AVERAGING OF NEUTRAL DIFFERENTIAL INCLUSIONS WHEN THE AVERAGE VALUE OF THE RIGHT-HAND SIDE DOES NOT EXIST

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ABSTRACT. We consider the problem of applying the averaging method to the asymptotic approximation of solutions of functional-differential equations of neutral type $\dot{x}(t) \in \varepsilon F(t, x_t, \dot{x}_t)$ in the case where the average of the right-hand side does not exist. Our paper generalizes results of W. A. Plotnikow, W. M. Sawczenko from [8] where the generalized system $\dot{x}(t) \in \varepsilon F(t, x)$ was investigated.

1. INTRODUCTION AND NOTATIONS

There has recently been a great deal of interest in the field of Bogolubov's type theorems. Many authors [1, 5, 6, 7] have discussed the averaging theorem for differential equations or functional-differential inclusions.

Let C_0 and L_0 denote the Banach spaces of all continuous and Lebesgue integrable functions of $[-r, 0]$ into R^n with the norms $\|x\|_0 = \sup_{-r \le t \le 0} |x(t)|$ and $|y|_0 = \int_{-r}^{0} |y(t)|dt$ for $x \in C_0$ and $y \in L_0$, respectively, where $|\cdot|$ denotes the Euclidean norm. For a given function $u : [-r, T] \to R^n$ and fixed $t \in [0, T]$, we denote $u_t(s) = u(t + s)$ and $\dot{u}_t(s) = \dot{u}(t + s)$ for $s \in [-r, 0]$, $r \ge 0$, $T > 0$. Finally let us denote by $(\operatorname{comp} R^n, H)$ and $(\operatorname{conv} R^n, H)$ the metric spaces of all nonempty compact and convex subsets of n-dimensional Euclidean space R^n with the Hausdorff metric H, respectively.

In this paper we shall study the functional-differential inclusions of the form:

$$(1) \qquad \begin{cases} \dot{x}(t) \in \varepsilon F(t, x_t, \dot{x}_t) & \text{for a.e. } t \ge 0 \\ x(t) = \varphi(t) & \text{for } t \in [-r, 0] \end{cases}$$

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where $\varepsilon > 0$ is a small parameter, $\varphi : [-r, 0] \to R^n$ is a given absolutely continuous function and $F : [0, \infty) \times C_0 \times L_0 \to \operatorname{conv} R^n$ satisfies the following conditions:

(a) $F(., u, v) : [0, \infty) \to \operatorname{conv} R^n$ is measurable for fixed $(u, v) \in C_0 \times L_0$;

(b) $F(t, ., .) : C_0 \times L_0 \to \operatorname{conv} R^n$ satisfies for fixed $t \in [0, \infty)$ the Lipschitz conditions of the form:

$$H(F(t,u,v), F(t,\overline{u},\overline{v})) \leq k\big(\|u-\overline{u}\|_0 + |v-\overline{v}|_0\big)$$

where $k > 0$, u, $\overline{u} \in C_0$ and v, $\overline{v} \in L_0$;

(c) there exists a $M > 0$ such that $H(F(t, u, v), \{0\}) \leq M$ for $(t, u, v) \in [0, \infty) \times C_0 \times L_0$.

By a solution of (1) we mean a function $x : [-r, \infty) \to R^n$ that is absolutely continuous on $[0, T]$ and satisfies the first of the conditions in (1) at almost all $t \in [0, \infty)$ such that $x(t) = \varphi(t)$ for $t \in [-r, 0]$.

In the method of averaging [3] we considered (1) together with the averaged inclusions:

$$\text{(2)} \qquad \begin{cases} \dot{y}(t) \in \varepsilon F_0(y_t, \dot{y}_t) & \text{for a.e. } t \geq 0, \\ y(t) = \varphi(t) & \text{for } t \in [-r, 0], \end{cases}$$

where $F_0 : C_0 \times L_0 \to \operatorname{conv} R^n$ and

$$\text{(3)} \qquad F_0(u.v) = \lim_{L\to\infty} \frac{1}{L} \int_0^L F(t,u,v)dt$$

uniformly with respect to $(u, v) \in C_0 \times L_0$, where the integral is meant in Aumann's-Hukuhara's sense.

The aim of this paper is to consider the application of the averaging method to the asymptotic approximation of solutions of (1) in the case where the limit (3) does not exist.

Let F^-, $F^+ : C_0 \times L_0 \to \operatorname{conv} R^n$ denote a multifunctions such that

(d) F^- and F^+ are Lipschitzean with respect to $(u, v) \in C_0 \times L_0$ with constant $k > 0$.

(e) F^- and F^+ satisfy the following conditions uniformly with respect to $(u, v) \in C_0 \times L_0$:

$$\text{(4)} \qquad \begin{aligned} &F^-(u,v) \subset \frac{1}{L}\int_0^L F(t,u,v)dt + S_\eta(0) \\ &\frac{1}{L}\int_0^L F(t,u,v)dt \subset F^+(u,v) + S_\eta(0) \end{aligned}$$

for every $\eta > 0$.

We shall consider (1) together with the following inclusions:

$$\begin{cases} y(t) = \varphi(t) & \text{for } t \in [-r, 0], \\ \dot{y}(t) \in \varepsilon F^{-}(y_t, \dot{y}_t) & \text{for a.e. } t \geq 0, \end{cases} \tag{5}$$

and

$$\begin{cases} z(t) = \varphi(t) & \text{for } t \in [-r, 0], \\ \dot{z}(t) \in \varepsilon F^{+}(z_t, \dot{z}_t) & \text{for a.e. } t \geq 0. \end{cases} \tag{6}$$

2. The theorem of averaging for neutral differential inclusions

The main result of this paper is contained in the following theorem.

Theorem 1. *Let $F : [0, \infty) \times C_0 \times L_0 \to conv\, R^n$ and F^{-}, $F^{+} : C_0 \times L_0 \to conv\, R^n$ satisfy the conditions* (a)-(c) *and* (d)-(c)*, respectively. Suppose that given are problems* (1), (5) *and* (6) *together with the initial conditions $x(t) = y(t) = z(t) = \varphi(t) = const$. Then for each $\eta > 0$ and $T > 0$ there exists a $\varepsilon^0(\eta, T) > 0$ such that for every $\varepsilon \in (0, \varepsilon^0)$ the following conditions are satisfied:*

(i) *for each solution $y(\cdot)$ of* (5) *there exists a solution $x(\cdot)$ of* (1) *such that*

$$|y(t) - x(t)| \leq \eta \quad \textit{for} \quad t \in \Big[-r, \frac{T}{\varepsilon}\Big], \tag{7}$$

(ii) *for each solution $x(\cdot)$ of* (1) *there exists a solution $z(\cdot)$ of* (6) *such that*

$$|x(t) - z(t)| \leq \eta \quad \text{for} \quad t \in \Big[-r, \frac{T}{\varepsilon}\Big]. \tag{8}$$

To prove this theorem we shall use a Filippov's type theorem for functional-differential equations of neutral type of the form:

Theorem 2 [4]. *Let $\delta : [0, T] \to R$ be a nonnegative Lebesgue integrable function and let $\varphi \in C_0$ be absolutely continuous. Suppose $F : [0, T] \times C_0 \times L_0 \to comp\, R^n$ satisfies* (a), (b) *and* (c) *of the form*

$$H(F(t, u, v), F(t, \overline{u}, \overline{v})) \leq k(t)\big(\|u - \overline{u}\|_0 + |v - \overline{v}|_0\big),$$

where $k : [0,T] \to R^+$ *is a Lebesgue integrable function,* u, $\overline{u} \in C_0$ *and* v, $\overline{v} \in L_0$. *Furthermore let* $z : [-r,T] \to R^n$ *be an absolutely continuous mapping such that:*

(f) $z(t) = \varphi(t)$ *for* $t \in [-r,0]$,

(g) $d(\dot{z}(t), F(t, z_t, \dot{z}_t)) \leq \delta(t)$ *for a.e.* $t \in [0,T]$.

Then there is a solution $x(\cdot)$ *of an initial-value problem*

$$\dot{x}(t) \in F(t, x_t, \dot{x}_t) \text{ for a.e. } t \in [0,T],$$
$$x(t) = \varphi(t) \text{ for } t \in [-r,0]$$

such that

$$|x(t) - z(t)| \leq \xi(t) \quad \text{for } t \in [0,T],$$
$$|\dot{x}(t) - \dot{z}(t)| \leq \delta(t) + 2k(t)\xi(t) \quad \text{for a.e. } t \in [0,T],$$

where $\xi(t) = \int\limits_0^t \delta(s) e^{2[m(t)-m(s)]} ds$ *and* $m(s) = \int\limits_0^t k(r)dr$.

Proof of the Theorem 1. Observe that the mappings F^-, F^+ are bounded, i.e. there exists a $M > 0$ such that $H(F^-(u,v), \{0\}) \leq M$ and $H(F^+(u,v), \{0\}) \leq M$ for $(u,v) \in C_0 \times L_0$.

Let $y(\cdot)$ be a solution of (5). To prove Theorem 1 we shall consider the solution $x(\cdot)$ of the inclusion (1) such that for $t \in [-r,0]$, $x(t) = y(t) = \varphi(t)$ (hence $|x(t) - y(t)| = 0 < \eta$) and, for $t \in \left[0, \frac{T}{\varepsilon}\right]$, the inequality (7) is satisfied too. To do this let us divide the interval $\left[0, \frac{T}{\varepsilon}\right]$ on m-subintervals $[t_i, t_{i+1}]$, where $t_i = \frac{iT}{\varepsilon m}$, $i = 0, 1, \ldots, m-1$, and write the solution $y(\cdot)$ in the form

$$(9) \qquad \begin{cases} y(t) = \text{const} & \text{for } t \in [-r,0], \\ y(t) = y(t_i) + \varepsilon \int\limits_{t_i}^{t} v(\tau) d\tau & \text{for } t \in [t_i, t_{i+1}], \end{cases}$$

where $v(t) \in F^-(y_t, \dot{y}_t)$.

Let us consider a function $y^1(\cdot)$ defined by

$$(10) \qquad \begin{cases} y^1(t) = \text{const} & \text{for } t \in [-r,0], \\ y^1(t) = y^1(t_i) + \varepsilon u^1(t_i)(t - t_i) & \text{for } t \in [t_i, t_{i+1}], \end{cases}$$

where $u^1(\cdot)$ is measurable functions such that $u^1(t) \in F^-(y^1_{t_i}, \dot{y}^1_{t_i})$ and

$$\Big|\frac{T}{\varepsilon m}u^1(t_i) - \int\limits_{t_i}^{t_{i+1}} v(t)dt\Big| = \min\Big\{\Big|\frac{T}{\varepsilon m}z - \int\limits_{t_i}^{t_{i+1}} v(t)dt\Big| \;\Big|\; z \in F^-(y^1_{t_i}, \dot{y}^1_{t_i})\Big\}.$$

The mapping u^1 exists because the set-valued function F^- is measurable and has compact and convex values [2]. Let $\delta_i = |y(t_i) - y^1(t_i)|$, $i = 1, 2, \dots, m-1$. Then, by virtue of (9), for every $t \in [t_i, t_{i+1}]$ we have

$$|y(t) - y^1(t_i)| = \Big|y(t_i) + \varepsilon\int\limits_{t_i}^{t} v(\tau)d\tau - y^1(t_i)\Big| \le \delta_i + \varepsilon M(t - t_i). \tag{11}$$

Furthermore,

$$|v(t) - u^1(t_i)| \le H\big(F^-(y_t, \dot{y}_t), F^-(y^1_{t_i}, \dot{y}^1_{t_i})\big) \le k\big(\|y_t - y^1_{t_i}\|_0 + |\dot{y}_t - \dot{y}^1_{t_i}|_0\big).$$

Adopting now the procedure presented in the proof of Theorem 1 of [3] we have

$$\begin{aligned}\|y_t - y^1_{t_i}\|_0 &\le \frac{MT}{m} + \delta_i + 2\varepsilon Mr,\\ |\dot{y}_t - \dot{y}^1_{t_i}|_0 &\le 2\varepsilon Mr.\end{aligned}$$

Hence for $t \in [t_i, t_{i+1}]$ we obtain

$$|v(t) - u^1(t_i)| \le k\Big(\frac{MT}{m} + \delta_i + 4\varepsilon Mr\Big). \tag{12}$$

By virtue of (9), (10) and (12) it follows

$$\begin{aligned}\delta_i &= |y(t_i) - y^1(t_i)| \le |y(t_{i-1}) - y^1(t_{i-1})| + \varepsilon\int\limits_{t_{i-1}}^{t_i} |v(\tau) - u^1(t_{i-1})|d\tau\\
&\le \delta_{i-1} + \varepsilon k\Big(\frac{MT}{m} + \delta_{i-1} + 4\varepsilon Mr\Big)(t_i - t_{i-1})\\
&= \delta_{i-1} + \varepsilon k\Big(\frac{MT}{m} + \delta_{i-1} + 4\varepsilon Mr\Big)\frac{T}{\varepsilon m}\\
&= \delta_{i-1}\Big(1 + \frac{a}{m}\Big) + \frac{b}{m},\end{aligned}$$

where $a = kT$ and $b = \dfrac{kMT}{m}(T + 4\varepsilon mr)$. Hence

$$
\begin{aligned}
\delta_i &\le \delta_{i-1}\Big(1 + \frac{a}{m}\Big) + \frac{b}{m} \\
&\le \Big(1 + \frac{a}{m}\Big)\Big[\delta_{i-2}\Big(1 + \frac{a}{m}\Big) + \frac{b}{m}\Big] + \frac{b}{m} \\
&= \Big(1 + \frac{a}{m}\Big)^2 \delta_{i-2} + \Big(1 + \frac{a}{m}\Big)\frac{b}{m} \\
&\le \dots \\
&\le \Big(1 + \frac{a}{m}\Big)^i \delta_0 + \Big(1 + \frac{a}{m}\Big)^{i-2}\frac{b}{m} + \dots + \frac{b}{m} \\
&= \frac{b}{m}\Big(1 + \big(1 + \frac{a}{m}\big) + \dots + \big(1 + \frac{a}{m}\big)^{i-1}\Big) \\
&= \frac{b}{a}\Big(\big(1 + \frac{a}{m}\big)^i - 1\Big) \\
&\le \frac{b}{a}(e^a - 1) \\
&= \frac{M}{m}(T + 4\varepsilon mr)(e^{kT} - 1),
\end{aligned}
\tag{13}
$$

where $i = 0, 1, \dots, m-1$. For $t \in [t_i, t_{i-1}]$ we have

$$|y(t) - y(t_i)| = \Big|\varepsilon \int_{t_i}^{t} v(\tau) d\tau\Big| \le \varepsilon M(t - t_i) \le \frac{MT}{m}$$

and

$$|y^1(t) - y^1(t_i)| = |\varepsilon u^1(t_i)(t - t_i)| \le \frac{MT}{m} \cdot$$

Hence we obtain

$$
\begin{aligned}
|y(t) - y^1(t)| &\le |y(t) - y(t_i)| + |y(t_i) - y^1(t_i)| + |y^1(t_i) - y^1(t)| \\
&\le \frac{2MT}{m} + \frac{M}{m}(T + 4\varepsilon mr)(e^{kT} - 1).
\end{aligned}
\tag{14}
$$

Now we shall consider the function

$$
\begin{cases}
y^2(t) = \text{const} & \text{for } t \in [-r, 0], \\
y^2(t) = y^2(t_i) + \varepsilon \displaystyle\int_{t_i}^{t} u^2(\tau) d\tau & \text{for } t \in [t_i, t_{i+1}],
\end{cases}
\tag{15}
$$

where $t_i = \dfrac{iT}{\varepsilon m}$, $i = 0, 1, \ldots, m-1$, and $u^2(t) \in F(t, y_{t_i}, \dot{y}_{t_i})$.

Let us notice that by virtue of condition (d), for each $\eta > 0$ there exists a $L_0(\eta)$ such that for every $L > L_0$ we have the inclusion

$$F^-(y^1_{t_i}, \dot{y}^1_{t_i}) \subset \frac{1}{L} \int\limits_0^L F(t, y^1_{t_i}, \dot{y}^1_{t_i}) dt + S_\eta(0).$$

In particular, for $\dfrac{T}{\varepsilon m} > L_0$ and for every $i \in \{0, 1, \ldots, m\}$ we have

$$F^-(y^1_{t_i}, \dot{y}^1_{t_i}) \subset \frac{\varepsilon m}{iT} \int\limits_0^{\frac{iT}{\varepsilon m}} F(t, y^1_{t_i}, \dot{y}^1_{t_i}) dt + S_\eta(0), \tag{16}$$

$$F^-(y^1_{t_i}, \dot{y}^1_{t_i}) \subset \frac{\varepsilon m}{(i+1)T} \int\limits_0^{\frac{(i+1)T}{\varepsilon m}} F(t, y^1_{t_i}, \dot{y}^1_{t_i}) dt + S_\eta(0). \tag{17}$$

Let us observe that $t_{i+1} = \dfrac{(1+i)T}{\varepsilon m}$ and $t_i = \dfrac{iT}{\varepsilon m}$. By (16), (17) and the Hausdorff metric condition (see Lemma 3 (vi) of [1]) we have

$$\begin{aligned}
&H\Big(\int\limits_{t_i}^{t_{i+1}} F(t, y^1_{t_i}, \dot{y}^1_{t_i}) dt, F^-(y^1_{t_i}, \dot{y}^1_{t_i}) \Big) \\
&\le H\Big(\int\limits_0^{t_i} F(t, y^1_{t_i}, \dot{y}^1_{t_i}) dt, \int\limits_0^{t_i} F^-(y^1_{t_i}, \dot{y}^1_{t_i}) \Big) + \\
&\quad + H\Big(\int\limits_0^{t_{i+1}} F(t, y^1_{t_i}, \dot{y}^1_{t_i}) dt, \int\limits_0^{t_i} F^-(y^1_{t_i}, \dot{y}^1_{t_i}) dt \Big) \\
&= \frac{iT}{\varepsilon m} H\Big(\frac{\varepsilon m}{iT} \int\limits_0^{t_i} F(t, y^1_{t_i}, \dot{y}^1_{t_i}) dt, F^-(y^1_{t_i}, \dot{y}^1_{t_i}) \Big) + \\
&\quad + \frac{(1+i)T}{\varepsilon m} H\Big(\frac{\varepsilon m}{(1+i)T} \int\limits_0^{t_{i+1}} F(t, y^1_{t_i}, \dot{y}^1_{t_i}) dt, F^-(y^1_{t_i}, \dot{y}^1_{t_i}) \Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{iT}{\varepsilon m}\eta + \frac{(1+i)T}{\varepsilon m}\eta = \frac{T\eta}{\varepsilon m}(2i+1) \\
&\leq \frac{T\eta}{\varepsilon m}(2m+1).
\end{aligned}
$$

Hence

$$
H\Big(\int\limits_{t_i}^{t_{i+1}} F(t, y^1_{t_i}, \dot{y}^1_{t_i})dt, F^{-}(y^1_{t_i}, \dot{y}^1_{t_i})\Big) \leq \frac{\eta_1 T}{\varepsilon m},
$$

where $\eta_1 = (2m+1)\eta$ for

$$
\frac{T}{\varepsilon m} > L_0\Big(\frac{\eta_1}{2m+1}\Big),
$$

$$
\varepsilon < \varepsilon_0(\eta_1, m) = \frac{T}{mL_0\Big(\dfrac{\eta_1}{2m+1}\Big)}\cdot
$$

It follows that

$$
\Big|\int\limits_{t_i}^{t_{i+1}} u^2(\tau) - u^1(\tau)dt\Big| \leq \frac{\eta_1 T}{\varepsilon m}\,, \tag{18}
$$

$$
\begin{aligned}
|y^1(t_{i+1}) - y^2(t_{i+1})| &\leq |y^1(t_i) - y^2(t_i)| + \varepsilon\Big|\int\limits_{t_i}^{t_{i+1}} u^1(t_i) - u^2(\tau)d\tau\Big| \\
&\leq |y^1(t_i) - y^2(t_i)| + \frac{\eta_1 T}{m} \\
&\leq \cdots \leq m\frac{\eta_1 T}{m} = \eta_1 T, \qquad (19)
\end{aligned}
$$

where $i = 0, 1, \ldots, m-1$.

Using the inequality (19) and the fact that for $t \in [t_i, t_{i+1}]$,

$$
\begin{aligned}
|y^2(t) - y^2(t_i)| &\leq \frac{MT}{m}\,, \\
|y^1(t) - y^1(t_i)| &\leq \frac{MT}{m}\,,
\end{aligned}
$$

we have

$$
\begin{aligned}
|y^1(t)-y^2(t)| &\le |y^1(t)-y^1(t_i)|+|y^1(t_i)-y^2(t_i)|+|y^2(t_i)-y^2(t)| \\
&\le \frac{2MT}{m}+\eta_1 T. \qquad (20)
\end{aligned}
$$

By the assumption (b) it follows that

$$H\big(F(t,y^2_t,\dot y^2_t),F(t,y^1_{t_i},\dot y^1_{t_i})\big)\le k\big(\|y^2_t-y^1_{t_i}\|_0+|\dot y^2_t-\dot y^1_{t_i}|_0\big).$$

Using the definitions of the norms $\|\cdot\|_0$, $|\cdot|_0$ and properties of F^- and making use of the inequality (20) we obtain

$$
\begin{aligned}
\|y^2_t-y^1_{t_i}\|_0 &\le \frac{3MT}{m}+\eta_1 T,\\
|\dot y^2_t-\dot y^1_{t_i}|_0 &\le 2\varepsilon M r,\\
H\big(F(t,y^2_t,\dot y^2_t),F(t,y^1_{t_i},\dot y^1_{t_i})\big) &\le k\big(\frac{3MT}{m}+\eta_1 T+2\varepsilon M r\big).
\end{aligned}
$$

Hence, by virtue of (15) we have

$$
\begin{aligned}
&d\big(\dot y(t),\varepsilon F(t,y^2_t,\dot y^2_t)\big)\\
&\le d\big(\dot y(t),\varepsilon F(t,y^1_{t_i},\dot y^1_{t_i})\big)+H\big(\varepsilon F(t,y^1_{t_i},\dot y^1_{t_i}),\varepsilon F(t,y^2_t,\dot y^2_t)\big)\\
&\le \varepsilon k\Big(\frac{3MT}{m}+\eta_1 T+2\varepsilon M r\Big).
\end{aligned}
$$

Now, on the ground of Theorem 2 there exists a solution $x(\cdot)$ of (1) such that for $t\in\Big[0,\dfrac{T}{\varepsilon}\Big]$,

$$
\begin{aligned}
|y^2(t)-x(t)| &\le \int_0^t \varepsilon k\Big(\frac{3MT}{m}+\eta_1 T+2\varepsilon M r\Big)e^{2\varepsilon k(t-s)}ds\\
&\le \Big(\frac{3MT}{\varepsilon m}+\frac{\eta_1 T}{2}+\varepsilon M r\Big)\big(e^{2kT}-1\big). \qquad (21)
\end{aligned}
$$

By the inequalities (21), (20) and (14) it follows that

$$
\begin{aligned}
&|x(t)-y(t)|\\
&\le |x(t)-y^2(t)|+|y^2(t)-y^1(t)|+|y^1(t)-y(t)|\\
&\le \Big(\frac{3MT}{2m}+\frac{\eta_1 T}{2}+\varepsilon M r\Big)\big(e^{2kT}-1\big)+\frac{2MT}{m}+\eta_1 T\\
&\quad+\frac{2MT}{m}+\frac{M}{m}(T+4\varepsilon m r)\big(e^{2kT}-1\big) \qquad (22)\\
&\le \frac{5MT}{2m}\big(1+e^{2kT}\big)+\frac{\eta_1 T}{2}\big(1+e^{kT}\big)+5M\varepsilon r\big(e^{2kT}-1\big).
\end{aligned}
$$

Therefore, choosing

$$m > \frac{15MT(1+e^{2kT})}{2\eta},$$
$$\eta_1 = \frac{2\eta}{3T(1+e^{kT})},$$
$$\varepsilon < \frac{\eta}{15Mr(e^{2kT}-1)},$$

we get the inequality $|x(t) - y(t)| \le \eta$ for $t \in \left[0, \frac{T}{\varepsilon}\right]$. The proof of the condition (i) is now complete.

Adopting the procedure presented above we will get condition (ii). □

Corollary 3. *In the case where there exists a limit*

$$F_0(u,v) = \lim_{L\to\infty} \int_0^L F(t,u,v)dt$$

we have $\overline{F^-(u,v)} = \overline{F^+(u,v)} = F_0(u,v)$, *where*

$$\overline{F^-(u,v)} = \underline{\lim}_{L\to\infty} \frac{1}{L} \int_0^L F(t,u,v)dt,$$

$$\overline{F^+(u,v)} = \overline{\lim}_{L\to\infty} \frac{1}{L} \int_0^L F(t,u,v)dt.$$

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