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STRONG CONVERGENCE OF NOOR ITERATION FOR GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract. In this paper, we establish strong convergence theorems of the Noor iteration for generalized asymptotically quasi-nonexpansive mappings in CAT(0) spaces. Our results improve and extend the corresponding results from the current literature.

1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [2]), $\mathbb{R}$-trees (see [17]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [9]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [2].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [16,17]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [1, 5–8, 10, 13, 14, 18–22] and the references therein). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see, e.g., [2]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly

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one geodesic joining x and y for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [2]).

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d(\overline{x}, \overline{y}). \tag{1.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [12]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the mid point of the segment $[y_1, y_2]$ which we will be denoted by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{1.2}$$

The inequality (1.2) is the (CN) inequality of Bruhat and Titz [4]. The above inequality has been extended in [7] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha)y) &\leq \alpha d^2(z, x) + (1-\alpha)d^2(z, y) \\ &\quad -\alpha(1-\alpha)d^2(x, y), \end{aligned} \tag{1.3}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [2, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z, x) + (1-\alpha)d(z, y), \tag{1.4}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

Let T be a self map on a nonempty subset C of X. Denote the set of fixed points of T by $F(T) = \{x \in C : T(x) = x\}$. We say that T is:

(1) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + r_n)d(x, y), \tag{1.5}$$

for all $x, y \in C$ and $n \geq 1$.

(2) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, p) \quad \leq \quad (1 + r_n) d(x, p), \tag{1.6}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

(3) generalized asymptotically quasi-nonexpansive [11] if $F(T) \neq \emptyset$ and there exist two sequences of real numbers $\{r_n\}$ and $\{s_n\}$ with $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} s_n$ such that

$$d(T^n x, p) \quad \leq \quad (1 + r_n) d(x, p) + s_n, \tag{1.7}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

(4) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \quad \leq \quad L\ d(x, y), \tag{1.8}$$

for all $x, y \in C$ and $n \geq 1$.

(5) semi-compact if for any bounded sequence $\{x_n\}$ in C with $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there is a convergent subsequence of $\{x_n\}$.

If in Definition (3), $s_n = 0$ for all $n \geq 1$, then T becomes asymptotically quasi-nonexpansive, and hence the class of generalized asymptotically quasi-nonexpansive maps includes the class of asymptotically quasi-nonexpansive maps.

Let $\{x_n\}$ be a sequence in a metric space (X, d), and let C be a subset of X. We say that $\{x_n\}$ is:

(6) of monotone type (A) with respect to C if for each $p \in C$, there exist two sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$ and

$$d(x_{n+1}, p) \leq (1 + a_n) d(x_n, p) + b_n, \tag{1.9}$$

(7) of monotone type (B) with respect to C if for each $p \in C$, there exist two sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$ and

$$d(x_{n+1}, C) \quad \leq \quad (1 + a_n) d(x_n, C) + b_n, \tag{1.10}$$

(see also [24]).

From of the above definitions, it is clear that a sequence of monotone type (A) is a sequence of monotone type (B) but the converse does not hold, in general.

Recently, Y. Niwongsa and B. Panyanak [20] studied the Noor iteration scheme in CAT(0) spaces and they proved some $\triangle$ and strong convergence theorems for asymptotically nonexpansive mappings which extend and improve some recent results from the literature.

The aim of this paper is to extend the corresponding results of [20] and to prove some convergence theorems of Noor iterations for generalized asymptotically quasi-nonexpansive mappings in CAT(0) spaces.

We need the following useful lemma to prove our convergence results.

Lemma 1.1 (see [23]). *Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three sequences of nonnegative real numbers satisfying the following conditions:*

$$p_{n+1} \leq (1+q_n)p_n + r_n, \quad n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then

(1) $\lim_{n\to\infty} p_n$ *exists.*

(2) *In addition, if* $\liminf_{n\to\infty} p_n = 0$, *then* $\lim_{n\to\infty} p_n = 0$.

2. Strong convergence theorems in CAT(0) spaces

We establish some convergence results of Noor iterations to a fixed point for generalized asymptotically quasi-nonexpansive self mappings in the general class of CAT(0) spaces.

Theorem 2.1. *Let (X,d) be a complete CAT(0) space, and let C be a nonempty closed convex subset of X. Let $T\colon C \to C$ be a generalized asymptotically quasi-nonexpansive mapping with $\{r_n\}, \{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F(T)$ is closed. Let $\{x_n\}$ be the Noor iteration sequence defined as: For a given $x_1 \in C$, define*

$$\begin{aligned} z_n &= \gamma_n T^n x_n \oplus (1-\gamma_n)x_n, \\ y_n &= \beta_n T^n z_n \oplus (1-\beta_n)x_n, \qquad n \geq 1, \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1-\alpha_n)x_n, \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in [0,1]. Then the sequence $\{x_n\}$ is of monotone type (A) and monotone type (B) with respect to $F(T)$. Moreover, $\{x_n\}$ converges strongly to a fixed point p of the mapping T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf_{p\in F(T)}\{d(x,p)\}$.

Proof. The necessity is obvious and so it is omitted. Now, we prove the sufficiency. For any $p \in F(T)$, from (1.4), (1.7) and (2.1), we have

$$\begin{aligned} d(z_n,p) &= d(\gamma_n T^n x_n \oplus (1-\gamma_n)x_n, p) \\ &\leq \gamma_n d(T^n x_n, p) + (1-\gamma_n)d(x_n,p) \\ &\leq \gamma_n[(1+r_n)d(x_n,p) + s_n] + (1-\gamma_n)d(x_n,p) \\ &\leq (1+r_n)[\gamma_n + 1 - \gamma_n]d(x_n,p) + \gamma_n s_n \\ &= (1+r_n)d(x_n,p) + \gamma_n s_n, \end{aligned} \tag{2.2}$$

and

$$
\begin{aligned}
d(y_n,p) &= d(\beta_n T^n z_n \oplus (1-\beta_n)x_n, p)\\
&\le \beta_n d(T^n z_n,p) + (1-\beta_n)d(x_n,p)\\
&\le \beta_n[(1+r_n)d(z_n,p)+s_n] + (1-\beta_n)d(x_n,p)\\
&\le \beta_n(1+r_n)d(z_n,p) + \beta_n s_n + (1-\beta_n)d(x_n,p). \qquad (2.3)
\end{aligned}
$$

Substituting (2.2) into (2.3), we get

$$
\begin{aligned}
d(y_n,p) &\le \beta_n(1+r_n)[(1+r_n)d(x_n,p)+\gamma_n s_n] + \beta_n s_n\\
&\quad +(1-\beta_n)d(x_n,p)\\
&\le (1+r_n)^2[\beta_n + 1 - \beta_n]d(x_n,p)\\
&\quad +\beta_n(1+r_n)s_n(1+\gamma_n)\\
&\le (1+r_n)^2 d(x_n,p) + 2\beta_n(1+r_n)s_n, \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
d(x_{n+1},p) &= d(\alpha_n T^n y_n \oplus (1-\alpha_n)x_n, p)\\
&\le \alpha_n d(T^n y_n,p) + (1-\alpha_n)d(x_n,p)\\
&\le \alpha_n[(1+r_n)d(y_n,p)+s_n] + (1-\alpha_n)d(x_n,p)\\
&\le \alpha_n(1+r_n)d(y_n,p) + \alpha_n s_n + (1-\alpha_n)d(x_n,p). \qquad (2.5)
\end{aligned}
$$

Substituting (2.4) into (2.5), we get

$$
\begin{aligned}
d(x_{n+1},p) &\le \alpha_n(1+r_n)[(1+r_n)^2 d(x_n,p) + 2\beta_n(1+r_n)s_n]\\
&\quad +\alpha_n s_n + (1-\alpha_n)d(x_n,p)\\
&\le (1+r_n)^3[\alpha_n + 1 - \alpha_n]d(x_n,p) + \alpha_n s_n\\
&\quad +2\alpha_n\beta_n(1+r_n)^2 s_n\\
&\le (1+r_n)^3 d(x_n,p) + (1+r_n)^2\alpha_n s_n(1+2\beta_n)\\
&\le (1+r_n)^3 d(x_n,p) + 3(1+r_n)^2\alpha_n s_n\\
&= (1+A_n)d(x_n,p) + B_n, \qquad (2.6)
\end{aligned}
$$

where $A_n = 3r_n + 3r_n^2 + r_n^3$ and $B_n = 3(1+r_n)^2\alpha_n s_n$. Since by hypothesis, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$. Now, from (2.6), we get

$$
d(x_{n+1},p) \le (1+A_n)d(x_n,p) + B_n, \qquad (2.7)
$$

$$
d(x_{n+1},F(T)) \le (1+A_n)d(x_n,F(T)) + B_n. \qquad (2.8)
$$

These inequalities, respectively, prove that $\{x_n\}$ is a sequence of monotone type (A) and monotone type (B) with respect to $F(T)$.

Now, we prove that $\{x_n\}$ converges strongly to a fixed point of the mapping T if and only if $\liminf_{n\to\infty} d(x_n,F(T)) = 0$.

If $x_n \to p \in F(T)$, then $\lim_{n\to\infty} d(x_n,p) = 0$. Since $0 \le d(x_n,F(T)) \le d(x_n,p)$, we have $\liminf_{n\to\infty} d(x_n,F(T)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Applying Lemma 1.1 to (2.8), we have that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Further, by hypothesis $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, we conclude that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence.

Since $1 + x \le e^x$ for $x \ge 0$, therefore from (2.7), we have

$$\begin{aligned} d(x_{n+m}, p) &\le (1 + A_{n+m-1})d(x_{n+m-1}, p) + B_{n+m-1} \\ &\le e^{A_{n+m-1}} d(x_{n+m-1}, p) + B_{n+m-1} \\ &\le e^{A_{n+m-1}}[e^{A_{n+m-2}} d(x_{n+m-2}, p) + B_{n+m-2}] \\ &\quad + B_{n+m-1} \\ &\le e^{(A_{n+m-1}+A_{n+m-2})} d(x_{n+m-2}, p) \\ &\quad + e^{A_{n+m-1}}[B_{n+m-2} + B_{n+m-1}] \\ &\le \dots \\ &\le e^{\sum_{k=n}^{n+m-1} A_k} d(x_n, p) + e^{\sum_{k=n+1}^{n+m-1} A_k} \Big(\sum_{k=n}^{n+m-1} B_k\Big) \\ &\le e^{\sum_{k=n}^{n+m-1} A_k} d(x_n, p) + e^{\sum_{k=n}^{n+m-1} A_k} \Big(\sum_{k=n}^{n+m-1} B_k\Big). \end{aligned} \tag{2.9}$$

Let $M = e^{\sum_{k=n}^{n+m-1} A_k}$. Then $0 < M < \infty$ and

$$d(x_{n+m}, p) \le M d(x_n, p) + M\Big(\sum_{k=n}^{n+m-1} B_k\Big), \tag{2.10}$$

for the natural numbers m, n and $p \in F(T)$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, therefore for any $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F(T)) < \varepsilon/8M$ and $\sum_{k=n}^{n+m-1} B_k < \varepsilon/4M$ for all $n \ge n_0$. So, we can find $p^* \in F(T)$ such that $d(x_{n_0}, p^*) < \varepsilon/4M$. Hence, for all $n \ge n_0$ and $m \ge 1$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\le d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\le M d(x_{n_0}, p^*) + M \sum_{k=n_0}^{\infty} B_k \\ &\quad + M d(x_{n_0}, p^*) + M \sum_{k=n_0}^{\infty} B_k \\ &= 2M\Big(d(x_{n_0}, p^*) + \sum_{k=n_0}^{\infty} B_k\Big) \\ &\le 2M\Big(\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\Big) = \varepsilon. \end{aligned} \tag{2.11}$$

This proves that $\{x_n\}$ is a Cauchy sequence. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = z$. Since C is closed, therefore $z \in C$. Next, we show that $z \in F(T)$. Now, the following two inequalities:

$$\begin{aligned} d(z,p) &\leq d(z,x_n) + d(x_n,p) \quad \forall p \in F(T),\ n \geq 1 \text{ and} \\ d(z,x_n) &\leq d(z,p) + d(x_n,p) \quad \forall p \in F(T),\ n \geq 1 \end{aligned} \tag{2.12}$$

give

$$-d(z,x_n) \leq d(z,F(T)) - d(x_n,F(T)) \leq d(z,x_n),\ n \geq 1. \tag{2.13}$$

That is,

$$|d(z,F(T)) - d(x_n,F(T))| \leq d(z,x_n),\ n \geq 1. \tag{2.14}$$

As $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n,F(T)) = 0$, we conclude that $z \in F(T)$. This completes the proof. □

We deduce some results from Theorem 2.1 as follows.

Corollary 2.2. *Let (X,d) be a complete $CAT(0)$ space, and let C be a nonempty closed convex subset of X. Let $T\colon C \to C$ be a generalized asymptotically quasi-nonexpansive mapping with $\{r_n\}, \{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F(T)$ is closed. Let $\{x_n\}$ be the Noor iteration sequence defined by (2.1). Then $\{x_n\}$ converges strongly to a fixed point p of the mapping T if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $p \in F(T)$.*

Corollary 2.3. *Let (X,d) be a complete $CAT(0)$ space, and C be a nonempty closed convex subset of X. Let $T\colon C \to C$ be an asymptotically quasi-nonexpansive mapping with $\{r_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$. Suppose that $F(T)$ is closed. Let $\{x_n\}$ be the Noor iteration sequence defined by (2.1). Then $\{x_n\}$ converges strongly to a fixed point p of the mapping T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof. Follows from Theorem 2.1 with $s_n = 0$ for all $n \geq 1$. □

Corollary 2.4. *Let X be a Banach space, and let C be a nonempty closed convex subset of X. Let $T\colon C \to C$ be an asymptotically quasi-nonexpansive mapping with $\{r_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$. Suppose that $F(T)$ is closed. Let $\{x_n\}$ be the Noor iteration sequence defined by (2.1). Then the sequence $\{x_n\}$ is of monotone type (A) and monotone type (B) with respect to $F(T)$. Moreover, $\{x_n\}$ converges strongly to a fixed point p of the mapping T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof. Take $\lambda x \oplus (1-\lambda)y = \lambda x + (1-\lambda)y$ in Corollary 2.3. □

Lemma 2.5. *Let (X,d) be a complete $CAT(0)$ space, and let C be a nonempty closed convex subset of X. Let $T\colon C\to C$ be a uniformly continuous generalized asymptotically quasi-nonexpansive mapping with $\{r_n\},\{s_n\}\subset[0,\infty)$ such that $\sum_{n=1}^{\infty} r_n<\infty$ and $\sum_{n=1}^{\infty} s_n<\infty$. Suppose that $F(T)\neq\emptyset$. Let $\{x_n\}$ be the Noor iteration sequence defined by (2.1). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\delta,1-\delta]$ for some $\delta\in(0,1)$. Then*

(a) $\lim_{n\to\infty} d(T^n y_n,x_n)=0$;

(b) $\lim_{n\to\infty} d(T^n z_n,x_n)=0$.

Proof. Let $p\in F(T)$. Then, by Theorem 2.1, we have $\lim_{n\to\infty} d(x_n,p)$ exists. Let $\lim_{n\to\infty} d(x_n,p)=a$. If $a=0$, then by the continuity of T the conclusion follows. Now suppose $a>0$. We claim that

$$\lim_{n\to\infty} d(T^n y_n,x_n)=0=\lim_{n\to\infty} d(T^n z_n,x_n). \tag{2.15}$$

Since $\{x_n\}$ is bounded, there exists $R>0$ such that $\{x_n\},\{y_n\},\{z_n\}\subset B_R(p)$ for all $n\geq 1$. Using (1.3) and (2.1), we have

$$\begin{aligned}
d^2(z_n,p) &= d^2(\gamma_n T^n x_n\oplus(1-\gamma_n)x_n,p)\\
&\leq \gamma_n d^2(T^n x_n,p)+(1-\gamma_n)d^2(x_n,p)\\
&\quad -\gamma_n(1-\gamma_n)d(T^n x_n,x_n)\\
&\leq \gamma_n[(1+r_n)d(x_n,p)+s_n]^2+(1-\gamma_n)d^2(x_n,p)\\
&\quad -\gamma_n(1-\gamma_n)d(T^n x_n,x_n)\\
&\leq \gamma_n[(1+r_n)^2 d^2(x_n,p)+\rho_n]+(1-\gamma_n)d^2(x_n,p)\\
&\quad -\gamma_n(1-\gamma_n)d(T^n x_n,x_n)\\
&= \gamma_n[(1+u_n)d^2(x_n,p)+\rho_n]+(1-\gamma_n)d^2(x_n,p)\\
&\quad -\gamma_n(1-\gamma_n)d(T^n x_n,x_n)\\
&\leq (1+u_n)[\gamma_n+1-\gamma_n]d^2(x_n,p)+\gamma_n\rho_n\\
&\quad -\gamma_n(1-\gamma_n)d(T^n x_n,x_n)\\
&\leq (1+u_n)d^2(x_n,p)+\rho_n, \qquad (2.16)
\end{aligned}$$

where $\rho_n=2(1+r_n)s_n d(x_n,p)+s_n^2$ and $u_n=2r_n+r_n^2$. Since the assumptions $\sum_{n=1}^{\infty} r_n<\infty$ and $\sum_{n=1}^{\infty} s_n<\infty$, it follows that $\sum_{n=1}^{\infty}\rho_n<\infty$ and $\sum_{n=1}^{\infty} u_n<\infty$. Again note that

$$\begin{aligned}
d^2(y_n,p) &= d^2(\beta_n T^n z_n\oplus(1-\beta_n)x_n,p)\\
&\leq \beta_n d^2(T^n z_n,p)+(1-\beta_n)d^2(x_n,p)\\
&\quad -\beta_n(1-\beta_n)d^2(T^n z_n,x_n)
\end{aligned}$$

$$\begin{aligned}
&\le \beta_n[(1+r_n)d(z_n,p)+s_n]^2+(1-\beta_n)d^2(x_n,p) \\
&\quad -\beta_n(1-\beta_n)d^2(T^n z_n,x_n) \\
&\le \beta_n[(1+r_n)^2 d^2(z_n,p)+\mu_n]+(1-\beta_n)d^2(x_n,p) \\
&\quad -\beta_n(1-\beta_n)d^2(T^n z_n,x_n) \\
&= \beta_n[(1+u_n)d^2(z_n,p)+\mu_n]+(1-\beta_n)d^2(x_n,p) \\
&\quad -\beta_n(1-\beta_n)d^2(T^n z_n,x_n) \\
&\le (1+u_n)[\beta_n+1-\beta_n]d^2(z_n,p)+\beta_n\mu_n \\
&\quad -\beta_n(1-\beta_n)d^2(T^n z_n,x_n) \\
&\le (1+u_n)d^2(z_n,p)+\beta_n\mu_n \\
&\quad -\beta_n(1-\beta_n)d^2(T^n z_n,x_n), \qquad (2.17)
\end{aligned}$$

where $\mu_n = 2(1+r_n)s_n d(z_n,p)+s_n^2$ and $u_n = 2r_n+r_n^2$. By assumptions $\sum\limits_{n=1}^{\infty} r_n < \infty$ and $\sum\limits_{n=1}^{\infty} s_n < \infty$, it follows that $\sum\limits_{n=1}^{\infty} \mu_n < \infty$ and $\sum\limits_{n=1}^{\infty} u_n < \infty$. Now (2.17) implies that

$$d^2(y_n,p) \le (1+u_n)d^2(z_n,p)+\beta_n\mu_n. \qquad (2.18)$$

Substituting (2.16) into (2.18) we see that

$$\begin{aligned}
d^2(y_n,p) &\le (1+u_n)[(1+u_n)d^2(x_n,p)+\rho_n] \\
&\quad +\beta_n\mu_n \\
&\le (1+u_n)^2 d^2(x_n,p)+(1+u_n)[\rho_n+\mu_n]. \qquad (2.19)
\end{aligned}$$

Again note that

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2(\alpha_n T^n y_n \oplus (1-\alpha_n)x_n,p) \\
&\le \alpha_n d^2(T^n y_n,p)+(1-\alpha_n)d^2(x_n,p) \\
&\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n,x_n) \\
&\le \alpha_n[(1+r_n)d(y_n,p)+s_n]^2+(1-\alpha_n)d^2(x_n,p) \\
&\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n,x_n) \\
&\le \alpha_n[(1+r_n)^2 d^2(y_n,p)+\nu_n]+(1-\alpha_n)d^2(x_n,p) \\
&\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n,x_n) \\
&= \alpha_n[(1+u_n)d^2(y_n,p)+\nu_n]+(1-\alpha_n)d^2(x_n,p) \\
&\quad -\alpha_n(1-\alpha_n)d(T^n y_n,x_n) \\
&\le (1+u_n)d^2(y_n,p)+\alpha_n\nu_n \\
&\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n,x_n) \\
&\le (1+u_n)d^2(y_n,p)+\nu_n \\
&\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n,x_n), \qquad (2.20)
\end{aligned}$$

where $\nu_n = 2(1+r_n)s_n d(y_n,p)+s_n^2$. By assumption $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, it follows that $\sum_{n=1}^{\infty} \nu_n < \infty$. Substituting (2.19) into (2.20) we get

$$\begin{aligned}
d^2(x_{n+1},p) &\leq (1+u_n)[(1+u_n)^2 d^2(x_n,p) + (1+u_n)(\rho_n+\mu_n)] \\
&\quad +\nu_n - \alpha_n(1-\alpha_n)d(T^n y_n, x_n) \\
&\leq (1+u_n)^3 d^2(x_n,p) + (1+u_n)^2(\rho_n+\mu_n+\nu_n) \\
&\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n) \\
&= (1+v_n)d^2(x_n,p) + (1+t_n)\theta_n \\
&\quad -\alpha_n(1-\alpha_n)d^2(T^n y_n, x_n), \qquad (2.21)
\end{aligned}$$

where $\theta_n = \rho_n+\mu_n+\nu_n$, $v_n = 3u_n+3u_n^2+u_n^3$ and $t_n = u_n^2+2u_n$. Since $\sum_{n=1}^{\infty} \rho_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$, it follows that $\sum_{n=1}^{\infty} \theta_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Observe that $\alpha_n(1-\alpha_n) \geq \delta^2$ and $\sum_{n=1}^{\infty} \theta_n < \infty$. For $m \geq 1$, (2.21) implies

$$\begin{aligned}
\sum_{n=1}^{m} d^2(T^n y_n, x_n) &\leq \frac{1}{\delta^2}\Big[d^2(x_1,p) - d^2(x_{m+1},p) + \sum_{n=1}^{m} v_n d^2(x_n,p) \\
&\quad + \sum_{n=1}^{m}\theta_n + \sum_{n=1}^{m} t_n\theta_n\Big] \\
&\leq \frac{1}{\delta^2}\Big[d^2(x_1,p) + R^2\sum_{n=1}^{m} v_n + \sum_{n=1}^{m}\theta_n \\
&\quad + \Big(\sum_{n=1}^{m}\theta_n^2\Big)^{1/2}\Big(\sum_{n=1}^{m} t_n^2\Big)^{1/2}\Big]. \qquad (2.22)
\end{aligned}$$

When $m \to \infty$, we have $\sum_{n=1}^{\infty} d^2(T^n y_n, x_n) < \infty$, since $\sum_{n=1}^{\infty} v_n < \infty$, $\sum_{n=1}^{\infty} \theta_n < \infty$, $\sum_{n=1}^{\infty} t_n < \infty$ and $d(x_n,p) \leq R \; \forall \; n$.

Hence

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0. \qquad (2.23)$$

Thus assertion (a) of the lemma is proved.

Now, we have

$$\begin{aligned}
d(x_n,p) &\leq d(x_n, T^n y_n) + d(T^n y_n, p) \\
&\leq d(x_n, T^n y_n) + (1+r_n)d(y_n,p) + s_n, \qquad (2.24)
\end{aligned}$$

from which we deduce that $a \leq \liminf_{n\to\infty} d(y_n, p)$. On the other hand, we have

$$\begin{aligned} d(y_n, p) &\leq \beta_n d(T^n z_n, p) + (1-\beta_n) d(x_n, p) \\ &\leq \beta_n [(1+r_n) d(z_n, p) + s_n] + (1-\beta_n) d(x_n, p) \\ &= \beta_n (1+r_n) d(z_n, p) + \beta_n s_n + (1-\beta_n) d(x_n, p), \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} d(z_n, p) &\leq \gamma_n d(T^n x_n, p) + (1-\gamma_n) d(x_n, p) \\ &\leq \gamma_n [(1+r_n) d(x_n, p) + s_n] + (1-\gamma_n) d(x_n, p) \\ &= \gamma_n (1+r_n) d(x_n, p) + \gamma_n s_n + (1-\gamma_n) d(x_n, p) \\ &\leq (1+r_n) d(x_n, p) + \gamma_n s_n. \end{aligned} \tag{2.26}$$

From (2.25) and (2.26), we have

$$\begin{aligned} d(y_n, p) &\leq \beta_n (1+r_n)[(1+r_n) d(x_n, p) + \gamma_n s_n] + \beta_n s_n \\ &\quad +(1-\beta_n) d(x_n, p) \\ &\leq (1+r_n)^2 d(x_n, p) + \beta_n s_n (1+r_n)(1+\gamma_n) \\ &\leq (1+r_n)^2 d(x_n, p) + 2\beta_n s_n (1+r_n), \end{aligned} \tag{2.27}$$

which implies $\limsup_{n\to\infty} d(y_n, p) \leq a$. Therefore, $\lim_{n\to\infty} d(y_n, p) = a$. Again consider (2.17) and using (2.16), we have

$$\begin{aligned} d^2(y_n, p) &\leq (1+u_n) d^2(z_n, p) + \beta_n \mu_n - \beta_n (1-\beta_n) d^2(T^n z_n, x_n) \\ &\leq (1+u_n)[(1+u_n) d^2(x_n, p) + \rho_n] + \beta_n \mu_n \\ &\quad -\beta_n (1-\beta_n) d^2(T^n z_n, x_n) \\ &\leq (1+t_n) d^2(x_n, p) + (1+u_n)(\rho_n + \beta_n \mu_n) \\ &\quad -\beta_n (1-\beta_n) d^2(T^n z_n, x_n) \\ &\leq (1+t_n) d^2(x_n, p) + (1+u_n)(\rho_n + \mu_n) \\ &\quad -\beta_n (1-\beta_n) d^2(T^n z_n, x_n) \\ &= (1+t_n) d^2(x_n, p) + \tau_n \\ &\quad -\beta_n (1-\beta_n) d^2(T^n z_n, x_n), \end{aligned} \tag{2.28}$$

where $t_n = u_n^2 + 2u_n$ and $\tau_n = (1+u_n)(\rho_n + \mu_n)$. Since $\sum_{n=1}^{\infty} u_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$, it follows that $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. From the assumption of the theorem, we have $\beta_n(1-\beta_n) \geq \delta^2$, $\sum_{n=1}^{\infty} \tau_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$.

For $m \geq 1$, (2.28) implies

$$\begin{aligned}\sum_{n=1}^{m} d^2(T^n z_n, x_n) &\leq \frac{1}{\delta^2}\Big[\sum_{n=1}^{m} t_n d^2(x_n, p) + \sum_{n=1}^{m} \tau_n\Big] \\ &\leq \frac{1}{\delta^2}\Big[R^2 \sum_{n=1}^{m} t_n + \sum_{n=1}^{m} \tau_n\Big]. \qquad (2.29)\end{aligned}$$

When $m \to \infty$, we have $\sum_{n=1}^{\infty} d^2(T^n z_n, x_n) < \infty$, since $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \tau_n < \infty$ and $d(x_n, p) \leq R \; \forall \; n$.

Hence

$$\lim_{n\to\infty} d(T^n z_n, x_n) = 0. \qquad (2.30)$$

Thus assertion (b) of the lemma is proved.

This completes the proof. □

Lemma 2.6. *Let (X, d) be a complete CAT(0) space, and C be a nonempty closed convex subset of X. Let $T: C \to C$ be a uniformly 1-Lipschitzian generalized asymptotically quasi-nonexpansive mapping with $\{r_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the Noor iteration sequence defined by (2.1). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Then $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.*

Proof. From Lemma 2.5, we have

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0 \;\text{ and }\; \lim_{n\to\infty} d(T^n z_n, x_n) = 0. \qquad (2.31)$$

Thus

$$d(T^n x_n, x_n) \leq d(T^n x_n, T^n y_n) + d(T^n y_n, x_n). \qquad (2.32)$$

Since T is uniformly 1-Lipschitzian and $d(x_n, y_n) \to 0$ as $n \to \infty$, it follows from (2.32) that

$$\lim_{n\to\infty} d(T^n x_n, x_n) = 0. \qquad (2.33)$$

Again since T is uniformly 1-Lipschitzian and $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$, we have

$$\begin{aligned} d(x_{n+1}, T^n x_{n+1}) &\leq d(x_{n+1}, x_n) + d(T^n x_n, x_n) + d(T^n x_{n+1}, T^n x_n) \\ &\leq d(x_{n+1}, x_n) + d(T^n x_n, x_n) + d(x_{n+1}, x_n) \\ &= 2d(x_{n+1}, x_n) + d(T^n x_n, x_n) \\ &\to 0 \text{ as } n \to \infty. \qquad (2.34)\end{aligned}$$

Since T is uniformly 1-Lipschitzian, from (2.33) and (2.34), we get

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &\leq d(x_{n+1}, T^{n+1} x_{n+1}) + d(T^{n+1} x_{n+1}, Tx_{n+1}) \\ &\leq d(x_{n+1}, T^{n+1} x_{n+1}) + d(T^n x_{n+1}, x_{n+1}) \\ &\to 0 \text{ as } n \to \infty, \qquad (2.35)\end{aligned}$$

which implies

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{2.36}$$

This completes the proof. □

Theorem 2.7. *Let (X, d) be a complete CAT(0) space, and C be a nonempty closed convex subset of X. Let $T\colon C \to C$ be a uniformly 1-Lipschitzian generalized asymptotically quasi-nonexpansive mapping with $\{r_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the Noor iteration sequence defined by (2.1). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Assume, in addition that T is compact. Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

Proof. By Lemmas 2.5 and 2.6, we have

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0, \quad \lim_{n\to\infty} d(T^n z_n, x_n) = 0, \tag{2.37}$$

and

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{2.38}$$

Again by Theorem 2.1, $\{x_n\}$ is bounded. It follows by our assumption that T is compact, then there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to x^* \in C$ as $k \to \infty$. Moreover, by (2.38), we have $d(Tx_{n_k}, x_{n_k}) \to 0$ which implies that $x_{n_k} \to x^*$ as $k \to \infty$. By (2.38) again, we have

$$d(x^*, Tx^*) = \lim_{k\to\infty} d(Tx_{n_k}, x_{n_k}) = 0. \tag{2.39}$$

It shows that $x^* \in F(T)$. Furthermore, since $\lim_{n\to\infty} d(x_n, x^*)$ exists, therefore $\lim_{n\to\infty} d(x_n, x^*) = 0$, that is, $\{x_n\}$ converges to some fixed point of T. Now, using (2.33) and (2.37) we have

$$d(y_n, x_n) \leq \beta_n d(T^n z_n, x_n) \to 0, \text{ as } n \to \infty, \tag{2.40}$$

and

$$d(z_n, x_n) \leq \gamma_n d(T^n x_n, x_n) \to 0, \text{ as } n \to \infty. \tag{2.41}$$

Therefore $\lim_{n\to\infty} y_n = x^* = \lim_{n\to\infty} z_n$. Thus $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T. This completes the proof. □

From Theorem 2.7, we obtain the following results.

Theorem 2.8. *Let (X, d) be a complete CAT(0) space, and let C be a nonempty closed convex subset of X. Let $T\colon C \to C$ be a uniformly 1-Lipschitzian generalized asymptotically quasi-nonexpansive mapping with $\{r_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be* *real sequences in* $[\delta, 1-\delta]$ *for some* $\delta \in (0,1)$*. For a given* $x_1 \in C$*, define*

$$\begin{aligned} y_n &= \beta_n T^n x_n \oplus (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1-\alpha_n)x_n, \quad n \geq 1. \end{aligned} \tag{2.42}$$

Assume, in addition that T *is compact. Then* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Theorem 2.9. *Let* (X,d) *be a complete* $CAT(0)$ *space, and* C *be a nonempty closed convex subset of* X*. Let* $T\colon C \to C$ *be a uniformly* 1*-Lipschitzian generalized asymptotically quasi-nonexpansive mapping with* $\{r_n\}, \{s_n\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} r_n < \infty$ *and* $\sum_{n=1}^{\infty} s_n < \infty$*. Suppose that* $F(T) \neq \emptyset$*. Let* $\{\alpha_n\}$ *be a real sequence in* $[\delta, 1-\delta]$ *for some* $\delta \in (0,1)$*. For a given* $x_1 \in C$*, define*

$$x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n)x_n, \quad n \geq 1. \tag{2.43}$$

Assume, in addition that T *is compact. Then* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Remark 2.10. Our results extend and improve the corresponding results of Niwongsa and Panyanak [20] to the case of more general class of asymptotically nonexpansive mappings considered in this paper.

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