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FURTHER RESULTS ON CONVEX FUNCTIONS AND SEPARABLE SEQUENCES WITH APPLICATIONS

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Abstract. In this paper, generalizations are given for some recent results of Niezgoda [M. Niezgoda, Remarks on convex functions and separable sequences, Discrete Math. 308 (2008) 1765-1773]. As applications, two mean value theorems are derived. Gram type inequality is proved. Exponential convexity for differences of power means is shown. Monotonicity of Cauchy type means is shown.

1. Preliminaries and summary

We start with some notation and definitions, quoted from [7, 8].

Throughout this paper, inner product on $\mathbb{R}^n$ is defined by

$$\langle a, b\rangle = \sum_{k=1}^{n} a_k b_k p_k \quad \text{for } a = (a_1, \ldots, a_n) \text{ and } b = (b_1, \ldots, b_n), \tag{1}$$

where $p_1, \ldots, p_n$ are positive numbers. We assume that $e = \{e_1, \ldots, e_n\}$ is a basis in $\mathbb{R}^n$, and $d = \{d_1, \ldots, d_n\}$ is the dual basis of e, that is $\langle e_i, d_j\rangle = \delta_{ij}$ (Kronecker delta).

We say that a vector $v \in \mathbb{R}^n$ is *e-positive* if $\langle e_i, v\rangle > 0$ for all $i = 1, \ldots, n$.

We denote $J = \{1, \ldots, n\}$. Let J_1 and J_2 be two sets of indices such that $J_1 \cup J_2 = J$. Let $v \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$. A vector $z \in \mathbb{R}^n$ is said to be *μ, v-separable* on J_1 and J_2 (with respect to the basis e), if

$$\langle e_i, z - \mu v\rangle \geq 0 \ \text{ for } i \in J_1, \ \text{ and } \ \langle e_j, z - \mu v\rangle \leq 0 \ \text{ for } j \in J_2 \tag{2}$$

(see [7]). If v is e-positive, then z is μ, v-separable on J_1 and J_2 w.r.t. e if and only if

$$\max_{j \in J_2} \frac{\langle e_j, z\rangle}{\langle e_j, v\rangle} \leq \mu \leq \min_{i \in J_1} \frac{\langle e_i, z\rangle}{\langle e_i, v\rangle}. \tag{3}$$

A vector $z \in \mathbb{R}^n$ is said to be *v-separable* on J_1 and J_2 (w.r.t. e), if z is μ, v-separable on J_1 and J_2 for some μ. By (3), z is v-separable on J_1 and J_2 w.r.t. e

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if and only if

$$\max_{j\in J_2}\frac{\langle e_j,z\rangle}{\langle e_j,v\rangle}\le\min_{i\in J_1}\frac{\langle e_i,z\rangle}{\langle e_i,v\rangle}\qquad\text{(provided } v \text{ is } e\text{-positive)}.\tag{4}$$

We say that a function $\varphi : I\subset\mathbb{R}\to\mathbb{R}$ *preserves v-separability* on J_1 and J_2 w.r.t. e, if $(\varphi(z_1),\varphi(z_2),\dots,\varphi(z_n))$ is v-separable on J_1 and J_2 w.r.t. e for each $z=(z_1,z_2,\dots,z_n)\in I^n$ such that z is v-separable on J_1 and J_2 w.r.t. e.

It is worth emphasising that depending on the choice of vector v and basis e, the class of separable vectors in $\mathbb{R}^n$ embraces, among others, monotone vectors, monotone in mean vectors, star-shaped vectors and convex vectors (see the discussions after each of Corollaries 2.4-2.7).

A vector $y\in\mathbb{R}^n$ is said to be *majorised* by $x\in\mathbb{R}^n$ (in symbol, $y\prec x$), if the sum of m largest entries of y does not exceed the sum of m largest entries of x for all $m=1,2,\dots,n$ with equality for $m=n$ [6, p. 7]. It is well known that

$$y\prec x\quad\text{if and only if}\quad\sum_{k=1}^n f(y_k)\le\sum_{k=1}^n f(x_k)$$

for all continuous convex functions $f : I\subset\mathbb{R}\to\mathbb{R}$ such that $x_k,y_k\in I$, $k=1,\dots,n$ [6, p. 108].

In [8] the following majorisation type theorem has been proved (cf. [4, 5]).

Theorem 1.1. (See [8, Theorem 2.2]). *Let $f : I\to\mathbb{R}$ be a convex function on an open interval $I\subset\mathbb{R}$. Assume $\varphi\in\partial f$, where ∂f is the subdifferential of f.*

Let $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ and $p=(p_1,\dots,p_n)$, where $x_i,y_i\in I$, $p_i>0$ for $i\in J=\{1,\dots,n\}$, and let $w,v\in\mathbb{R}^n$ with $\langle w,v\rangle>0$.

If there exist index sets J_1 and J_2 with $J_1\cup J_2=J$ such that

(i) *y is v-separable on J_1 and J_2 w.r.t. e,*
(ii) *$x-y$ is λ,w-separable on J_1 and J_2 w.r.t. d, where $\lambda=\langle x-y,v\rangle/\langle w,v\rangle$,*
(iii) *$\langle x-y,v\rangle=0$, or $\langle x-y,v\rangle\langle z,w\rangle\ge 0$, where $z=(\varphi(y_1),\dots,\varphi(y_n))$,*
(iv) *φ preserves v-separability on J_1 and J_2 w.r.t. e,*

then

$$\sum_{k=1}^n p_k f(y_k)\le\sum_{k=1}^n p_k f(x_k).\tag{5}$$

Remark 1.2. Theorem 1.1 remains valid for arbitrary interval $I\subset\mathbb{R}$ whenever f and φ are continuous on I (e.g., $f\in C^1(I)$).

Remark 1.3. It is not hard to check that the quadratic function $f(t):=t^2$, $t\in I$, satisfies condition (iv). So, it follows from Theorem 1.1 that

$$\sum_{k=1}^n p_k y_k^2\le\sum_{k=1}^n p_k x_k^2,\tag{6}$$

provided x,y,p,w,v satisfy the above conditions (i)-(ii) and (iii) for $z=2y$.

Remark 1.4. For some bases e and d and vectors w and v in $\mathbb{R}^n$ (see Corollaries 2.4 and 2.5), condition (iv) is satisfied automatically, since $\varphi \in \partial f$ is nondecreasing by the convexity of f. In such cases, (iv) can be dropped from Theorem 1.1.

In this paper, we give some extensions of Theorem 1.1 (see Theorem 2.1 and Corollaries 2.4, 2.5, 2.6 and 2.7). As applications, we derive some mean value theorems (see Theorems 3.1 and 3.2). This allows to introduce a class of Cauchy type means (see Section 3). In Section 4, we derive a Gram type inequality and prove exponential convexity for differences of power means. Finally, we prove monotonicity of Cauchy type means (see Section 5).

2. Refinements for twice differentiable functions

We now give a refinement of Theorem 1.1 for twice differentiable functions (not necessarily convex) (cf. [4]).

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on an open interval I. Assume that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with the property that*

$$\gamma \leq f''(t) \leq \Gamma \quad \text{for all } t \in I. \tag{7}$$

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $p = (p_1, \ldots, p_n)$, where $x_i, y_i \in I$, $p_i > 0$ for $i \in J = \{1, \ldots, n\}$, and let $w, v \in \mathbb{R}^n$ with $\langle w, v\rangle > 0$.

If there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that

(i) *y is v-separable on J_1 and J_2 w.r.t. e,*

(ii) *$x - y$ is λ, w-separable on J_1 and J_2 w.r.t. d, where $\lambda = \langle x - y, v\rangle / \langle w, v\rangle$,*

(iii') *$\langle x - y, v\rangle = 0$, or $\langle x - y, v\rangle\langle z, w\rangle \geq 0$ for*

$$z = (\varphi_\gamma(y_1), \ldots, \varphi_\gamma(y_n)) \quad \text{and} \quad z = (\varphi_\Gamma(y_1), \ldots, \varphi_\Gamma(y_n)), \tag{8}$$

where

$$\varphi_\gamma(t) := f'(t) - \gamma t \quad \text{and} \quad \varphi_\Gamma(t) := \Gamma t - f'(t), \quad t \in I, \tag{9}$$

(iv') *φ_γ and φ_Γ preserve v-separability on J_1 and J_2 w.r.t. e,*

then

$$\frac{1}{2}\gamma \sum_{k=1}^{n} p_k (x_k^2 - y_k^2) \leq \sum_{k=1}^{n} p_k f(x_k) - \sum_{k=1}^{n} p_k f(y_k) \leq \frac{1}{2}\Gamma \sum_{k=1}^{n} p_k (x_k^2 - y_k^2). \tag{10}$$

Proof. Similarly as in the proof of [4, Proposition 1], it is sufficient to apply Theorem 1.1 and (5) to the convex functions $f_\gamma(t) := f(t) - \frac{1}{2}\gamma t^2$ and $f_\Gamma(t) := \frac{1}{2}\Gamma t^2 - f(t)$, $t \in I$. □

Remark 2.2. Theorem 2.1 remains valid for an arbitrary interval I whenever f and f' are defined and continuous on I.

Remark 2.3. For some bases e and d and vectors w and v (see Corollaries 2.4 and 2.5), condition (iv') holds automatically, since the functions φ_γ and φ_Γ are nondecreasing by (7).

In the rest of this section, we demonstrate special cases of Theorem 2.1 for various vectors w and v and bases e and d in $\mathbb{R}^n$. This leads to generalizations of [8, Corollaries 2.3, 2.6, 2.10, 2.11].

Corollary 2.4. *Under the assumptions of Theorem 2.1, let $w = v = (1, \ldots, 1)$ and let $e = d$ be the basis in $\mathbb{R}^n$ (orthonormal w.r.t. inner product (1)) given by*

$$e_i = d_i = \frac{1}{\sqrt{p_i}}(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, 0, \ldots, 0), \quad i = 1, \ldots, n. \tag{11}$$

Denote

$$\lambda = \langle x - y, v\rangle / \langle w, v\rangle = \frac{1}{P_n}\sum_{k=1}^{n}(x_k - y_k)p_k, \quad \text{where } P_n = \sum_{k=1}^{n} p_k. \tag{12}$$

If there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that

(i) *y is v-separable on J_1 and J_2 w.r.t. e, i.e.,*

$$y_j \leq y_i \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{13}$$

(ii) *$x - y$ is λ, w-separable on J_1 and J_2 w.r.t. $d = e$, i.e.,*

$$x_j - y_j \leq \lambda \leq x_i - y_i \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{14}$$

(iii') *$\langle x - y, v\rangle = 0$, or $\langle x - y, v\rangle\langle z, v\rangle \geq 0$ where z and φ_γ and φ_Γ are defined by (8)-(9),*

then (10) holds.

Proof. It is sufficient to show that condition (iv') in Theorem 2.1 is fulfilled.

Since $f_\gamma(t) := f(t) - \frac{1}{2}\gamma t^2$, $t \in I$, is a convex function (see (7)), $\varphi_\gamma(t) = f'_\gamma(t)$ is a nondecreasing function. If $a = (a_1, \ldots, a_n)$ is a v-separable vector on J_1 and J_2 w.r.t. e, then $a_j \leq a_i$ for $i \in J_1$ and $j \in J_2$ (see (4), (1) and (11)). Consequently,

$$\varphi_\gamma(a_j) \leq \varphi_\gamma(a_i) \quad \text{for } i \in J_1 \text{ and } j \in J_2.$$

Therefore the vector $(\varphi_\gamma(a_1), \ldots, \varphi_\gamma(a_n))$ is v-separable on J_1 and J_2 w.r.t. e. Thus φ_γ preserves v-separability on J_1 and J_2.

In a similar way it can be proved that φ_Γ preserves v-separability on J_1 and J_2 w.r.t. e.

In summary, condition (iv') is satisfied, as required. □

Observe that conditions (13)-(14) are satisfied for

$$J_1 = \{1, 2, \ldots, m\} \text{ and } J_2 = \{m + 1, \ldots, n\}$$

for some $m \in J$, if both y and $x - y$ are monotonic nonincreasing vectors, i.e.,

$$y_1 \geq \ldots \geq y_n \quad \text{and} \quad x_1 - y_1 \geq \ldots \geq x_n - y_n.$$

Corollary 2.5. *Under the assumptions of Theorem 2.1, let $w = v = (1, \ldots, 1)$ and let λ be as in (12). Suppose that e is the basis in $\mathbb{R}^n$ consisting of the vectors*

$$e_i = (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, \frac{1}{p_i}, -\frac{1}{p_{i+1}}, 0, \ldots, 0), \quad i = 1, \ldots, n-1, \quad \text{and} \tag{15}$$

$$e_n = (0, \ldots, 0, \frac{1}{p_n}). \tag{16}$$

Let d be the dual basis of e, that is

$$d_i = (\underbrace{1, \ldots, 1}_{i \ times}, 0, \ldots, 0), \quad i = 1, \ldots, n. \tag{17}$$

If there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that

(i) *y is v-separable on J_1 and J_2 w.r.t. e, i.e., there exists $\mu \in \mathbb{R}$ satisfying*

$$y_j - y_{j+1} \leq 0 \leq y_i - y_{i+1} \quad \text{for } i \in J_1 \text{ and } j \in J_2 \tag{18}$$

with the convention $y_{n+1} = \mu$,

(ii) *$x - y$ is λ, w-separable on J_1 and J_2 w.r.t. d, i.e.,*

$$\frac{1}{P_j} \sum_{k=1}^{j} (x_k - y_k) p_k \leq \lambda \leq \frac{1}{P_i} \sum_{k=1}^{i} (x_k - y_k) p_k \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{19}$$

where $P_l = \sum_{k=1}^{l} p_k$ for $l = 1, 2, \ldots, n$,

(iii') *$\langle x - y, v\rangle = 0$, or $\langle x - y, v\rangle \langle z, v\rangle \geq 0$ where z and φ_γ and φ_Γ are defined by (8)-(9),*

then (10) holds.

Proof. It is not hard to check that condition (iv') of Theorem 2.1 is met (see the proof of Corollary 2.4). Now, Corollary 2.5 follows from Theorem 2.1. □

If y is monotonic nondecreasing, i.e., $y_1 \leq y_2 \leq \ldots \leq y_n$, and $x - y$ is *monotonic nondecreasing in P-mean* [11, p. 318], i.e.,

$$\frac{1}{P_l} \sum_{k=1}^{l} (x_k - y_k) p_k \leq \frac{1}{P_{l+1}} \sum_{k=1}^{l+1} (x_k - y_k) p_k \ , \quad l = 1, 2, \ldots, n-1, \tag{20}$$

then conditions (18)-(19) are satisfied for

$$J_1 = \{n\} \text{ and } J_2 = \{1, 2, \ldots, n-1\}.$$

Moreover, (20) can be replaced by

$$\frac{1}{P_l} \sum_{k=1}^{l} (x_k - y_k) p_k \leq \frac{1}{P_n} \sum_{k=1}^{n} (x_k - y_k) p_k \ , \quad l = 1, 2, \ldots, n-1.$$

Corollary 2.6. *Under the assumptions of Theorem 2.1, let $w = v = (1, 2, \ldots, n)$ and let $e = d$ be the basis in $\mathbb{R}^n$ given by (11). Denote*

$$\lambda = \langle x - y, v\rangle / \langle w, v\rangle = \frac{1}{\widetilde{P_n}} \sum_{k=1}^{n} (x_k - y_k) k p_k \ , \quad \text{where } \widetilde{P_n} = \sum_{k=1}^{n} k^2 p_k. \tag{21}$$

If there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that

(i) *y is v-separable on J_1 and J_2 w.r.t. e, i.e.,*

$$\frac{y_j}{j} \leq \frac{y_i}{i} \quad \textit{for } i \in J_1 \textit{ and } j \in J_2, \tag{22}$$

(ii) *$x - y$ is λ, w-separable on J_1 and J_2 w.r.t. $d = e$, i.e.,*

$$\frac{x_j - y_j}{j} \leq \lambda \leq \frac{x_i - y_i}{i} \quad \textit{for } i \in J_1 \textit{ and } j \in J_2, \tag{23}$$

(iii') *$\langle x - y, v\rangle = 0$, or $\langle x - y, v\rangle\langle z, v\rangle \geq 0$ where z and φ_γ and φ_Γ are defined by* (8)-(9),

(iv') *φ_γ and φ_Γ preserve v-separability on J_1 and J_2 w.r.t. e, i.e.,* (22) *implies*

$$\frac{\varphi_\gamma(y_j)}{j} \leq \frac{\varphi_\gamma(y_i)}{i} \quad \textit{and} \quad \frac{\varphi_\Gamma(y_j)}{j} \leq \frac{\varphi_\Gamma(y_i)}{i} \quad \textit{for } i \in J_1 \textit{ and } j \in J_2, \tag{24}$$

then (10) *holds.*

Proof. Apply Theorem 2.1. □

A vector $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is said to be *star-shaped* [11, p. 318], if

$$\tfrac{y_l}{l} \leq \tfrac{y_{l+1}}{l+1} \text{ for } l = 1, 2, \ldots, n-1. \tag{25}$$

A function $\varphi : I \to \mathbb{R}$, $t \in I$, where $I \subset \mathbb{R}_+$, is said to be *star-shaped*, if the function $t \to \frac{\varphi(t)}{t}$ is nondecreasing.

It has been proved in [8] that if $\varphi : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable nondecreasing convex and star-shaped function on an open interval I, then φ preserves star-shapeness of vectors, i.e., (25) implies

$$\frac{\varphi(y_l)}{l} \leq \frac{\varphi(y_{l+1})}{l+1} \quad \text{for } l = 1, 2, \ldots, n-1. \tag{26}$$

If y and $x - y$ are star-shaped vectors, and φ_γ and φ_Γ preserve star-shaped vectors, then conditions (22)-(24) are satisfied for the index sets

$$J_1 = \{m+1, \ldots, n\} \quad \text{and} \quad J_2 = \{1, 2, \ldots, m\}$$

for some m.

Corollary 2.7. *Under the assumptions of Theorem 2.1, let $w = v = (1, 2, \ldots, n)$ and let λ be as in* (21). *Assume that e and d are the bases in $\mathbb{R}^n$ defined by* (15)-(17).

If there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that

(i) *y is v-separable on J_1 and J_2 w.r.t. e, i.e., there exists $\mu \in \mathbb{R}$ satisfying*

$$y_{j+1} - y_j \geq \mu \geq y_{i+1} - y_i \quad \textit{for } i \in J_1 \textit{ and } j \in J_2 \tag{27}$$

with the convention $y_{n+1} = \mu(n+1)$,

(ii) *$x - y$ is λ, w-separable on J_1 and J_2 w.r.t. d, i.e.,*

$$\frac{1}{\widehat{P}_j} \sum_{k=1}^{j} (x_k - y_k) p_k \leq \lambda \leq \frac{1}{\widehat{P}_i} \sum_{k=1}^{i} (x_k - y_k) p_k \quad \textit{for } i \in J_1 \textit{ and } j \in J_2, \tag{28}$$

where $\widehat{P}_l = \sum\limits_{k=1}^{l} k p_k$, $l = 1, \ldots, n$, (iii') $\langle x-y, v\rangle = 0$, *or* $\langle x-y, v\rangle\langle z, v\rangle \geq 0$ *where* z *and* φ_γ *and* φ_Γ *are defined by* (8)-(9),

(iv') φ_γ *and* φ_Γ *preserve v-separability on* J_1 *and* J_2 *w.r.t.* e, *i.e.,* (27) *implies that there exist* $\nu, \rho \in \mathbb{R}$ *satisfying*

$$\varphi_\gamma(y_{j+1}) - \varphi_\gamma(y_j) \geq \nu \geq \varphi_\gamma(y_{i+1}) - \varphi_\gamma(y_i) \quad \textit{for } i \in J_1 \textit{ and } j \in J_2, \tag{29}$$

$$\varphi_\Gamma(y_{j+1}) - \varphi_\Gamma(y_j) \geq \rho \geq \varphi_\Gamma(y_{i+1}) - \varphi_\Gamma(y_i) \quad \textit{for } i \in J_1 \textit{ and } j \in J_2 \tag{30}$$

with the convention $\varphi_\gamma(y_{n+1}) = \nu(n+1)$ *and* $\varphi_\Gamma(y_{n+1}) = \rho(n+1)$,

then (10) *holds.*

Proof. Use Theorem 2.1. □

A vector $y = (y_1, \ldots, y_n)$ is said to be *convex* [11, p. 318], if

$$y_2 - y_1 \leq y_3 - y_2 \leq \ldots \leq y_n - y_{n-1}. \tag{31}$$

Equivalently, (31) says that

$$y_l \leq \frac{y_{l-1} + y_{l+1}}{2} \quad \text{for } l = 2, \ldots, n-1. \tag{32}$$

In consequence, a function $\varphi : I \to \mathbb{R}$ preserves convex vectors if (32) implies

$$\varphi(y_l) \leq \frac{\varphi(y_{l-1}) + \varphi(y_{l+1})}{2} \quad \text{for } l = 2, \ldots, n-1. \tag{33}$$

For instance, if φ is nondecreasing and convex, then (33) is met.

Conditions (27)-(30) are fulfilled for the index sets

$$J_1 = \{1, 2, \ldots, m\} \text{ and } J_2 = \{m+1, \ldots, n\}$$

for some m depending on λ, whenever φ_γ and φ_Γ are nondecreasing convex functions with $\varphi_\gamma(0) = 0$ and $\varphi_\Gamma(0) = 0$, and $x - y$ is *monotonic nonincreasing in* $\widehat{P}$*-mean*, i.e.,

$$\frac{1}{\widehat{P}_l} \sum_{k=1}^{l} (x_k - y_k) p_k \geq \frac{1}{\widehat{P}_{l+1}} \sum_{k=1}^{l+1} (x_k - y_k) p_k \quad \text{for } l = 1, 2, \ldots, n-1,$$

and, in addition, $y = (y_1, \ldots, y_n)$ is a decreasing convex vector such that $y_1 \leq n(y_2 - y_1)$ (e.g., $y = -(n+1, n+2, \ldots, 2n)$).

3. Mean value theorems

We are now in a position to give a mean value theorem.

Theorem 3.1. *Let* $f \in C^2(I)$, *where* I *is a closed interval in* $\mathbb{R}$, *and let* $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ *and* $p = (p_1, \ldots, p_n)$, *where* $x_i, y_i \in I$, $p_i > 0$ *for* $i \in J = \{1, \ldots, n\}$, *and* $w, v \in \mathbb{R}^n$ *with* $\langle w, v\rangle > 0$.

Suppose that x, y, p, w, v *satisfy conditions (i)- (iii) from Theorem 1.1, where* $z = 2y$ *and conditions (iii')- (iv') from Theorem 2.1, where* $\gamma := \min_{t \in I} f''(t)$ *and* $\Gamma := \max_{t \in I} f''(t)$.

Then there exists $\xi \in I$ such that

$$\sum_{k=1}^{n} p_k \left[f(y_k) - f(x_k)\right] = \frac{f''(\xi)}{2} \sum_{k=1}^{n} p_k \left(y_k^2 - x_k^2\right). \tag{34}$$

Proof. By Theorem 2.1 and Remark 2.2, we conclude that (10) holds. If

$$\sum_{k=1}^{n} p_k \left(x_k^2 - y_k^2\right) = 0$$

then (10) gives

$$\sum_{k=1}^{n} p_k \left[f(x_k) - f(y_k)\right] = 0.$$

Taking any number ξ in I we obtain (34).

Let us consider the case when

$$\sum_{k=1}^{n} p_k \left(x_k^2 - y_k^2\right) \neq 0.$$

Applying Remark 1.3 gives

$$\sum_{k=1}^{n} p_k \left(x_k^2 - y_k^2\right) > 0.$$

Now by Theorem 2.1 we have

$$\frac{1}{2}\gamma \sum_{k=1}^{n} p_k \left(x_k^2 - y_k^2\right) \leq \sum_{k=1}^{n} p_k \left[f(x_k) - f(y_k)\right] \leq \frac{1}{2}\Gamma \sum_{k=1}^{n} p_k \left(x_k^2 - y_k^2\right).$$

In consequence, we obtain

$$\gamma \leq \frac{2\sum_{k=1}^{n} p_k \left[f(x_k) - f(y_k)\right]}{\sum_{k=1}^{n} p_k \left(x_k^2 - y_k^2\right)} \leq \Gamma.$$

Making use of the fact that for each $\rho \in [\gamma, \Gamma]$ there exists $\xi \in I$ such that $f''(\xi) = \rho$, we get (34). □

Theorem 3.2. *Let $f, g \in C^2(I)$, where I is a closed interval in $\mathbb{R}$, and let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $p = (p_1, \ldots, p_n)$, where $x_i, y_i \in I$, $p_i > 0$ for $i \in J = \{1, \ldots, n\}$, and $w, v \in \mathbb{R}^n$ with $\langle w, v\rangle > 0$, $\langle x - y, v\rangle = 0$ and $\sum_{k=1}^{n} p_k \left(y_k^2 - x_k^2\right) \neq 0$.*

Suppose that x, y, p, w, v satisfy conditions (i)- (ii) from Theorem 2.1 for some index sets J_1 and J_2 ($J_1 \cup J_2 = J$), and all nondecreasing functions defined on I preserve v-separability on J_1 and J_2 w.r.t. e.

Then there exists $\xi \in I$ such that

$$\frac{f''(\xi)}{g''(\xi)} = \frac{\sum_{k=1}^{n} p_k \left[f(y_k) - f(x_k)\right]}{\sum_{k=1}^{n} p_k \left[g(y_k) - g(x_k)\right]}, \tag{35}$$

provided that the denominators are non-zero.

Proof. Define

$$h := c_1 f - c_2 g,$$

where

$$c_1 := \sum_{k=1}^{n} p_k \left[g(y_k) - g(x_k)\right] \quad \text{and} \quad c_2 := \sum_{k=1}^{n} p_k \left[f(y_k) - f(x_k)\right]. \tag{36}$$

Clearly, $h \in C^2(I)$.

Observe that the assumptions of Theorem 2.1 are fulfilled for the function h in place of f. Using (34) we obtain

$$\sum_{k=1}^{n} p_k \left[h(y_k) - h(x_k)\right] = \frac{h''(\xi)}{2} \sum_{k=1}^{n} p_k \left(y_k^2 - x_k^2\right) \tag{37}$$

for some $\xi \in I$. It is easy to verify that $\sum_{k=1}^{n} p_k \left[h(y_k) - h(x_k)\right] = 0$. Therefore (37) reduces to

$$0 = \left(\frac{c_1 f''(\xi)}{2} - \frac{c_2 g''(\xi)}{2}\right) \sum_{k=1}^{n} p_k \left(y_k^2 - x_k^2\right), \tag{38}$$

which gives

$$\frac{c_2}{c_1} = \frac{f''(\xi)}{g''(\xi)}.$$

Combining this with (36) implies (35). □

Corollary 3.3. *Under the assumptions of Theorem 3.2, set $f(x) = x^a$ and $g(x) = x^b$, for $b \neq a \neq 0, 1$, with $I \subset \mathbb{R}_+^n$.*

Then there exists $\xi \in I$ such that

$$\xi^{a-b} = \frac{b(b-1) \sum_{k=1}^{n} p_k \left(y_k^a - x_k^a\right)}{a(a-1) \sum_{k=1}^{n} p_k \left(y_k^b - x_k^b\right)}. \tag{39}$$

Proof. Apply Theorem 3.2. □

Remark 3.4. Since the function $\xi \to \xi^{a-b}$, $b \neq a \neq 0, 1$, is invertible, then from (39) we have

$$\min_k \{x_k, y_k\} \leq \left\{\frac{b(b-1) \sum_{k=1}^{n} p_k \left(y_k^a - x_k^a\right)}{a(a-1) \sum_{k=1}^{n} p_k \left(y_k^b - x_k^b\right)}\right\}^{\frac{1}{a-b}} \leq \max_k \{x_k, y_k\} \tag{40}$$

with $I = [\min_k \{x_k, y_k\}, \max_k \{x_k, y_k\}]$.

In fact, similar result can also be given for (35). Namely, suppose that $\frac{f''}{g''}$ has the inverse function. Then from (35) we have

$$\xi = \left(\frac{f''}{g''}\right)^{-1} \left(\frac{\sum_{k=1}^{n} p_k \left[f(y_k) - f(x_k)\right]}{\sum_{k=1}^{n} p_k \left[g(y_k) - g(x_k)\right]}\right). \tag{41}$$

So, the expression on the right hand side of (41) is a mean (see Section 5).

4. Exponential convexity and Gram's inequality

In this section we develop some ideas given in [3].

A continuous function $h : (a, b) \to \mathbb{R}$ is said to be *exponentially convex* if

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j h(x_i + x_j) \geq 0$$

for all $n \in \mathbb{N}$ and $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$, and x_i, x_j such that $x_i + x_j \in (a, b)$, $i, j = 1, \ldots, n$.

Equivalently, a continuous function $h : (a, b) \to \mathbb{R}$ is exponentially convex if and only if

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j h\Big(\frac{x_i + x_j}{2}\Big) \geq 0 \tag{42}$$

for all $\alpha_i \in \mathbb{R}$ and $x_i, x_j \in (a, b)$, $i = 1, \ldots, n$ (see [3]).

It is known (see [3]) that each exponentially convex positive function h : $(a, b) \to (0, \infty)$ is log-convex:

$$h^2\Big(\frac{x_i + x_j}{2}\Big) \leq h(x_i)h(x_j) \quad \text{for } x_i, x_j \in (a, b). \tag{43}$$

Let us define the function

$$\varphi_s(u) := \begin{cases} \frac{u^s}{s(s-1)}\ , & s \neq 0, 1; \\ -\log u\ , & s = 0; \\ u \log u\ , & s = 1 \end{cases} \qquad \text{for } u > 0. \tag{44}$$

It is easily seen that $\frac{d^2 \varphi_s(u)}{du^2} = u^{s-2}$ for $u > 0$, that is the function $u \mapsto \varphi_s(u)$ is convex on $(0, \infty)$.

Remind that e and d are dual bases in $\mathbb{R}^n$ with respect to the inner product given by (1).

Theorem 4.1. *Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in I^n$ with $I \subset \mathbb{R}_+$, $p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$ and $v, w \in \mathbb{R}^n$ with $\langle v, w\rangle > 0$ and $\langle x - y, v\rangle = 0$. Assume that x, y, p, w, v satisfy the conditions (i)- (ii) of Theorem 1.1 for some index sets J_1 and J_2 ($J_1 \cup J_2 = J$), and that all nondecreasing functions defined on I preserve v-separability on J_1 and J_2 w.r.t. e.*

Denote

$$\Gamma_s = \Gamma_s\,(x, y; p) := \sum_{k=1}^{n} p_k \left[\varphi_s(x_k) - \varphi_s(y_k)\right] \quad \textit{for } s \in \mathbb{R}. \tag{45}$$

Then the following two statements hold.

(i) *The matrix*

$$\left[\Gamma_{\frac{s_i + s_j}{2}}\right]_{i,j=1,\ldots,n} \qquad \textit{for } s_i, s_j \in \mathbb{R}$$

is positive semi-definite.

In consequence, the following Gram's inequality holds

$$\det\left[\Gamma_{\frac{s_i+s_j}{2}}\right]_{i,j=1,\ldots,m} \geq 0 \quad \text{for } s_i, s_j \in \mathbb{R},\ m = 1, \ldots, n. \tag{46}$$

(ii) *If the function $s \mapsto \Gamma_s$ is continuous (e.g., if $v = (1, \ldots, 1)$ and $\sum\limits_{k=1}^{n} p_k x_k = \sum\limits_{k=1}^{n} p_k y_k$) then it is exponentially convex.*

Proof. (Based on the proof of [3, Theorem 3].)

(i) Setting

$$f(u) := \sum_{i,j=1}^{n} \alpha_i \alpha_j \varphi_{s_{ij}}(u) \quad \text{for } u > 0, \tag{47}$$

where $\alpha_i, \alpha_j \in \mathbb{R}$ and $s_{ij} = \frac{s_i+s_j}{2}$, we derive

$$\frac{d^2 f(u)}{du^2} = \sum_{i,j=1}^{n} \alpha_i \alpha_j u^{s_{ij}-2} = \left(\sum_{i=1}^{n} \alpha_i u^{\frac{s_i}{2}-1}\right)^2 \geq 0 \quad \text{for } u > 0.$$

So, f is a convex function on $(0, \infty)$. Thus the derivative $\frac{df}{du}$ is nondecreasing. In consequence, it preserves v-separability on J_1 and J_2 w.r.t. e. Using Theorem 1.1 and Remark 1.2 for f, we obtain

$$\sum_{k=1}^{n} p_k f(x_k) \geq \sum_{k=1}^{n} p_k f(y_k). \tag{48}$$

Now from (45), (47) and (48) we get that

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j \Gamma_{s_{ij}} = \sum_{k=1}^{n} p_k [f(x_k) - f(y_k)] \geq 0.$$

Therefore the matrix $\left[\Gamma_{s_{ij}}\right]_{i,j=1,\ldots,n}$ is positive semi-definite, as required.

Clearly, (46) is a consequence of the semi-definiteness of the matrix.

(ii) Assume the function $s \mapsto \Gamma_s$ is continuous. By the proved positive semi-definiteness of the matrix $\left[\Gamma_{\frac{s_i+s_j}{2}}\right]_{i,j=1,\ldots,n}$ we obtain

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j \Gamma_{\frac{s_i+s_j}{2}} \geq 0$$

for all $\alpha_i \in \mathbb{R}$ and $s_i, s_j \in \mathbb{R}$. This implies exponential convexity of the function $s \mapsto \Gamma_s$ (see (42)). $\square$

5. Cauchy type means

Let $x, y, p \in \mathbb{R}_+^n$ and $w, v \in \mathbb{R}^n$ with $\langle v, w\rangle > 0$, $\langle x - y, v\rangle = 0$ and $\sum_{k=1}^{n} p_k(x_k^2 - y_k^2) \neq 0$ as in Theorem 4.1. We give the following definition.

$$M_{a,b} := \left(\frac{\Gamma_a}{\Gamma_b}\right)^{\frac{1}{a-b}} \quad \text{for } a, b \in \mathbb{R} \text{ such that } a \neq b, \tag{49}$$

where Γ_s is defined by (45) (see also (44)). By Remark 3.4 these expressions are means. For example,

$$M_{a,b} = \left\{\frac{b(b-1)\sum_{k=1}^n p_k\left(y_k^a - x_k^a\right)}{a(a-1)\sum_{k=1}^n p_k\left(y_k^b - x_k^b\right)}\right\}^{\frac{1}{a-b}} \quad \text{for } a \neq b,\ a, b \neq 0, 1.$$

Moreover, by limit we also have

$$\begin{aligned}
M_{a,a} &= \exp\left(\frac{\sum_{k=1}^n p_k\left[y_k^a \log y_k - x_k^a \log x_k\right]}{\sum_{k=1}^n p_k\left[y_k^a - x_k^a\right]} - \frac{2a-1}{a(a-1)}\right) \quad \text{for } a \neq 0, 1,\\
M_{0,0} &= \exp\left(\frac{\sum_{k=1}^n p_k\left[\log^2 y_k - \log^2 x_k\right]}{2\sum_{k=1}^n p_k\left[\log y_k - \log x_k\right]} + 1\right),\\
M_{1,1} &= \exp\left(\frac{\sum_{k=1}^n p_k\left[y_k \log^2 y_k - x_k \log^2 x_k\right]}{2\sum_{k=1}^n p_k\left[y_k \log y_k - x_k \log x_k\right]} - 1\right).
\end{aligned}$$

Theorem 5.1. *Under the assumptions of Theorem 4.1, let the function $s \mapsto \Gamma_s$ be continuous and positive. Then the following inequality is valid.*

$$M_{r,t} \leq M_{a,b} \quad \textit{for } r, t, a, b \in \mathbb{R} \textit{ such that } r \leq a \textit{ and } t \leq b. \tag{50}$$

Proof. Since the function $s \mapsto \Gamma_s$ is continuous, it is exponential convex by Theorem 4.1. Consequently, Γ_s is log-convex (see (43)). So, we have

$$\frac{\log \Gamma_t - \log \Gamma_r}{t - r} \leq \frac{\log \Gamma_b - \log \Gamma_a}{b - a} \quad \text{for } r \neq t \text{ and } a \neq b$$

(see [9, p. 2]). That is,

$$\log\left(\frac{\Gamma_t}{\Gamma_r}\right)^{\frac{1}{t-r}} \leq \log\left(\frac{\Gamma_a}{\Gamma_b}\right)^{\frac{1}{a-b}} \quad \text{for } r \neq t \text{ and } a \neq b. \tag{51}$$

From (51) and (49) we get (50) for $r \neq t$ and $a \neq b$. For $r = t$ or $a = b$, we have the limiting case. □

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