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THE COMPARISON OF THE CONVERGENCE SPEED BETWEEN PICARD, MANN, ISHIKAWA AND TWO-STEP ITERATIONS IN BANACH SPACES

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Abstract. The purpose of this paper is to compare convergence speed of the Picard and two-step iterations, Mann and two-step iterations, Ishikawa and two-step iterations for the class of Zamfirescu operators.

1. Introduction

Let E be a real Banach space, D a closed convex subset of E, and $T : D \to D$ a self-map. Let p_0, v_0, u_0, x_0 be arbitrary in D. The sequence $\{p_n\}_{n=0}^{\infty} \subset D$ defined by

$$p_{n+1} = Tp_n, \quad n \geqslant 0, \tag{1.1}$$

is called the Picard iteration. Let $\{a_n\}$ be a sequence of real numbers in $[0, 1]$. The sequence $\{v_n\}_{n=0}^{\infty} \subset D$ defined by

$$v_{n+1} = (1 - a_n)v_n + a_nTv_n, \quad n \geqslant 0, \tag{1.2}$$

is called the Mann iteration. The sequence $\{u_n\}_{n=0}^{\infty} \subset D$ defined by

$$\begin{aligned} &u_0 \in D, \\ &z_n = (1 - b_n)u_n + b_nTu_n, \quad n \geqslant 0, \\ &u_{n+1} = (1 - a_n)u_n + a_nTz_n, \quad n \geqslant 0, \end{aligned} \tag{1.3}$$

is called the Ishikawa iteration, where $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers in $[0, 1]$.

Recently, Thianwan [4] introduced a new two-step iteration as follows

$$\begin{aligned} &x_0 \in D, \\ &y_n = (1 - b_n)x_n + b_nTx_n, \quad n \geqslant 0, \\ &x_{n+1} = (1 - a_n)y_n + a_nTy_n, \quad n \geqslant 0, \end{aligned} \tag{1.4}$$

where $\{a_n\}, \{b_n\} \subset [0, 1]$.

In the last twenty years, many authors have studied the convergence of the sequence of Picard, Mann, Ishikawa, and two-step iterations of a mapping T to a

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fixed point of T, under various contractive conditions. In such situations, it is of theoretical and practical importance to compare these iteration methods in order to estabilsh which one converges faster if possible.

Definition 1.1. (see [9]). The operator $T : X \to X$ satisfies condition Zamfirescu (condition Z, for short) if and only if there exist real numbers a, b, c satisfying $0 < a < 1, 0 < b, c < \frac{1}{2}$ such that for each pair x, y in X, at least one of the following conditions is true:

(1) $||Tx - Ty|| \leq a||x - y||$;
(2) $||Tx - Ty|| \leq b(||x - Tx|| + ||y - Ty||)$;
(3) $||Tx - Ty|| \leq c(||x - Ty|| + ||y - Tx||)$.

Obviously, we could obtain that every Zamfirescu operator T satisfies the inequality

$$||Tx - Ty|| \leq \delta||x - y|| + 2\delta||x - Tx|| \tag{1.5}$$

for all $x, y \in X$, where $\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$ with $0 < \delta < 1$.

In 1972, Zamfirescu [9] obtained a very interesting fixed point theorem for Zamfirescu operator.

Theorem 1.2. (see [9]). *Let (X, d) be a complete metric space and $T : X \to X$ a Zamfirescu operator. Then T has a unique fixed point q and the Picard iteration (1.1) converges to q.*

Later on, Berinde [2, 3] improved and extended the above-mentioned theorem and the results in the papers [2, 3] as follows.

Theorem 1.3. (see [3]). *Let E be an arbitrary Banach space, D a closed convex subset of E, and $T : D \to D$ an operator satisfying condition Z. Let $\{v_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (1.2) *for $u_0 \in D$, with $\{a_n\} \subset [0, 1]$ satisfying $\sum_{n=0}^{\infty} a_n = \infty$. Then $\{v_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

Theorem 1.4. (see [2]). *Let E be an arbitrary Banach space, D a closed convex subset of E, and $T : D \to D$ an operator satisfying condition Z. Let $\{u_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by (1.3) for $u_0 \in D$, with $\{a_n\}, \{b_n\}$ being sequences of positive numbers in $[0, 1]$ and $\{a_n\}$ satisfying $\sum_{n=0}^{\infty} a_n = \infty$. Then $\{u_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

Recently, Isa Yildirim, Murat Ozdemir and Hukmi Kiziltunc [8] proved that the two-step iteration converges strongly to the fixed point of T with the following result.

Theorem 1.5. (see [8]). *Let E be an arbitrary Banach space, D a closed convex subset of E, and $T : D \to D$ an operator satisfying condition Z. Let $\{x_n\}_{n=0}^{\infty}$ be* *two-step iteration defined by (1.4) and* $x_0 \in D$*, where* $\{a_n\}$ *and* $\{b_n\}$ *are sequences of positvie numbers in* $[0,1]$ *satisfying*

$$\sum_{n=0}^{\infty} a_n = \infty.$$

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the fixed point of* T.

In order to compare the speed of fixed point iterative methods, Berinde [3], see also [Vasile Berinde, Iterative Approximation of Fixed Points, Spinger Verlag, Lectures Notes in Mathematics, 1912, 2007] introduced the following concept of rate of convergence.

Definition 1.6. (see [6]). Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively. Then $\{a_n\}$ is said to converge faster than $\{b_n\}$ if

$$\lim_{n\to\infty} \left| \frac{a_n - a}{b_n - b} \right| = 0$$

.

The purpose of this paper is to show that the Picard iteration converges faster than the two-step iteration and two-step converges faster than Mann and Ishikawa iterations for the class of Zamfirescu operators of an arbitrary closed convex subset of a Banach space.

2. Main results

In the sequel, suppose that δ is the constant from (1.5).

Theorem 2.1. *Let* E *be an arbitrary real Banach space,* D *a closed convex subset of* E*, and* $T : D \to D$ *a Zamfirescu operator. Let* $\{p_n\}_{n=0}^{\infty}$ *be defined by (1.1) for* $p_0 \in D$*, and let* $\{x_n\}_{n=0}^{\infty}$ *be defined by (1.4) for* $x_0 \in D$ *with* $\{a_n\}, \{b_n\}$ *in* $\left[0, \dfrac{1}{1+\delta}\right)$ *and satisfying*

(i) $\sum\limits_{n=0}^{\infty} a_n = \infty$;

(ii) $\lim\limits_{n\to\infty} a_n = 0$;

(iii) $\lim\limits_{n\to\infty} b_n = 0$.

Then the Picard iteration converges faster that the two-step iteration to the fixed point of T.

Proof. By Theorem 1.2, T has a unique fixed point, denote it by q. Moreover, Picard's iteration $\{p_n\}_{n=0}^{\infty}$ defined by (1.1) converges to q, for any $p_0 \in D$ and

$$||p_{n+1} - q|| = ||Tp_n - q||.$$

Take $x = q$ and $y = p_n$ in (1.5), then we get

$$||p_{n+1} - q|| \leq \delta ||p_n - q|| \leq \delta^{n+1} ||p_0 - q||, n \geqslant 0. \tag{2.1}$$

Now, by two-step iteration in (1.4) and (1.5),

$$\begin{aligned}||x_{n+1}-q|| &\geqslant (1-a_n)||y_n-q||-a_n||Ty_n-q||\\ &\geqslant [1-(1+\delta)a_n]||y_n-q||.\end{aligned}$$

On the other hand

$$\begin{aligned}||y_n-q|| &\geqslant (1-b_n)||x_n-q||-b_n||Tx_n-q||\\ &\geqslant [1-(1+\delta)b_n]||x_n-q||.\end{aligned}$$

Thus

$$\begin{aligned}||x_{n+1}-q|| &\geqslant [1-(1+\delta)a_n][1-(1+\delta)b_n]||x_n-q||\\ &\geqslant \prod_{k=0}^{n}[1-(1+\delta)a_k][1-(1+\delta)b_k]||x_0-q||.\end{aligned} \tag{2.2}$$

From (2.1) and (2.2), it follows that

$$\frac{||p_{n+1}-q||}{||x_{n+1}-q||} \leq \frac{\delta^{n+1}}{\prod\limits_{k=0}^{n}[1-(1+\delta)a_k][1-(1+\delta)b_k]} \to 0 \text{ as } n\to\infty.$$

Indeed, put $w_n = \dfrac{\delta^{n+1}}{\prod\limits_{k=0}^{n}[1-(1+\delta)a_k][1-(1+\delta)b_k]}$, we have

$$\frac{w_{n+1}}{w_n} = \frac{\delta}{[1-(1+\delta)a_{n+1}][1-(1+\delta)b_{n+1}]} \to \delta < 1 \text{ as } n\to\infty.$$

Applying reatio test, we get $\sum\limits_{n=0}^{\infty} w_n < \infty$, so $w_n \to 0$ as $n\to\infty$, that is, $||p_n-q|| = o(||x_n-q||)$. By Definition 1.6, we obtain the conculusion of Theorem 2.1. □

Now we will give an example satisfying the hypotheses of Theorem 2.1.

Example 2.2. Suppose

$$T:[0,1]\to[0,1], \quad Tx=\frac{1}{2}x,$$

$$a_n=b_0=0, \quad n=0,1,...,n_0; \quad a_n=b_n=\frac{1}{n}, n\geqslant n_0+1.$$

It is clear that T is a Zamfirescu operator with a unique fixed point $x^*=0$. And it is easy to see that T, a_n, b_n satisfy all the conditions of Theorem 2.1.

Theorem 2.3. *Let E be an arbitrary real Banach space, D a closed convex subset of E, and $T: D\to D$ a Zamfirescu operator. Let $\{v_n\}_{n=0}^{\infty}$ be defined by (1.2) for $v_0\in D$, and let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.4) for $x_0 \in D$ with $\{a_n\}$ in $\left[0, \dfrac{1}{1+\left(1+\frac{2}{m}\right)\delta}\right]$ for some $m>0$ and satisfying*

(i) $\sum\limits_{n=0}^{\infty} a_n = \infty$;

(ii) $\sum_{n=0}^{\infty} b_n = \infty$.

Then the two-step iteration converges faster that the Mann iteration to the fixed point of T.

Proof. By Theorem 1.3 and Theorem 1.5 $\{x_n\}$ and $\{v_n\}$ converge strongly to a fixed point of T, denoted by q. Now, by Mann's iteration in (1.2) and (1.5),

$$\begin{aligned} ||v_{n+1} - q|| &\geqslant (1 - a_n)||u_n - q|| - a_n||Tv_n - Tq|| \\ &\geqslant [1 - (1 + \delta)a_n]||v_n - q|| \\ &\geqslant \prod_{k=0}^{n}[1 - (1 + \delta)a_k]||v_0 - q||. \end{aligned} \tag{2.3}$$

Now by using two-step iteration (1.4), we have

$$\begin{aligned} ||x_{n+1} - q|| &\leq (1 - a_n)||y_n - q|| + a_n||Ty_n - q|| \\ &\leq (1 - a_n)||y_n - q|| + a_n\delta||y_n - q|| \\ &= [1 - (1 - \delta)a_n]||y_n - q||. \end{aligned} \tag{2.4}$$

On the other hand, we have

$$\begin{aligned} ||y_n - q|| &\leq (1 - b_n)||x_n - q|| + b_n||Tx_n - q|| \\ &\leq (1 - b_n)||x_n - q|| + b_n\delta||x_n - q|| \\ &= [1 - (1 - \delta)b_n]||x_n - q||. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we get the following inequality

$$\begin{aligned} ||x_{n+1} - q|| &\leq [1 - (1 - \delta)a_n][1 - (1 - \delta)b_n]||x_n - q|| \\ &\leq \prod_{k=0}^{n}[1 - (1 - \delta)a_k][1 - (1 - \delta)b_k]||x_0 - q||. \end{aligned} \tag{2.6}$$

From (2.3) and (2.6) it follows that

$$\frac{||x_{n+1} - q||}{||v_{n+1} - q||} \leq \prod_{k=0}^{n} \frac{[1 - (1 - \delta)a_k][1 - (1 - \delta)b_k]}{[1 - (1 + \delta)a_k]} \frac{||x_0 - q||}{||v_0 - q||}. \tag{2.7}$$

Here we observe that

$$\frac{1 - (1 - \delta)a_k}{1 - (1 + \delta)a_k} \leq 1 + m, \quad \text{for all } k = 0, 1, 2, ...$$

Thus

$$\frac{||x_{n+1} - q||}{||v_{n+1} - q||} \leq \prod_{k=0}^{n}(1 + m)[1 - (1 - \delta)b_k]\frac{||x_0 - q||}{||v_0 - q||}. \tag{2.8}$$

Since $0 \leq \delta < 1$, $b_n \in [0, 1]$ and $\sum_{n=0}^{\infty} b_n = \infty$ then

$$\lim_{n\to\infty} \prod_{k=0}^{n}(1 + m)[1 - (1 - \delta)b_k] = 0,$$

it follows from (2.8) that

$$\lim_{n\to\infty}\frac{||x_{n+1}-q||}{||v_{n+1}-q||}=0.$$

Therefore the two-step iteration converges faster than the Mann iteration to the fixed point of T. □

Remark 2.4. It is easy to see that Example 2.2 satisfies all the conditions of Theorem 2.2.

Theorem 2.5. *Let E be an arbitrary real Banach space, D a closed convex subset of E, and $T : D \to D$ a Zamfirescu operator. Let $\{u_n\}_{n=0}^{\infty}$ be defined by (1.3) for $u_0 \in D$, and let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.4) for $x_0 \in D$ with $\{a_n\}$ in $\left[0, \dfrac{1}{1+\left(1+\frac{2}{m}\right)\delta}\right]$ for some $m > 0$ and satisfying*

(i) $\sum\limits_{n=0}^{\infty} a_n = \infty$;

(ii) $\sum\limits_{n=0}^{\infty} b_n = \infty$.

Then the two-step iteration converges faster that the Ishikawa iteration to the fixed point of T.

Proof. By Theorem 1.4 $\{u_n\}$ converges strongly to the fixed point of T, denoted by q. Now by using Ishikawa iteration (1.3) and (1.5), we have

$$\begin{aligned}||u_{n+1}-q|| &\geqslant (1-a_n)||x_n-q||-a_n||Ty_n-q||\\ &\geqslant (1-a_n)||x_n-q||-a_n\delta||y_n-q||.\end{aligned}\tag{2.9}$$

Moreover

$$\begin{aligned}||y_n-q|| &\leq (1-b_n)||x_n-q||+b_n||Tx_n-q||\\ &\leq (1-b_n)||x_n-q||+b_n\delta||x_n-q||\\ &= [1-(1-\delta)b_n]||x_n-q||.\end{aligned}\tag{2.10}$$

From (2.9) and (2.10) we get

$$\begin{aligned}||x_{n+1}-q|| &\geqslant [1-a_n-\delta a_n+a_nb_n\delta(1-\delta)]||x_n-q||\\ &\geqslant [1-(1+\delta)a_n]||x_n-q||\\ &\geqslant \prod_{k=0}^{n}[1-(1+\delta)a_k]||x_0-q||.\end{aligned}\tag{2.11}$$

From (2.6) and (2.11) it follows that

$$\begin{aligned}\frac{||x_{n+1}-q||}{||u_{n+1}-q||} &\leq \prod_{k=0}^{n}\frac{[1-(1-\delta)a_k][1-(1-\delta)b_k]}{[1-(1+\delta)a_k]}\frac{||x_0-q||}{||u_0-q||}\\ &\leq \prod_{k=0}^{n}(1+m)[1-(1-\delta)b_k]\frac{||x_0-q||}{||u_0-q||}\to 0 \quad \text{as } n\to\infty.\end{aligned}\tag{2.12}$$

Thus we obtain the conclusion of Theorem 2.5. □

Remark 2.6. Theorem 2.1 provides a direct comparison of the rate of convergence of Picard and two-step iterations in the class of Zamfirescu operators, while Theorem 2.3 and Theorem 2.5 obtain similar results for Mann, Ishikawa and two-step iterations. However, we do not have an answer for the rate of convergence in the case of Mann and Ishikawa iterations in the same class of mappings (see [5, 6]).

References

[1] G. V. R. Babu and K. N. V. V. Vara Prasad, Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators, *Fixed Point Theory Appl.* **2006**, Article ID 49615, 6 pages, 2006.

[2] V. Berinde, On the convergence of the Ishikawa iteration in the class of quasi contractive operators, *Acta Math. Univ. Comenian.* **73** (2004), 119-126.

[3] V. Berinde, Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators, *Fixed Point Theory Appl.* **2** (2004), 97-105.

[4] S. Thianwan, Common fixed points of new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space, *J. Comput. Appl. Math.* **2004** (2009), 688-695.

[5] O. Popescu, Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators, *Math. Commun.* **12** (2007), 195-202.

[6] Y. Qing and B. E. Rhoades, Comments on the rate of convergence between Mann and Ishikawa iterations applied to Zamfirescu operators, *Fixed Point Theory Appl.* **2008**, Article ID 387504, 3 pages, 2008.

[7] Z. Xue, The comparison of the convergence speed between Picard, Mann, Krasnoselskij and Ishikawa iterations in Banach spaces, *Fixed Point Theory Appl.* **2008**, Article ID 387056, 5 pages, 2008.

[8] I. Yildirim, M. Ozdemir and H. Kiziltunc, On the convergence of a new two-step iteration in the class of quasi-contractive operators, *Int. J. Math. Anal.* **3** (2009), 1881-1892.

[9] T. Zamfirescu, Fix point theorems in metric spaces, *Arch. Math.* **23** (1972), 292-298.

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