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GLAESKE-KILBAS-SAIGO FRACTIONAL INTEGRATION AND FRACTIONAL DIXMIER TRACE

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Abstract. The Dixmier trace was introduced by Jacques Dixmier in 1966 and its key role in noncommutative geometry was discovered by Connes around 1990 during his development of non-commutative infinitesimal calculus. Remarkably, the Dixmier trace is used to define dimension, integration and has been used along with heat kernel type expansions, to define 'spectral actions' for noncommutative quantum field theories. This work concerns a generalization of the Dixmier trace to its fractional counterpart. Some new properties are raised and explored in some details.

The pursuit for an ultimate theory of quantum gravity, in particular the understanding of the spacetime organization at Planck's distance, is actually one of the major objectives in contemporary mathematical physics. As it is usually believed that the spacetime at very tiny distance may not be described by a physical manifold of any type, the conventional geometrical setting of Einstein's General Relativity seems to be deficient to elucidate the non-manifold spacetime structure at very short distances. So it is normal to ask about a generalization of the standard geometry. One beautiful attempt is Connes's Noncommutative Geometry (CNG) which attracts an ever increasing attention of researchers especially after the greatest success of unifying the forces of nature into a single gravitational action-the spectral action in a purely algebraic way, rather than a completely new formalism. It has introduced a new twist in the search for a quantum theory of gravity. Therefore, the possibility that our spacetime is a noncommutative one should be taken seriously.

In CNG, differential structures augmented by spin structure are used to recover geometry from the spectrum of a differential operator [6]. The points of the manifold defined as a noncommutative algebra $'\mathcal{A}'$ of $N \times N$ matrices with entries as functions on spacetime acting in addition to the hermitian self-adjoint Dirac operator $'\mathcal{D}'$ on a Hilbert space $'\mathcal{H}'$, generally unbounded. $'\mathcal{H}'$ is in fact a vector space of $N \times N$ matrices with entries as spinors. Thus, a CNG is defined by a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where the algebra $\mathcal{A} = C^{\infty}(\mathcal{M})$ is the pre-C^* algebra of smooth functions on $\mathcal{M}$ with respect to the C^0-norm acting in $\mathcal{H}$ by multiplication operators as follows: $(fg)(x) = f(x)g(x)$, $\forall x \in \mathcal{M}$. $\mathcal{M}$ is an orientable, c-

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onnected, compact, N-dimensional differentiable unbounded manifold which may not be a manifold. The total Riemannian spin geometry of $\mathcal{M}$ can be reconstructed from $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The set of manifolds which allow to be described by a spectral triple must be Riemannian, i.e. of Euclidian signature, and they have to admit a Dirac operator, which is not true for any manifold. More generally, CNG generalizes $(C^{\infty}(\mathcal{M}), L^2(\mathcal{M}, \mathbb{S}), \mathbb{D}) \to (\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathbb{D}$ is an unbounded Dirac operator acting on $\mathcal{H} = L^2(\mathcal{M}, \mathbb{S})$ of square-integrable spinors with positive-definite signature specifying the metric and $C^{\infty}(\mathcal{M})$ acts on $'\mathcal{H}'$ by multiplication operators with $||[\mathcal{D}, \pi(x)]|| = ||\mathrm{grad}\pi(x)||_{\infty}$, $\pi \in C(\mathcal{M})$. A positive functional on the affine space set F containing all the possible Dirac operators on $\mathcal{M}$ is needed to obtain the dynamics on the gravitational field. In fact, once we operate the original Riemannian geometry for its corresponding commutative triple we need a replacement for the Einstein-Hilbert action. The so-called spectral action is one possible candidate [4]. It depends only on the eigenvalues of the Dirac operators and contains the Einstein-Hilbert action as a dominant term. It was observed that the existence of Dirac operators is determined by topological properties of the manifold and therefore contains information about the geometry of the manifold. This feature was exploited in the use of Dirac operators in the Seiberg-Witten theory of differential topological invariants for 4-manifolds.

For a definite choice of matrix algebras, one obtains in the limit of high eigenvalues of the Dirac operator on $\mathcal{M}$ and in particular in the asymptotic expansion of the spectral triples action, the pure Einstein-Hilbert gravity including the cosmological constant term. In contrast to the diffeomorphisms of the geometrical manifold, the automorphisms of the algebra permit to be extended to comprise compact Lie groups. As the entire notion of a spectral triple is autonomous, i.e. independent of the commutativity of the algebra, it is possible to bond the algebra of functions over the space-time manifold with an algebra being the sum of simple matrix algebras automorphisms by simply tensorising. These techniques are entitled "almost-commutative geometries" and the part of the spectral triple based on the matrix algebra is often called the "finite or internal part". Choosing as matrix algebras $\mathbb{C} \oplus H \oplus \mathbb{M}_3(\mathbb{C})$, where H are the quaternions, one recovers with a suitable choice for the Hilbert space, the Einstein-Hilbert action and the Yang-Mills theory for the gauge group $SU(N)$ of the standard model of elementary particle physics with the spontaneous symmetry-breaking Higgs potential action of the standard model. The Higgs scalar together with its potential emerges naturally as the "Einstein-Hilbert action" in the noncommutative part of the algebra. At this point, it has become potential for the first time to give the Higgs scalar a geometrical interpretation. The Yang-Mills gauge potential emerges as the inner part of the spacetime metric in a similar way as the group of gauge transformations for $SU(N)$ appears as the group of inner diffeomorphisms. They provide a natural elucidation of the Higgs boson as a connection in the noncommutative part of the geometry.

The most general form of the bosonic action is given by $S = \mathrm{Tr}[\mathbb{F}[\mathcal{D}^2]/\Lambda^2] : \mathbb{F} \to \mathbb{R}^+$. Here $\mathbb{F} : \mathbb{R}^+ \to \mathbb{R}^+$ is any regular and fast decreasing function at infinity for which the Hilbert space trace exists and $\Lambda \in \mathbb{R}$ is a cut-off homogeneous to the mass of order of Planck's mass, e.g. it fixes the mass scale. The bosonic action counts the Dirac operator eigenvalues smaller than Λ. Note that S is spectral invariant, i.e. it is invariant under all unitaries on $'\mathcal{H}'$ and thus in particular under all diffeomorphisms. It depends merely on the eigenvalues of the Dirac operators and contains the Einstein-Hilbert action as a dominant term. While occasionally pure mathematicians are motivated from theoretical physics, more frequently what induce mathematical progress and enhancement is the internal logic of the theoretical investigated subjects. Understanding deeply the mathematical formalism, one can overpass a connection between the diverse theoretical frameworks by modifying or creating new ideas that will be able to build a new link that the old one couldn't succeed to do. This kind of research and explorations require us to pay profound awareness to the subject of precisely what it is that one understands.

Recently, in an attempt to investigate about the characteristic properties of the triplet action satisfied by a class of fractional integrals and derivatives, namely the Riemann-Liouville (RL) and the Erdelyi-Kober (EK) fractional operators of a function $G \in L_p(0,\infty)$ defined respectively by:

$$\mathbb{I}_\alpha^{RL}(s) = \lim_{t\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t G(\tau)(t-\tau)^{\alpha-1} e^{-s\tau} d\tau, \quad N-2 < 2k \leq N, \tag{1}$$

$$\mathbb{I}_\alpha^{EK}(s) = \lim_{t\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t G(\tau)\tau^{N/2-k-1}(t^m - \tau^m)^{\alpha-1} e^{-s\tau} d\tau, \tag{2}$$

$N-2m < 2k \leq N$, where $0 < \alpha \leq 1, m \in \mathbb{R}^+, n$ is the number of spacetime dimensions, τ is the intrinsic time and t is the observer time ($t \neq \tau$), remarkably the fractional spectral triplet action was found to be complexified [8]. The complexified spectral action is similar in form to the canonical/loop approach [27] with the major differentiation: the gravitational coupling constant is complexified in the fractional structure. The emergent imaginary spectral triplet action will contribute as a corrector to the real part when applied to a physical problem, i.e. the Yang-Mills-Higgs theory. Accordingly, the fractional spectral triplet action is expected to bring new topological terms to the standard theory. In particular, the ghost solutions arising from the real part do not survive in the classical phase space, but somewhat in the space of the complexified metric and consequently complex periodic orbits may take place. In the semi-classical limit ($h \to 0$), their contributions to the fractional path integral are insignificant, but rather become significant for not too small h. Even in the limit $h \to \infty$, the vanishing of the complex part of such an orbit would lead to a non-vanishing contribution to the propagator which invalidates standard semiclassical quantization involving only real orbits. Moreover, within the same fractional approach, it was argued that the Connes 2-points space distance problem amazingly is finite even at the classical level and differs to some extent from the quantum results.

In fact, fractional integral and derivative approaches have been shown to be useful in the study of several complex dynamical systems [26, 29, 31-33], in particular quantum field theory [9-11, 14-16, 20]. Today, fractional integration appears in various fields, some in the form of not-so-subtle variation and generalizations. It is noteworthy that the main argument for dealing with fractional operators concerns the fact that it may represent an analytic framework suitable for the description of physical phenomena that are likely to arise in the TeV realm of particle physics. For example strong-gravity effects emerging from the short distance behavior of quantum field theory necessitate the use of fractional operators. Moreover, the macroscopic description of phenomena in terms of conventional differential and integral operators breaks down due of dynamical instabilities developed on long time scales, i.e. unstable vacuum fluctuations leading to self-organized criticality and therfore, this is one of the main arguments for using fractional differential and integral operators within the context of field theory [10, 12, 13, 19, 20, 25].

Motivated by all these, we would like in this work to enlarge our search to investigate about some properties of the fractional spectral triples, namely the fractional integration and Dixmier trace. The Dixmier trace, in a broad sense, by taking a class of compact operators for which the common trace diverges at a given rate. In this paper, we will explore the fractional aspects in noncommutative geometry making use of the Riemann-Liouville fractional integral approach and we left the Erdélyi-Kober approach for a future work. Our main aim is to introduce the basic settings. It is notable that one can associate spectral triples to certain fractal sets and calculate their spectra. Moreover it has inspired the designation of fractional dimension and of Hausdorff (and Hausdorff-Besicovitch) measure in the abstract setting of spectral triples, because of the strong analogies with the fractal case. We start by proposing the following definition:

Definition 1. If $\mathcal{D}$ is the Dirac operator and λ its corresponding eigenvalue, we define its matching fractional RL-Mellin integral transformation through this work and in particular after limitation to the λ-eigenspace by:

$$|\mathcal{D}|^{-\omega} = \frac{1}{\Gamma((\omega+2\alpha-2)/2)}|\lambda|^{-\omega}\int_0^\infty (t-\tau)^{(\omega-2)/2}e^{t-\tau}d\tau. \tag{3}$$

Here α and ω could take any fractional value and they could have complex values as well. The operator $e^{t-\tau} \equiv e^{-T}, T = \tau - t$, namely $e^{-T} : f \to g$ is the solution of the operator for the heat equation $\partial_T g + g = 0$ with initial value $g(t=0) = f$. In the standard noncommutative approach, the condition $||[\mathcal{D}, f]||_{C^0} \leq 1$ which means that the gradient of f is bounded by one is significant since it implies that we can renovate the distance function and therefore the metric on $\mathcal{M}$ from the spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. In order to get a fractional noncommutative generalization, we have to express classical geometric fractional operators in terms of the triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ [1]. We will perform this for fractional integration over $\mathcal{M}$. For this, we let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of the generalized Dirac operator ordered by increasing absolute values, $|\lambda_1| \leq |\lambda_2| \leq \ldots \nearrow \infty$. Besides, 0 is not an eigenvalue of $\mathcal{D}$. It is notable that the square $\mathcal{D}$ is a generalized Laplacian with eigenvalues $0 < \lambda_1^2 \leq \lambda_2^2 \leq \ldots \nearrow \infty$.

As in general the RL fractional operator has the effect of increasing the dimension of the manifold and turns it into a fractional dimension by its relation to the index of fractional integration we can introduce the following proposition:

Proposition 1. *The Dirac operator* $|\mathcal{D}|^{-\dim \mathcal{M}}$, $\dim \mathcal{M} = n$ *is fractionalized as follows*

$$|\mathcal{D}|^{-\dim \mathcal{M}}, \dim \mathcal{M} = n \to |\mathcal{D}|^{-(n+1-\alpha)}, n < \dim \mathcal{M} = n+1-\alpha \leq n+1. \tag{4}$$

Here $|\mathcal{D}|^{-1}$ has order $1/(n+1-\alpha)$ so that $|\mathcal{D}|^{-(n+1-\alpha)}$ has order one, and hence $|\mathcal{D}|^{-(n+1-\alpha)} \in \mathcal{L}^{1,\infty}(\mathcal{H})$ and since

$$\mathcal{L}^{1,\infty}(\mathcal{H}) := \{\mathcal{D} \in \mathcal{K}(\mathcal{H}) : \sum_{i=1}^{n} \lambda_k(\mathcal{D}) = 0(\log n)\}$$

is an ideal, then for any compact operator $T \in \mathcal{A}$ on a finite dimensional Hilbert space $\mathcal{H}$, $T|\mathcal{D}|^{-(n+1-\alpha)} \in \mathcal{L}^{1,\infty}(\mathcal{H}), (n+1-\alpha) \in (0,\infty)$. When $\mathcal{A}$ consists of measurable operators, we can define a fractional trace on $\mathcal{A}$ by setting $\int T = \mathrm{Tr}_\omega(T|\mathcal{D}|^{-(n+1-\alpha)})$. Here Tr_ω is a logarithmic Dixmier trace [7], i.e. a singular trace summing logarithmic divergences. This is to say that

$$\mathrm{Tr}_\omega(T|\mathcal{D}|^{-(n+1-\alpha)}_{\text{fractional}}) \longleftrightarrow \text{volume of the space that has dimensions } (n+1-\alpha).$$

Lemma 1. *A spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *has dimension* $n > \alpha - 1$ *if the infinitesimal unit of length* $L = |\mathcal{D}|^{-1}$ *has order* $1/(n+1-\alpha)$.

Definition 2. We call the $(n+1-\alpha)$-dimensional function the map:

$$T \to \mathrm{Tr}_\omega(T|\mathcal{D}|^{-(n+1-\alpha)})$$

and the fractional dimension of the spectral triples, the number

$$\begin{aligned} d(\mathcal{A}, \mathcal{H}, \mathcal{D}) &= \inf\{n > \alpha - 1 : |\mathcal{D}|^{-(n+1-\alpha)} \in \mathcal{L}_0^{1,\infty}(\mathcal{H})\}, \\ &= \sup\{n > \alpha - 1 : |\mathcal{D}|^{-(n+1-\alpha)} \notin \mathcal{L}^{1,\infty}(\mathcal{H})\}. \end{aligned}$$

Here $\mathcal{L}_0^{1,\infty}(\mathcal{H}) := \{\mathcal{D} \in \mathcal{K}(\mathcal{H}) : \sum\limits_{i=1}^{n} \lambda_k(\mathcal{D}) = o(\log n)\}$ and ω is a generalized limit [24].

Theorem 2. *Given the spectral triples* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. *Let* $E \to \mathcal{M}$ *be a Riemannian or Hermitian vector bundle over* $\mathcal{M}$ *and* $\Psi = f \cdot \mathbb{I} \in C^\infty(\mathcal{M})(\mathrm{End}\,(E))$ *and* $f = 1$, *then*

$$\begin{aligned} \mathrm{Tr}_\omega(|\mathcal{D}|^{\alpha-n-1}) = {} & \frac{\cos(((n+\alpha-3)/2)[2m+1]\pi)}{\Gamma((n+3\alpha-3)/2)}(4\pi)^{-n/2}\frac{2rk(E)}{(n+\alpha-1)}\mathrm{Vol}\,(\mathcal{M}) \\ & + j\frac{\sin(((n+\alpha-3)/2)[2m+1]\pi)}{\Gamma((n+3\alpha-3)/2)}(4\pi)^{-n/2}\frac{2rk(E)}{(n+\alpha-1)}\mathrm{Vol}\,(\mathcal{M}), \end{aligned}$$

$j = \sqrt{-1}$.

Proof. By Weyl's theorem, $\exists C \in \mathbb{R}^+ / |\lambda_i| \geq C_i^{1/(n-\alpha+1)}$,

$$C = [rk(E) \cdot \mathrm{Vol}\,(\mathcal{M})]/[(4\pi)^{n/2}\Gamma((n/2)+1)], \quad i = (1, 2, \ldots, N).$$

Therefore, the Dixmier trace of the fractional operator $|\mathcal{D}|^{1-\alpha-n}$ assumed to have non-zero eigenvalue is given by

$$\mathrm{Tr}_\omega(|\mathcal{D}|^{1-\alpha-n}) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{i=1}^{N} |\lambda_i|^{1-\alpha-n} \leq \overline{C} \cdot \lim_{N\to\infty} \frac{1}{\log N} \sum_{i=1}^{N} i^{-1} < \infty,$$

where $\overline{C} \in \mathbb{R}^+$. Let $E \to \mathcal{M}$ be a Riemannian or Hermitian vector bundle over $\mathcal{M}$ and $\Psi \in C^\infty(\mathcal{M})(\mathrm{End}(E))$ be an endomorphism field, i.e. $\Psi = f \cdot \mathbb{I}$ with $f \in C^\infty(\mathcal{M})$. First, we have to manage the integral kernel of $|\mathcal{D}|^{1-\alpha-p}, p > n$, making use of the fractional Mellin transformation. After restriction to the λ-eigenspace of $\mathcal{D}$, we may write simple algebra, making use of the fractional RL-Mellin transformation (3)

$$\begin{aligned}
|\mathcal{D}|^{1-\alpha-p} &= \frac{1}{\Gamma((p+3\alpha-3)/2)} |\lambda|^{1-\alpha-p} \int_0^\infty (t-\tau)^{(p+\alpha-3)/2} e^{t-\tau} d\tau \\
&= \frac{1}{\Gamma((p+3\alpha-3)/2)} |\lambda|^{1-\alpha-p} \int_0^\infty (-T)^{(p+\alpha-3)/2} e^{-T} dT, \qquad (T = \tau - t) \\
&= \frac{1}{\Gamma((p+3\alpha-3)/2)} \int_0^\infty (-s)^{(p+\alpha-3)/2} e^{-\lambda^2 s} ds \\
&= \frac{1}{\Gamma((p+3\alpha-3)/2)} \int_0^\infty (-T)^{(p+\alpha-3)/2} e^{-T\mathcal{D}^2} dT,
\end{aligned}$$

$\dfrac{p-3}{2} < \dfrac{p+\alpha-3}{2} \leq \dfrac{p}{2}$. Therefore the kernel of $|\mathcal{D}|^{1-\alpha-p}$ is

$$k(x, y; |\mathcal{D}|^{1-\alpha-p}) = \frac{(-1)^{(p+\alpha-3)/2}}{\Gamma((p+3\alpha-3)/2)} \int_0^\infty T^{(p+\alpha-3)/2} k_T(x, y) dT,$$

k_T is the heat kernel of the generalized Laplacian $\mathcal{D}^2$. Using the relation:

$$(-1)^\alpha = \exp(j\alpha[2m+1]\pi), \qquad m \in \mathbb{N},$$

we may write the previous equation as

$$\begin{aligned}
k(x, y; |\mathcal{D}|^{1-\alpha-p}) =& \frac{e^{j(p+\alpha-3)/2[2m+1]\pi}}{\Gamma((p+3\alpha-3)/2)} \int_0^\infty T^{(p+\alpha-3)/2} k_T(x, y) dT \\
=& \frac{\cos(((p+\alpha-3)/2)[2m+1]\pi)}{\Gamma((p+3\alpha-3)/2)} \int_0^\infty T^{(p+\alpha-3)/2} k_T(x, y) dT
\end{aligned}$$

$$+j\frac{\sin(((p+\alpha-3)/2)[2m+1]\pi)}{\Gamma((p+3\alpha-3)/2)}\int_0^\infty T^{(p+\alpha-3)/2}k_T(x,y)dT.$$

Obviously, the integral kernel is complexified. Then, $\Psi\circ|\mathcal{D}|^{1-\alpha-p}$ has integral complexified kernel

$$k(x,y;\Psi\circ|\mathcal{D}|^{1-\alpha-p})=\frac{\cos(((p+\alpha-3)/2)[2m+1]\pi)}{\Gamma((p+3\alpha-3)/2)}\int_0^\infty T^{(p+\alpha-3)/2}\Psi\circ k_T(x,y)dT$$
$$+j\frac{\sin(((p+\alpha-3)/2)[2m+1]\pi)}{\Gamma((p+3\alpha-3)/2)}\int_0^\infty T^{(p+\alpha-3)/2}\Psi\circ k_T(x,y)dT,$$

and therefore

$$\mathrm{Tr}(\Psi\circ|\mathcal{D}|^{1-\alpha-p})=\int_{\mathcal{M}}\mathrm{Tr}(k(x,x;\Psi\circ|\mathcal{D}|^{1-\alpha-p}))dV(x)$$
$$=\frac{\cos(((p+\alpha-3)/2)[2m+1]\pi)}{\Gamma((p+3\alpha-3)/2)}\int_0^\infty T^{(p+\alpha-3)/2}\int_{\mathcal{M}}\mathrm{Tr}(\Psi(x)k_T(x,x))dV(x)dT$$
$$+j\frac{\cos(((p+\alpha-3)/2)[2m+1]\pi)}{\Gamma((p+3\alpha-3)/2)}\int_0^\infty T^{(p+\alpha-3)/2}\int_{\mathcal{M}}\mathrm{Tr}(\Psi(x)k_T(x,x))dV(x)dT.$$

For $0<T<T_0$, using the relation $k_T(x,x)=(4\pi T)^{-n/2}\mathbb{I}+\mathrm{O}(T^{-(n/2)+1}), n=\dim\mathcal{M}$, we can write

$$\int_0^{T_0}T^{(p+\alpha-3)/2}\int_{\mathcal{M}}\mathrm{Tr}(\Psi(x)k_T(x,x))dV(x)dT$$
$$=(4\pi)^{-n/2}\int_0^{T_0}T^{(p-n+\alpha-3)/2}\int_{\mathcal{M}}\mathrm{Tr}(\Psi(x)k_T(x,x))dV(x)+\mathrm{O}(T^{(p-n)/2})dT$$
$$=(4\pi)^{-n/2}\frac{2}{p-n+\alpha-1}T_0^{(p-n+\alpha-1)/2}\int_{\mathcal{M}}\mathrm{Tr}(\Psi(x)k_T(x,x))dV(x)+\mathrm{O}(1).$$

Making use of the fractional version of the Connes' Trace Theorem which states that the Dixmier trace is a residue and in particular for

$$\Psi=f\cdot\mathbb{I}\in C^\infty(\mathcal{M})(\mathrm{End}\ (E))$$

and $f=1$, i.e.

$$\mathrm{Tr}(\Psi\circ|\mathcal{D}|^{1-\alpha-n})=\frac{1}{n+\alpha-1}\lim_{p\to n-\alpha+1}(p-n+\alpha-1)\mathrm{Tr}(\Psi\circ|\mathcal{D}|^{-p}),$$

we obtain

$$\begin{aligned}\mathrm{Tr}_\omega(|\mathcal{D}|^{\alpha-n-1}) &= \frac{\cos(((n+\alpha-3)/2)[2m+1]\pi)}{\Gamma((n+3\alpha-3)/2)}(4\pi)^{-n/2}\frac{2rk(E)}{(n+\alpha-1)}\mathrm{Vol}\,(\mathcal{M}) \\ &\quad + j\frac{\sin(((n+\alpha-3)/2)[2m+1]\pi)}{\Gamma((n+3\alpha-3)/2)}(4\pi)^{-n/2}\frac{2rk(E)}{(n+\alpha-1)}\mathrm{Vol}\,(\mathcal{M}) \\ &= \frac{e^{j((n+\alpha-3)/2)[2m+1]\pi}}{\Gamma((n+3\alpha-3)/2)}(4\pi)^{-n/2}\frac{2rk(E)}{(n+\alpha-1)}\mathrm{Vol}\,(\mathcal{M}).\end{aligned}$$

□

Remark 1. It is noticeable that for $\alpha = 1$, we find

$$\begin{aligned}\mathrm{Tr}_\omega(|\mathcal{D}|^{-n}) &= e^{j((n-2)/2)[2m+1]\pi}(4\pi)^{-n/2}\frac{2rk(E)}{n\Gamma(n/2)}\mathrm{Vol}\,(\mathcal{M}) \\ &= \begin{cases}(4\pi)^{-n/2}\dfrac{2rk(E)}{n\Gamma(n/2)}\mathrm{Vol}\,(\mathcal{M}), & n = \text{even} \\ j(4\pi)^{-n/2}\dfrac{2rk(E)}{n\Gamma(n/2)}\mathrm{Vol}\,(\mathcal{M}), & n = \text{odd},\end{cases}\end{aligned} \tag{5}$$

while for $\alpha = 3 + n$, we obtain

$$\mathrm{Tr}_\omega(|\mathcal{D}|^2) \equiv \frac{1}{\Gamma(3+2n)}(4\pi)^{-n/2} rk(E)\cdot \mathrm{Vol}\,(\mathcal{M}). \tag{6}$$

The case where $\alpha = 1$ is remarkable because it shows that the operator theoretic volume element depends on the nature of the dimension of $\mathcal{M}$ with $n < n+1-\alpha \leq n+1$, which may take in our arguments, complex values.

In this fractional-theoretic framework, the Connes' Trace Theorem allows to reconstruct from the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ data, and given the Riemann-measure dm_g over $\mathcal{M}$ on the Riemann spin manifold $(\mathcal{M}, g)$, the Riemann fractional measure of $\mathcal{M}$ [24]:

$$\int_{\mathcal{M}} T dm_g = C(n+1-\alpha)\mathrm{Tr}_\omega(T|\mathcal{D}|^{-(n+1-\alpha)}), \tag{7}$$

where $C(n+1-\alpha)$ is a constant which depends on $n+1-\alpha$. All the above discussions are limited to $n \neq \alpha$.

Remark 2. If we consider a compact $\mathbb{F}$ totally disconnected subset of $\mathbb{R}$ without isolated points, then within our arguments, $\mathbb{F}$ may represents a Minkowski measurable subset with box fractional dimension $0 < \alpha \leq 1$ where $|\mathcal{D}|^{-\alpha} \in \mathcal{L}^{1,\infty}(\mathcal{H})$ and

$$\mathrm{Tr}_\omega(|\mathcal{D}|^{-\alpha}) = 2^\alpha(1-\alpha)\mathcal{M}_\alpha(\mathbb{F}), \tag{8}$$

which follows from Lapidus and Pomerance [28]. See also [5, 17, 18, 21-23].

Remark 3. It has been recently argued that the temporal Riemann-Liouville fractional integral with "complex" fractional exponent has a physical significance. In reality, for true regular discrete fractals, the imaginary part of the fractional integral could be realized and in turn can be observed in disordered material sciences. Recent numerical advances showed that the imaginary part of the complex fractional exponent can be estimated by finite grouping of the primary sine/cosine log-periodical functions with period $\ln \xi, \xi$ is a scaling parameter. The consequent Fourier components give a couple of complex conjugated exponents defining the imaginary part of the complex fractional integral. For random fractals, where invariant scaling properties are recognized merely in the statistical sense the imaginary part of the complex exponent is averaged and the result is reduced to the conventional Riemann-Liouville integral [30].

Motivated by these results, we may replace the real fractional exponent α by its complexified counterpart, i.e. $\alpha \to \alpha - j\epsilon$. The Dirac operator $|\mathcal{D}|^{-\dim \mathcal{M}}, \dim \mathcal{M} = n$ is now fractionalized as follows

$$|D|^{-\dim \mathcal{M}}, \dim \mathcal{M} = n \to |\mathcal{D}|^{-(n+1-\alpha+j\epsilon)}, n < \dim \mathcal{M} = n+1-\alpha+j\epsilon \leq n+1. \tag{9}$$

Accordingly, the following theorem holds:

Theorem 3. *Given the spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Let $E \to \mathcal{M}$ be a Riemannian or Hermitian vector bundle over $\mathcal{M}$ and $\Psi = f \cdot \mathbb{I} \in C^\infty(\mathcal{M})(\mathrm{End}\ (E))$ and $f = 1$, then*

$$\begin{aligned} \mathrm{Tr}_\omega(|\mathcal{D}|^{\alpha - j\epsilon - n - 1}) = &\frac{e^{j((n+\alpha-j\epsilon-3)/2)[2m+1]\pi}}{\Gamma((n+3\alpha-3j\epsilon-3)/2)}(4\pi)^{-n/2} \\ &\times \frac{2rk(E)}{(n+\alpha-j\epsilon-1)}\mathrm{Vol}\ (\mathcal{M}). \end{aligned} \tag{10}$$

The proof is direct.

Notice that for $n = 0$ and $\alpha = 1/2$, then $\dim \mathcal{M} = n + 1 - \alpha + j\epsilon = 1/2 + j\epsilon$ coincides with the critical zeros the Riemann zeta function. More surprisingly, if we replace the complex fractional exponent $\alpha - j\epsilon \to \phi^p - j\epsilon, [p \in \mathbb{N}, \epsilon = \mathfrak{J}(\phi^p - j\epsilon)]$ where $\phi = 1/(1+\phi) = (\sqrt{5}-1)/2 \approx 0.618\ldots$ is the Golden mean, then

$$\begin{aligned} \mathrm{Tr}_\omega(|\mathcal{D}|^{\phi^p - j\epsilon - n - 1}) = &\frac{e^{j((n+\phi^p-j\epsilon-3)/2)[2m+1]\pi}}{\Gamma((n+3\phi^p-3j\epsilon-3)/2)}(4\pi)^{-n/2} \\ &\times \frac{2rk(E)}{(n+\phi^p-j\epsilon-1)}\mathrm{Vol}\ (\mathcal{M}), \end{aligned} \tag{11}$$

and hence, for $n = 1$, we have

$$\mathrm{Tr}_\omega(|\mathcal{D}|^{\phi^p - j\epsilon - 2}) = \frac{e^{j((\phi^p-j\epsilon-2)/2)[2m+1]\pi}}{\Gamma((3\phi^p-3j\epsilon-2)/2)}(4\pi)^{-1/2}\frac{2rk(E)}{(\phi^p - j\epsilon)}\mathrm{Vol}\ (\mathcal{M}). \tag{12}$$

It would be interesting to examine in the future the behavior of the fractional Dixmier trace at $(\phi + j\epsilon), (\phi^2 + j\epsilon), \ldots$ and its connection to fractal strings [3, 18]. For $p = 1$ and $p = 2$, we obtain respectively

$$\begin{aligned}\mathrm{Tr}_\omega(|\mathcal{D}|^{\phi-j\epsilon-2}) &= \frac{e^{j((\phi-j\epsilon-2)/2)[2m+1]\pi}}{\Gamma((3\phi-3j\epsilon-2)/2)}(4\pi)^{-1/2}\frac{2rk(E)}{\phi-j\epsilon}\mathrm{Vol}\,(\mathcal{M})\\
&= \frac{e^{j((\phi-2)/2)[2m+1]\pi}e^{\epsilon[2m+1]\pi/2}}{\Gamma((3\phi-3j\epsilon-2)/2)}(4\pi)^{-1/2}\frac{2rk(E)}{(\phi-j\epsilon)}\mathrm{Vol}\,(\mathcal{M}), \qquad (13)\end{aligned}$$

(14)

$$\begin{aligned}\mathrm{Tr}_\omega(|\mathcal{D}|^{\phi^2-j\epsilon-2}) &= \mathrm{Tr}_\omega(|\mathcal{D}|^{-\phi-j\epsilon-1})\\
&= \frac{e^{j((-\phi-j\epsilon-1)/2)[2m+1]\pi}}{\Gamma((1-3\phi-3j\epsilon)/2)}(4\pi)^{-1/2}\frac{2rk(E)}{1-\phi-j\epsilon}\mathrm{Vol}\,(\mathcal{M})\\
&= \frac{e^{-j((\phi+1)/2)[2m+1]\pi}e^{\epsilon[2m+1]\pi/2}}{\Gamma((1-3\phi-3j\epsilon)/2)}(4\pi)^{-n/2}\frac{2rk(E)}{(1-\phi-j\epsilon)}\mathrm{Vol}\,(\mathcal{M}).\end{aligned}$$

This suggests that the complex fractional exponents could have a geometrical meaning.

Remark 4. We may replace as well $\alpha-j\epsilon \to \alpha+1-j\epsilon$ so that $\dim\mathcal{M} = n-\alpha+j\epsilon$ and hence for $n=1$ and $\alpha=1/2$, then $\dim\mathcal{M} = 1/2+j\epsilon$ as well. In this way, we avoid the special value $\dim\mathcal{M} = 0$ discussed in the previous case.

The results obtained here turn out to be useful to explore many new novel properties and to build bridges between fractal/fractional calculus and noncommutative geometry. However, our main goal in the future is to enlarge the class of discussions with applications to the standard model of particle physics. The fractional formalism we use is extremely recent, and it is still in its infancy. Indeed, all the available inquiries together with the subsequent work are theoretial. Our contribution is, however, only theoretical and, in that sense, more modest. We anticipate that many new interesting features will arise and that will have significant outcomes in quantum field theory and gravity theory. Various other constructions of fractional Dixmier traces on fractals and aperiodic structures [2] are under progress.

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