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SOME NOTES ON WEAKLY REVERSIBLE RINGS

JUNCHAO WEI

Abstract. In this paper, we give some properties of weakly reversible rings, such as weakly reversible rings are left min-abel and directly finite. Next, we show that R is a strongly regular ring if and only if R is a weakly reversible regular ring. Finally, we give some applications: Let R be a weakly reversible ring. Then (1) R is clean if and only if R is exchange; (2) If R is exchange, then R has stable range one; (3) If R is a clean ring, then R is a tb-ring.

1. Introduction

Throughout this paper, all rings are associative with identity. Let R be a ring, we use $N_*(R), N^*(R)$, $N(R)$, $E(R)$ and $U(R)$ to denote the prime radical (i.e., the intersection of all prime ideals), the nilradical (i.e., the sum of all nil ideals), the set of all nilpotent elements in R, the set of all idempotent elements of R and the set of all invertible elements of R, respectively. Note $N_*(R) \subseteq N^*(R) \subseteq N(R)$. According to Cohn [3], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson-Camillo [1], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible; while Krempa-Niewieczerzal [4] took the term C_0 for it. In [14] a generalization of reversible rings is given, that is, a ring R is called *weakly reversible* if $ab = 0$ implies that $Rbra$ is a nil left ideal of R for all $a, b, r \in R$. Clearly semicommutative rings (e.g., $ab = 0$ implies $aRb = 0$ for all $a, b \in R$) are weakly reversible. A ring R is *Abelian* if every idempotent element of R is contained in the central of R. Evidently semicommutative rings are Abelian.

According to [11], an element k of a ring R is called $left\ minimal$ if Rk is a minimal left ideal of R, and an idempotent e of R is said to be $left\ minimal$ *idempotent* if e is a left minimal element of R. We use $ME_l(R)$ to denote the set of all left minimal idempotent elements of R.

According to [11], A ring R is
$left\ min-abel$ if every element of $ME_l(R)$ is left semicentral in R,
$strongly\ left\ min-abel$ if every element of $ME_l(R)$ is central in R,
$left\ MC2$ if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in ME_l(R)$.

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[11, Theorem 1.8] shows that R is a strongly left min-abel ring if and only if R is a left min-abel left $MC2$ ring. Clearly Abelian rings and so semicommutative rings are strongly left min-abel. In this paper, an example is given to show that weakly reversible rings are not left $MC2$. Hence weakly reversible rings are not necessarily Abelian and so not necessarily semicommutative. But we show that weakly reversible rings are left min-abel and directly finite (e.g., $ab = 1$ implies $ba = 1$ for all $a, b \in R$).

Call a left $R-$module M

$nil-injective$ [12] if for each $k \in N(R)$ and any left $R-$morphism $Rk \longrightarrow M$ extends to R.

$GP-injective$ [5] if for each $k \in R$, there exists a positive integer n such that $k^n \neq 0$ and any left $R-$morphism $Rk^n \longrightarrow M$ extends to R.

Clearly, left $GP-$injective modules are $nil-$injective. [5, Lemma 3] shows that if R is a semicommutative ring whose every simple singular left $R-$ module is $GP-$ injective, then R is a reduced ring (e.g., $N(R) = 0$). We show that a ring R is a reduced ring if and only if R is a left $MC2$ weakly reversible ring whose every simple singular left $R-$module is $nil-$injective.

2. Main results

We begin with the following lemma.

Lemma 2.1. *The following conditions are equivalent for a ring R:*

(1) *R is a weakly reversible ring.*
(2) *$al(a) \subseteq N^*(R)$ for any $a \in R$.*
(3) *$r(a)a \subseteq N^*(R)$ for any $a \in R$.*

Proof. (1) $\Longrightarrow$ (2) Let $a \in R$ and $b \in l(a)$. Then $ba = 0$. Since R is a weakly reversible ring, $Rarb \subseteq N^*(R)$ for any $r \in R$. Hence $aRb \subseteq N^*(R)$ for any $b \in l(a)$, so we have $al(a) = aRl(a) \subseteq N^*(R)$.

(2) $\Longrightarrow$ (3) Let $c \in r(a)$. Then $a \in l(c)$. By (2), $cRl(c) \subseteq N^*(R)$. Hence $cRa \subseteq N^*(R)$ for any $c \in r(a)$. This implies $r(a)a = r(a)Ra \subseteq N^*(R)$.

(3) $\Longrightarrow$ (1) Assume that $ab = 0$ in R. By (3), $bRa \subseteq N^*(R)$, so we have $Rbra \subseteq N^*(R)$ for any $r \in R$, which shows that R is a weakly reversible ring. □

Proposition 2.2. *Let R be a weakly reversible ring. Then R is a left min-abel ring.*

Proof. Let $e \in ME_l(R)$ and $a \in R$. Write $h = ae - eae$. If $h \neq 0$, then $eh = 0$, $he = h$ and $Rh = Re$. Since R is a weakly reversible ring, by Lemma 2.1, $h = he \in hl(h) \subseteq N^*(R)$. Especially $Re = Rh \subseteq N^*(R)$, which is a contradiction. Hence $h = 0$, which implies e is left semicentral in R, so R is left min-abel. □

Let D be a division ring. Then the $2-$by-2 upper triangular ring $T_2(D) = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ is a weakly reversible ring by [14, Proposition 2.3]. Clearly $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in ME_l(T_2(D))$. Since $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$ and

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \neq 0$, by [11], we know that $T_2(D)$ is not a left $MC2$ ring. Hence weakly reversible rings are not necessarily strongly left min-abel by [11, Theorem 1.8], so weakly reversible rings are not necessarily Abelian. On the other hand, it is well known that Abelian rings are directly finite, where a ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for all $a, b \in R$. We have the following proposition.

Proposition 2.3. *Weakly reversible rings are directly finite.*

Proof. Let R be a weakly reversible ring and $ab = 1$. Write $e = ba$. Then $(1-e)b = 0$, so, by Lemma 2.1, we have $b(1-e) \in bl(b) \subseteq N^*(R)$ because R is a weakly reversible ring. Hence $1-e = ab(1-e) \subseteq N^*(R)$, which implies $1-e = 0$, that is $ba = e = 1$. Thus R is a directly finite ring. □

A ring R is called
regular if $a \in aRa$ for all $a \in R$.
unit − regular if for any $a \in R$, $a = aua$ for some $u \in U(R)$, where $U(R)$ denotes the group of units of R.
strongly regular if $a \in a^2R$ for all $a \in R$.
$n-$*regular* [12] if $a \in aRa$ for all $a \in N(R)$.
weakly regular if $a \in RaRa \cap aRaR$ for all $a \in R$.
Clearly, strongly regular $\Longrightarrow$ unit-regular $\Longrightarrow$regular $\Longrightarrow$ $n-$ regular;
regular $\Longrightarrow$ weakly regular and strongly regular $\Longrightarrow$ reduced $\Longrightarrow$ $n-$ regular.

Theorem 2.4. *The following conditions are equivalent for a ring R.*

(1) *R is a strongly regular ring.*
(2) *R is a weakly reversible ring and unit-regular ring.*
(3) *R is a weakly reversible ring and regular ring.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are trivial.

(3) $\Longrightarrow$ (1) We only need to show that R is reduced because reduced regular rings are strongly regular. Let $a^2 = 0$ for $a \in R$. Since R is a regular ring, $a = aba$ for some $b \in R$. Let $e = ba$ and $g = ab$, then $e^2 = e, g^2 = g$, $ae = a = ga$ and $eg = ba^2b = 0$, so we obtain $a(1-e) = 0$. Since R is a weakly reversible ring, by Lemma 2.1, $(1-e)a \in (1-e)l(1-e) \subseteq N^*(R)$. Hence $(1-e)g = (1-e)ab \in N^*(R)$. Since $eg = 0$, $g \in N^*(R)$, which implies $g = 0$, so $a = ga = 0$. □

By the proof of Theorem 2.4, we have the following corollary.

Corollary 2.5. *R is a reduced ring if and only if R is a weakly reversible $n-$regular ring.*

A left $MC2$ ring R is called strongly left $MC2$ if R is also a weakly reversible ring. Clearly semicommutative rings are strongly left $MC2$ and strongly left $MC2$ rings are strongly left min-abel. [5, Lemma 3] shows that if R is a semicommutative ring whose simple singular left modules are $GP-$ injective, then R is a reduced ring. We can generalize this result as follows.

Theorem 2.6. *R is a reduced ring if and only if R is a strongly left $MC2$ ring whose simple singular left modules are nil−injective.*

Proof. The necessity is evident.

Conversely, let $a^2 = 0$. Suppose that $a \neq 0$. Then there exists a maximal left ideal M of R containing $l(a)$. First obverse that M is an essential left ideal of R. If not, then $M = l(e)$ for some $e \in ME_l(R)$. Since R is a strongly left $MC2$ ring, R is a strongly left min-abel ring by Proposition 2.2 and [11, Theorem 1.8], so we obtain that e is central in R. Using $a \in l(a)$, we get $ea = ae = 0$. Hence $e \in l(a) \subseteq M = l(e)$, which is a contradiction. Therefore M must be an essential left ideal of R. Thus R/M is $nil-$injective and so any $R-$homomorphism of Ra into R/M extends to the one of R into R/M. Let $f : Ra \longrightarrow R/M$ be defined by $f(ra) = r + M$. Note that f is a well-defined $R-$homomorphism. Since R/M is $nil-$injective, there exists $c \in R$ such that $1 + M = f(a) = ac + M$. Since $a^2 = 0$, $Raca \subseteq N^*(R) \subseteq N(R)$. Hence $ac \in N(R)$ and so $1 - ac \in U(R)$, which implies $M = R$. This contradiction shows that $a = 0$. □

The following corollary generalizes [5, Theorem 4].

Corollary 2.7. *Let R be a strongly left $MC2$ ring. If every simple singular left $R-$module is $GP-$injective, then R is a reduced weakly regular ring.*

Proof. By hypothesis and Theorem 2.6, we obtain that R is a reduced ring. Hence R is a semicommutative ring. Thus, by [5, Theorem 4], R is a weakly regular ring. □

Call an idempotent e of R a *left weakly corner element* if $ReN = N$ for any left $R-$submodule N of Re. Clearly any central idempotent of a ring R is left weakly corner element. Let $e \in E(R)$ such that $ReR = R$, then e is also a left weakly corner element of R.

Theorem 2.8. *Let R be a strongly left $MC2$ ring with $e \in E(R)$. If e satisfies one of the following conditions, then $S = eRe$ is strongly left $MC2$.*

(1) *e is a left weakly corner element of R.*
(2) *e is contained in central of R.*
(3) *$ReR = R$.*

Proof. (1) Note that any subring of a weakly reversible ring is weakly reversible, so if R is weakly reversible, then so is S. Now let $g \in ME_l(S)$ and $a \in S$ such that $aSg = 0$. Then $aRg = aeReg = aSg = 0$. We claim that $g \in ME_l(R)$. In fact for any $x \in R$, if $xg \neq 0$, then $Rxg \subseteq Re$, so we have $ReRxg = Rxg$ because e is a left weakly corner element of R. Hence $eRxeg = eRxg \neq 0$. Since $eRxeg$ is a left ideal of $S = eRe$ and $g \in ME_l(S)$, $eRxg = eReg$. Therefore $Rg = Reg = ReReg = ReRxg \subseteq Rxg \subseteq Rg$, this means that $g \in ME_l(R)$. Since R is a left $MC2$ ring and $aRg = 0$, $gRa = 0$. Hence $gSa = 0$, this implies S is left $MC2$.

(2) and (3) are immediate results of (1). □

Let R be a ring and $a \in R$. Then a is called $\Pi - regular$, if there exists $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$, in case of $n = 1$, a is called *von Neumann regular*, and a is said to be *strongly* $\Pi - regular$, if $a^n = a^{n+1}b$, and in case of $n = 1$, a is called *strongly regular*. A ring R is called $\Pi - regular$ and *strongly* $\Pi - regular$, if every element of R is $\Pi-$regular and strongly $\Pi-$regular, respectively.

Proposition 2.9. *Let R be a weakly reversible ring and $x \in R$. Then*

(1) *If x is von Neumann regular, then x is strongly regular.*
(2) *If x is $\Pi-$regular, then there exists an $e \in E(R)$ such that ex is von Neumann regular.*
(3) *R is $\Pi-$regular if and only if R is strongly $\Pi-$regular.*

Proof. (1) Let $x = xyx$ for some $y \in R$. Write $e = yx$. Then $e^2 = e \in R$ and $x = xe$. Since $x(1 - e) = 0$, $R(1 - e)x$ is a nil left ideal of R. Hence there exists a $n \geq 1$ such that $(y(1 - e)x)^n = 0$. Since $y(1 - e)x = e - yex$ and $x = xe$, there exists $a \in R$ such that $0 = (y(1 - e)x)^n = e - aex$. Therefore $x = xe = xaex = xayx^2 \in Rx^2$. Similarly, we can show that $x \in x^2R$. Hence x is strongly regular.

(2) By hypothesis, there exists a positive integer n such that x^n is regular. By (1), x^n is strongly regular. By [8], $x^n = x^n u x^n$ and $x^n u = u x^n$ for some $u \in U(R)$. Let $e = x^n u$. Then $e \in E(R)$, $x^n = ex^n$ and $x^n = ev$, where $v = u^{-1}$. Since $(ex)(x^{n-1}u)(ex) = ex^n uex = evuex = ex$, ex is von Neumann regular.

(3) Follows from (1). □

A module $_RM$ has the *finite exchange property* if for every module $_RA$ and any two decompositions $A = M^{/} \oplus N = \oplus_{i \in I} A_i$ with $M^{/} \cong M$ and I finite set, there exist submodules $A_i^{/} \subseteq A_i$ such that $A = M^{/} \oplus (\oplus_{i \in I} A_i^{/})$.

In [10], Warfield called a ring R an *exchange* if $_RR$ has the finite exchange property and showed that this definition is left-right symmetric. In [7], Nicholson showed that R is an exchange ring if and only if idempotents can be lifted modulo every left (equivalently, right) ideal of R.

Lemma 2.10. *Let R be a weakly reversible exchange ring. Then $\bar{R} = R/J(R)$ is an Abelian ring.*

Proof. Let $g \in E(\bar{R})$. For any $u \in \bar{R}$, set $v = ug - gug$. Since R is an exchange ring, there exists $e \in E(R)$ such that $\bar{e} = e + J(R) = g$. Let $a \in R$ such that $\bar{a} = u$. Then $\bar{h} = v$, where $h = ae - eae$. Since $eh = 0$ and R is a weakly reversible ring, by Lemma 2.1, $Rh = Rhe \subseteq Rhl(h) \subseteq N^*(R)$. Therefore $h \in J(R)$ which implies $v = 0$, so $ug = gug$. Hence every idempotent element of $\bar{R}$ is left semicentral, we have $\bar{R}$ is Abelian. □

Following [7], a ring R is called *clean* if every element of R is a sum of a unit and an idempotent and clean rings are always exchange rings, but the converse is not true unless R is an Abelian ring. We have the following theorem.

Theorem 2.11. *Let R be a weakly reversible ring. Then R is clean if and only if R is exchange.*

Proof. One direction is trivial.

For the other direction, let R be an exchange ring, then by Lemma 2.10, $R/J(R)$ is Abelian. Therefore $R/J(R)$ is clean by [7], so, by [2, Proposition 7], R is a clean ring. □

Since $\Pi-$ regular rings are exchange, by Theorem 2.11, we have the following corollary.

Corollary 2.12. *Let R be a weakly reversible $\Pi-$regular ring. Then R is clean.*

Recall that a ring R is said to *have stable range* 1 [9] if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. Clearly, R has stable range 1 if and only if $R/J(R)$ has stable range 1. [13, Theorem 6] showed that exchange rings with all idempotents central have stable range 1.

Theorem 2.13. *Weakly reversible exchange rings have stable range 1.*

Proof. Let R be a weakly reversible exchange ring. By Lemma 2.10, then $R/J(R)$ is exchange with all idempotents central. So, by [13, Theorem 6], $R/J(R)$ has stable range 1. Therefore R has stable range 1. □

Naturally, we have the following corollary.

Corollary 2.14. (1) *Weakly reversible $\Pi-$regular rings have stable range 1.*
(2) *Weakly reversible clean rings have stable range 1.*

Proposition 2.15. *Let R be a weakly reversible ring and M a maximal left ideal of R. Then*

(1) *For any $e \in E(R)$, $eR(1-e) \subseteq J(R)$.*
(2) *For any $e \in E(R)$, either $e \in M$ or $1-e \in M$.*
(3) *For any $e \in E(R)$, $Me \subseteq M$.*
(4) *For any $a \in R$ and $e \in E(R)$, $Ra + R(ae-1) = R$.*
(5) *If $a \in N_1(R) = \{x \in R | x^2 = 0\}$, then $Ma \subseteq M$.*
(6) *For any $a \in R$ and $b \in N_1(R)$, $Ra + R(ab-1) = R$.*
(7) *$N_1(R) \subseteq J(R)$.*
(8) *If $e \in E(R)$ and $ReR = R$, then $e = 1$.*

Proof. (1) By definition of weakly reversible rings, (1) is clear.

(2) If $e \notin M$, then $Re + M = R$. By (1), $(1-e)Re \subseteq J(R) \subseteq M$. Thus $1-e \in (1-e)R = (1-e)Re + (1-e)M \subseteq M$.

(3) If $Me \nsubseteq M$, then $Me + M = R$ and $e \notin M$. By (2), $1-e \in M$. Write $me + n = 1$ for some $m, n \in M$. Thus $1-m = me+n-m = m(1-e)+n \in M$, so $1 \in m$, which is a contradiction. Hence $Me \subseteq M$ for any $e \in E(R)$.

(4) If $Ra + R(ae-1) \neq R$ for some $a \in R$ and $e \in E(R)$, then there exists a maximal left ideal N of R containing $Ra + R(ae-1)$. By (3), $Ne \subseteq N$. Hence $ae \in N$ because $a \in N$. Therefore $1 \in N$ because $ae - 1 \in N$. This is impossible, so $Ra + R(ae-1) = R$.

(5) Let $a \in N_1(R)$. If $Ma \nsubseteq M$, then $Ma + M = R$. Write $1 = xa + y$, where $x, y \in M$. Since R is a weakly reversible ring and $a^2 = 0$, $Rara \subseteq N^*(R) \subseteq$

$J(R) \subseteq M$. Thus $axa \in M$, this gives $a = axa + ay \in M$. Hence $Ma \subseteq M$, a contradiction. Thus $Ma \subseteq M$.

(6) This is an immediate result of (5).

(7) If $N_1(R) \nsubseteq J(R)$, then there exists $a \in N_1(R)$ such that $a \notin J(R)$. Thus there exists a maximal left ideal K of R such that $a \notin K$. This gives $Ra+K = R$, so $Ra = Ra^2 + Ka = Ka \subseteq K$ by (5), which contradicts $a \notin K$. Hence $N_1(R) \subseteq J(R)$.

(8) Follows from (1). □

A ring R is called *left topologically boolean*, or a $left\ tb-ring$ for short, if for every pair of distinct maximal left ideals of R there is an idempotent in exactly one of them.

Theorem 2.16. *Let R be a weakly reversible clean ring. Then R is a left tb−ring.*

Proof. Suppose that M and N are distinct maximal left ideals of R. Let $a \in M \backslash N$. Then $Ra + N = R$ and $1 - xa \in N$ for some $x \in R$. Clearly, $xa \in M \backslash N$. Since R is clean, there exists an idempotent $e \in E(R)$ and a unit u in R such that $xa = e + u$. If $e \in M$, then $u = xa - e \in M$. It follows that $R = M$, a contradiction. Thus $e \notin M$. If $e \notin N$, then $1 - e \in N$ by Proposition 2.15 and hence $u = (1-e)+(xa-1) \in N$. It follows that $N = R$ which is also impossible. We thus have that e belongs to N only. □

It is known that the Jacobson radical of $\Pi-$regular ring is nil. Hence we have the following proposition.

Proposition 2.17. *Let R be a weakly reversible $\Pi-$regular ring. Then $N(R) = J(R)$, so R is a NI ring.*

Proof. By Lemma 2.10, $R/J(R)$ is an Abelian ring. By [6, Theorem 4.6], $R/J(R)$ is a reduced ring. Hence $N(R) \subseteq J(R)$. Since $J(R)$ is nil, $J(R) \subseteq N(R)$. Thus $J(R) = N(R)$. □

According to [15], a ring R is called NCI if $N(R) = 0$ or there exists a nonzero ideal of R contained in $N(R)$.

Proposition 2.18. *Weakly reversible rings are NCI.*

Proof. Let R be a weakly reversible ring. If $N(R) = 0$, we are done. If $N(R) \neq 0$, then there exists $0 \neq a \in N(R)$ such that $a^2 = 0$. Since R is a weakly reversible ring, $al(a) \subseteq N^*(R)$ by Lemma 2.1. Hence $Ral(a)R \subseteq N^*(R) \subseteq N(R)$. If $al(a) \neq 0$, then $N(R)$ contains a nonzero ideal $Ral(a)R$. If $al(a) = 0$, then $aRa \subseteq al(a) = 0$ because $Ra \subseteq l(a)$, so $N(R)$ contains a nonzero ideal RaR. Therefore R is a NCI ring. □

By [15], we know that there exists a NCI ring which is not directly finite. Hence, the converse of Proposition 2.18 is not true, in general. Moreover, Proposition 2.3 and Proposition 2.18 imply that there exists a NCI ring which is not weakly reversible.

A ring R is called left *pp* if $_RRa$ is projective for each $a \in R$. In terms of left *pp* rings, we have the following theorem.

Theorem 2.19. *The following conditions are equivalent for a left pp ring R:*

(1) *R is a weakly reversible ring;*
(2) *$ae = 0$ implies $Rera \subseteq N^*(R)$ for all $a, r \in R$ and $e \in E(R)$;*
(3) *$eR(1-e) \subseteq N^*(R)$ for each $e \in E(R)$;*
(4) *$ae = 0$ implies $Rera^n \subseteq N^*(R)$ for all $a, r \in R$, $e \in E(R)$ and $n \in Z^+$.*

Proof. (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3) Let $e \in E(R)$. Since $(1-e)e = 0$, $Rer(1-e) \subseteq N^*(R)$ for each $r \in R$ by (2). Hence $eR(1-e) \subseteq N^*(R)$.

(3) $\Longrightarrow$ (4) Let $ae = 0$. Then for any $n \geq 1$, $a^n = a^n(1-e)$, by (3), $eRa^n = eRa^n(1-e) \subseteq eR(1-e) \subseteq N^*(R)$. Hence for each $r \in R$, $Rera^n \subseteq N^*(R)$.

(4) $\Longrightarrow$ (1) Let $ab = 0$. Since R is a left *pp* ring, $_RRb$ is projective. Hence there exists $e \in E(R)$ such that $l(b) = l(e)$, so $ae = 0$ and $b = eb$. By (4), $Rexa \subseteq N^*(R)$ for any $x \in R$. Especially, $Rbra = Re(br)a \subseteq N^*(R)$. Hence R is a weakly reversible ring. □

By Theorem 2.19, we have the following corollary.

Corollary 2.20. *The following conditions are equivalent for a left pp ring R:*

(1) *R is a weakly reversible ring;*
(2) *$ae = 0$ implies $Rera \subseteq N^*(R)$ for all $a \in N(R)$, $r \in R$ and $e \in E(R)$;*
(3) *$eN(R)(1-e) \subseteq N^*(R)$ for each $e \in E(R)$.*

Proof. (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3) Let $e \in E(R)$. For any $x \in N(R)$, $ex(1-e) \in N(R)$ and $(ex(1-e))e = 0$, by (2), $Rerex(1-e) \subseteq N^*(R)$ for any $r \in R$. Especially, $Rex(1-e) = Reeex(1-e) \subseteq N^*(R)$. Hence $eN(R)(1-e) \subseteq N^*(R)$.

(3) $\Longrightarrow$ (1). For any $e \in E(R)$, $eR(1-e) = e(eR(1-e)(1-e) \subseteq eN(R)(1-e) \subseteq N^*(R)$ by (3), so, by Theorem 2.19, R is a weakly reversible ring. □

Theorem 2.21. *Let R be a ring and Δ be a multiplicatively closed subset of R consisting of central regular elements. Then R is weakly reversible if and only if $\Delta^{-1}R$ is weakly reversible.*

Proof. Note that the class of weakly reversible rings is closed under subrings, so if $\Delta^{-1}R$ is weakly reversible then R is weakly reversible.

Conversely, suppose R is weakly reversible. Put $(u^{-1}a)(v^{-1}b) = 0$, $a, b \in R$ and $u, v \in \Delta$. Then $ab = 0$ and so $Rbra \subseteq N^*(R)$ for each $r \in R$. For any $x = (s^{-1}d)(v^{-1}b)(w^{-1}r)(u^{-1}a) \in \Delta^{-1}R(v^{-1}b)(w^{-1}r)(u^{-1}a)$, $x = (uwvs)^{-1}dbra$. Since $dbra \in N^*(R) \subseteq N(R)$, $x \in \Delta^{-1}N(R) = N(\Delta^{-1}R)$. Hence $\Delta^{-1}R(v^{-1}b)(w^{-1}r)(u^{-1}a) \subseteq N^*(\Delta^{-1}R)$ for any $w^{-1}r \in \Delta^{-1}R$, which implies $\Delta^{-1}R$ is weakly reversible. □

Corollary 2.22. *For a ring R, $R[x]$ is weakly reversible if and only if $R[x; x^{-1}]$ is weakly reversible.*

Proof. Clearly, it suffices to establish necessity. Let $\Delta = \{1; x; x^2; \cdots\}$, then clearly Δ is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = \Delta^{-1}R[x]$, it follows that $R[x; x^{-1}]$ is weakly reversible by Theorem 2.21. □

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School of Mathematics, Yangzhou University,
Yangzhou, 225002, P. R. China
E-mail address: jcweiyz@yahoo.com.cn