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AN APPLICATION OF ZBAGĂNU CONSTANT IN FIXED POINT THEORY

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Abstract. Some sufficient conditions for Banach spaces have normal structure were gotten in terms of Zbagănu constant, the weak orthogonality coefficient and the weakly convergent sequence coefficient, which imply the existence of fixed points for single valued and multivalued nonexpansive mappings.

1. Introduction

We assume that X and X^* stand for a Banach space and its dual space, respectively. By S_X and B_X we denote the unit sphere and the unit ball of a Banach space X, respectively. Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be nonexpansive provided the inequality

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for every $x, y \in C$. A Banach space X is said to have the fixed point property if every nonexpansive mapping $T : C \to C$ has a fixed point, where C is a nonempty bounded closed convex subset of X.

Recall that a Banach space X is called to be uniformly non-square if there exists $\delta > 0$ such that $\|x + y\|/2 \leq 1 - \delta$ or $\|x - y\|/2 \leq 1 - \delta$ whenever $x, y \in S_X$(see [6]). A bounded convex subset K of a Banach space X is said to have normal structure (see [5]) if for every convex subset H of K that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

A Banach space X is said to have weak normal structure if every weakly compact convex subset of X that contains more than one point has normal structure. In reflexive spaces, both notions coincide. A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \sup\{\|x - y\| : x, y \in K\}.$$

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It was proved by W. A. Kirk that every reflexive Banach space with normal structure has the fixed point property (see [10]).

The weakly convergent sequence coefficient $WCS(X)$ of X is defined as follows: $WCS(X) = \inf\{\lim_{n\neq m} \|x_n - x_m\|\}$, where the infimum is taken over all weakly null sequences $\{x_n\}$ in X such that $\lim_{n\to\infty} \|x_n\| = 1$ and $\lim_{n,m\to\infty, n\neq m} \|x_n - x_m\|$ exist (see [2]).

The WORTH property was introduced by B. Sims in [11] as follows: a Banach space X has the WORTH property if

$$\lim_{n\to\infty} \Big| \|x_n + x\| - \|x_n - x\| \Big| = 0$$

for all $x \in X$ and all weakly null sequences $\{x_n\}$. In [7], Jiménez-Melado and Llorens-Fuster defined the coefficient of weak orthogonality, which measures the degree of WORTH whileness, by

$$\mu(X) = \inf\{\lambda : \limsup_{n\to\infty} \|x_n + x\| \leq \lambda \limsup_{n\to\infty} \|x_n - x\|\},$$

where the infimum is taken over all $x \in X$ and all weakly null sequence $\{x_n\}$. It is known that X has the WORTH property if and only if $\mu(X) = 1$.

The following Zbagănu constant $C_Z(X)$, which is a quadratic one introduced in [12] in order to characterize inner product spaces, is defined by

$$C_Z(X) = \sup\left\{ \frac{\|x+y\|\|x-z\|}{\|x\|^2 + \|y\|^2} : x, y \in X, \|x\| + \|y\| > 0 \right\}.$$

It is easy to see that for any normed space

$$1 \leq C_Z(X) \leq C_{\mathrm{NJ}}(X) \leq 2$$

where $C_{\mathrm{NJ}}(X)$ is the von Neumann-Jordan constant (see [9]) defined as

$$C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ and } \|x\| + \|y\| > 0 \right\}.$$

Conditions $C_Z(X) = 1$ as well as $C_{\mathrm{NJ}}(X) = 1$ characterizes inner product spaces. Recently, Alonso and Martin showed that there exists a Banach space X such that $C_Z(X) < C_{\mathrm{NJ}}(X)$.

Let C be a nonempty subset of a Banach space X. We shall denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T : C \to CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, x, y \in C$$

where $H(.,.)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) := \max\{\sup_{x\in A} \inf_{y\in B} \|x - y\|, \sup_{y\in B} \inf_{x\in A} \|x - y\|\}, A, B \in CB(X).$$

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r(C, \{x_n\})$ and the asymptotic center $A(C, \{x_n\})$ of $\{x_n\}$ in C are defined by

$$r(C, \{x_n\}) = \inf\{\limsup_n \|x_n - x\| : x \in C\}$$

and

$$A(C, \{x_n\}) = \{x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\})\},$$

respectively. It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex set whenever C is. The sequence $\{x_n\}$ is called regular with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined by

$$r_C(D) = \inf_{x \in C} \sup_{y \in D} \|x - y\|.$$

S. Dhompongsa et al. [4] introduced the property (D) if there exists $\lambda \in [0, 1)$ such that for any nonempty weakly compact convex subset C of X, any sequence $\{x_n\} \subset C$ which is regular asymptotically uniform relative to C, and any sequence $\{y_n\} \subset A(C, \{x_n\})$ which is regular asymptotically uniform relative to X we have

$$r(C, \{y_n\}) \leq \lambda r(C, \{x_n\}).$$

The Domínguez-Lorenzo condition ((DL)-condition, in short) introduced in [5] is defined as follows: if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C which is regular with respect to C,

$$r_C(A(C, \{x_n\})) \leq \lambda r(C, \{x_n\}).$$

It is clear from the definition that property (D) is weaker than the (DL)-condition. The next results show that property (D) is stronger than weak normal structure and also the existence of fixed points for multivalued nonexpansive mappings [4].

Theorem 1.1. *Let X be a Banach space satisfying property* (D). *Then X has weak normal structure.*

Theorem 1.2. *Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies the property* (D). *Let $T : C \to KC(C)$ be a nonexpansive mapping, then T has a fixed point.*

2. Main results

Theorem 2.1. *Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ be a bounded sequence in C regular with respect to C, then*

$$r_C(A(C, \{x_n\})) \leq C_Z(X) \frac{\mu(X)^2}{\mu(X)^2 + 1} r(C, \{x_n\}).$$

Proof. Denote $r = r(C, \{x_n\})$ and $A = A(C, \{x_n\})$. We can assume $r > 0$. By passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x \in C$. Since $\{x_n\}$ is regular with respect to C, passing to a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Let $z \in A$, then we have

$$\limsup_n \|x_n - z\| = r.$$

Denote $\mu = \mu(X)$. By the definition of μ we have

$$\begin{aligned}\limsup_n \|x_n - 2x + z\| &= \limsup_n \|(x_n - x) + (z - x)\| \\ &\leq \mu \limsup_n \|(x_n - x) - (z - x)\| = \mu r.\end{aligned}$$

The convexity of C implies that $\frac{2}{1+\mu^2}x + \frac{\mu^2-1}{1+\mu^2}z \in C$ and by the definition of r, we obtain

$$\limsup_n \|x_n - (\frac{2}{1+\mu^2}x + \frac{\mu^2-1}{1+\mu^2}z)\| \geq r.$$

On the other hand, by the weak lower semicontinuity of the norm, we have

$$\liminf_n \|(\mu^2 - 1)(x_n - x) - (\mu^2 + 1)(z - x)\| \geq (\mu^2 + 1)\|z - x\|.$$

For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \leq r + \varepsilon$.
(2) $\|x_N - 2x + z\| \leq \mu(r + \varepsilon)$.
(3) $\left\|x_N - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\right)\right\| \geq r - \varepsilon$.
(4) $\|(\mu^2 - 1)(x_N - x) - (\mu^2 + 1)(z - x)\| \geq (\mu^2 + 1)\|z - x\|(\frac{r-\varepsilon}{r})$.

Now, put $u = \mu^2(x_N - z)$, $v = (x_N - 2x + z)$ and use the above estimates to obtain $\|u\| \leq \mu^2(r + \varepsilon)$, $\|v\| \leq \mu(r + \varepsilon)$ and so that

$$\begin{aligned}\|u + v\| &= \|\mu^2((x_N - x) - (z - x)) + (x_N - x) + (z - x)\| \\ &= (\mu^2 + 1)\left\|(x_N - x) - \frac{\mu^2 - 1}{\mu^2 + 1}(z - x)\right\| \\ &= (\mu^2 + 1)\left\|x_N - \left(\frac{2}{\mu^2 + 1}x + \frac{\mu^2 - 1}{\mu^2 + 1}z\right)\right\| \\ &\geq (\mu^2 + 1)(r - \varepsilon),\end{aligned}$$

$$\begin{aligned}\|u - v\| &= \left\|\mu^2((x_N - x) - (z - x)) - \Big((x_N - x) + (z - x)\Big)\right\| \\ &= \left\|(\mu^2 - 1)(x_N - x) - (\mu^2 + 1)(z - x)\right\| \\ &\geq (\mu^2 + 1)\|z - x\|(\frac{r - \varepsilon}{r}).\end{aligned}$$

By the definition of $C_Z(X)$ we get

$$\begin{aligned} C_Z(X) &\geq \left\{ \frac{\|u+v\|\|u-v\|}{\|u\|^2+\|v\|^2} \right\} \\ &\geq \frac{\mu^2+1}{\mu^2}\frac{\|z-x\|}{r}\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^2. \end{aligned}$$

Let $\varepsilon \to 0^+$, we obtain that $C_Z(X) \geq \frac{\mu^2+1}{\mu^2}\frac{\|z-x\|}{r}$. Then we have

$$\|z-x\| \leq C_Z(X)\frac{\mu^2}{\mu^2+1}r.$$

This holds for arbitrary $z \in A$, hence we have

$$r_C(A(C,\{x_n\})) \leq C_Z(X)\frac{\mu(X)^2}{\mu(X)^2+1}r(C,\{x_n\}).$$

□

Corollary 2.1. *If $C_Z(X) < 1 + \frac{1}{\mu(X)^2}$, then X have normal structure.*

Proof. If $C_Z(X) < 1 + \frac{1}{\mu(X)^2}$, then X is a reflexive Banach space. So X has normal structure by Theorem 1.1 and Theorem 2.1. □

Corollary 2.2. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_Z(X) < 1 + \frac{1}{\mu(X)^2}$ and $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

Proof. If $C_Z(X) < 1 + \frac{1}{\mu(X)^2}$, then X satisfies the (DL)-condition by Theorem 2.1. So T has a fixed point by Theorem 1.2. □

Remark. Corollaries 2.1 and 2.2 strengthen the results in [8].

Theorem 2.2. *A Banach space X has property* (D) *whenever $C_Z(X) < WCS(X)$.*

Proof. Let C be a nonempty weakly compact convex subset of X. Suppose that $\{x_n\} \subset C$ and $\{y_n\} \subset A(C,\{x_n\})$ are regular asymptotically uniform relative to C. Passing to a subsequence, we may assume that $\{y_n\}$ is weakly convergent to a point $y_0 \in C$ and $\lim_{n\neq m} \|y_n - y_m\|$ exists. Let $r = r(C,\{x_n\})$. Again, passing to a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, we assume in addition that

$$\lim_{n\neq m} \|y_n - y_m\| = d \geq 0.$$

We can assume that $r > 0$ and $d > 0$, otherwise the conclusion is obvious. For every $\varepsilon > 0$ and $0 < \varepsilon < d \wedge r$ (the minimum of d and r), there exist n, m $(n \neq m)$ such that

$$\Big| \|y_n - y_m\| - d \Big| < \varepsilon, n \neq m.$$

Since $y_n, y_m \in A(C,\{x_n\})$ and $A(C,\{x_n\})$ is a convex set, there exists $N \in \mathbb{N}$ such that

$$\|x_N - y_m\| < r+\varepsilon, \|x_N - y_n\| < r+\varepsilon, \|x_N - \frac{y_n+y_m}{2}\| > r-\varepsilon.$$

By the definition of $C_Z(X)$ we get

$$\begin{aligned} C_Z(X) &\geq \frac{\|2x_N-(y_m+y_n)\|\|y_n-y_m\|}{\|x_N-y_m\|^2+\|x_N-y_n\|^2} \\ &\geq \frac{(r-\varepsilon)(d-\varepsilon)}{(r+\varepsilon)^2}. \end{aligned}$$

Let $\varepsilon \to 0^+$, we obtain that $C_Z(X) \geq \frac{d}{r}$. By the definition of $WCS(X)$, we get

$$C_Z(X) \geq \frac{WCS(X)\limsup_{n,m}\|y_n-y_m\|}{r} \geq \frac{WCS(X)r(C,\{y_n\})}{r}.$$

Therefore

$$r(C,\{y_n\}) \leq \frac{C_Z(X)}{WCS(X)} r(C,\{x_n\}).$$

□

Corollary 2.3. *If $C_Z(X) < WCS(X)$, then X has normal structure.*

Proof. If $C_Z(X) < WCS(X)$, then X is a reflexive Banach space. So X has normal structure by Theorems 1.1 and 2.2. □

Corollary 2.4. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_Z(X) < WCS(X)$ and $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

Proof. If $C_Z(X) < WCS(X)$, then X satisfies the property (D) by Theorem 2.2. So T has a fixed point by Theorem 1.2. □

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