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A NOTE ON THE INEQUALITIES FOR A POLYNOMIAL AND ITS DERIVATIVE

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Abstract. In this paper, we obtain new bounds concerning the maximum moduli of complex polynomials of lacunary type and their derivatives with restricted zeros. These results generalize and refine upon all the earlier results.

1. Introduction

The study on the maximum-minimum modulus theorem and the related inequalities on complex polynomials is a fertile field for researchers, especially mathematical analysts. The growth of the topic is very well explained in [8]. In this paper, a new inequality concerning the maximum moduli of complex polynomials of lacunary type and their derivatives with restricted zeros is obtained. We begin our discussion with some fundamental and important inequalities. In the wake of this, the well-known theorem of Turan [9] on the complex polynomials having all its zeros in the unit disc can be stated as follows.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is sharp and the equality holds in (1.1) *if all zeros of $p(z)$ lie on the unit circle.*

More generally, if the polynomial $p(z)$ has all its zeros in $|z| \leq K \leq 1$, then Malik [7] proved the following inequality.

Theorem 1.2. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The result is sharp and the equality holds in (1.2) *if $p(z) = z^n + K$.*

On the other hand, there is an interesting inequality due to Govil [6] concerning the estimate of $|p'(z)|$ having all its zeros in $|z| \leq K,\ K \geq 1$.

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Theorem 1.3. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K, \ K \geq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |p(z)|. \tag{1.3}$$

The result is sharp and the equality holds in (1.3) if $p(z) = z^n + K^n$.

At the outset, the inequality (1.3) seems sharp. But it has left considerable space for improvement. Note that the bound in (1.3) depends only on the zeros of largest modulus and not on any other zeros, even if many of them are of smaller moduli. This case was very well handled by Govil [5]. Another important point here is, since the extremal polynomial (1.3) is $z^n + K^n$, it must be possible to obtain a sharpest bound for polynomials whenever some of their coefficients are zero or they are of lacunary type, $p(z) = a_n z^n + \sum_{k=0}^{t} a_k z^k, \quad t \leq n-1$. So the inequality (1.3) can be bettered for such polynomials if we obtain a bound which depends on the nonzero coefficients of $p(z)$, the location of the zeros of the given polynomial and more importantly, the new parameter t. At this stage, it is quite interesting and natural to ask how does the inequality look if none of the zeros are within the unit disc. In this connection, this paper makes an attempt to prove a new inequality which overcomes some drawbacks appearing in earlier papers. In fact, we state a theorem more generally as follows.

Theorem 1.4. *If $p(z) = a_n z^n + \sum_{k=0}^{t} a_k z^k = a_n \prod_{k=1}^{n} (z - z_k), \quad t \leq n-1$ with $a_0 \neq 0$, is a polynomial of degree $n > 3$ having all its zeros in $|z| \leq K, \ K \geq 1$, then*

$$\begin{aligned}
\max_{|z|=1} |p'(z)| \geq & \frac{2K^n}{(K^{n-1}+1)B} \sum_{k=1}^{n} \frac{1}{1+|z_k|} \max_{|z|=1} |p(z)| \\
& + \frac{nK^n \left(\frac{K^n-1}{n} + K - 1\right) \left(\frac{1+\frac{(n-t)|a_t|}{n(|a_n|-m)}}{2+\frac{(n-t)|a_t|}{n(|a_n|-m)}}\right)}{(K^{n-1}+1)B} \sum_{k=1}^{n} \frac{1}{1+|z_k|} m \\
& + \frac{2|a_{n-2}|K^n \left[\frac{K^n-1}{n(n-1)} - \frac{K^{n-2}-1}{(n-2)(n-3)} + \frac{2(K-1)}{(n-1)(n-3)}\right]}{(K^{n-1}+1)B} \sum_{k=1}^{n} \frac{1}{1+|z_k|} \\
& + \frac{4}{K^{n-1}+1} \left(\frac{K^{n-1}-1}{n-1} - \frac{K^{n-3}-1}{n-3}\right) |a_2|
\end{aligned} \tag{1.4}$$

where

$$B = 1 + \frac{n}{2} \left(\frac{K^n-1}{n} + K - 1\right) \left(\frac{1+\frac{(n-t)|a_t|}{n(|a_n|-m)}}{2+\frac{(n-t)|a_t|}{n(|a_n|-m)}}\right), \quad m = \min_{|z|=1} |p(z)|.$$

The result is sharp and the equality holds in (1.4) if $p(z) = z^n + K^n$.

2. Lemmas

For the proof of the theorem, we need the following lemmas. The first lemma is the famous Gauss-Lucas theorem [1], a consequence of which is used in this article quite a few times.

Lemma 2.1. *If all the zeros of a polynomial $p(z)$ lie in a half plane, then all the zeros of the derivative of $p(z)$ also lie in the same half plane.*

The next lemma is due to Girox, Rahman and Schmeisser [4].

Lemma 2.2. *If $p(z) = a_n \prod_{k=1}^{n} (z - z_k)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \sum_{k=1}^{n} \frac{1}{1 + |z_k|} \max_{|z|=1} |p(z)|. \tag{2.1}$$

Equality in (2.1) holds if every z_k is positive real.

The following lemma is due to Frappier, Rahman and Ruscheweyh [2].

Lemma 2.3. *If $p(z)$ is a polynomial of degree $n \geq 2$, then for all $R > 1$,*

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - (R^n - R^{n-2})|p(0)|.$$

Govil [5] derived the general version of the above theorem for the polynomials having no zeros in the unit disc, which is stated below.

Lemma 2.4. *If $p(z)$ is a polynomial of degree $n > 2$, having no zeros in $|z| < 1$, then for any $R \geq 1$,*

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right) |p'(0)|. \tag{2.2}$$

Equality holds in (2.2) if $p(z) = z^n + 1$.

Recently Gardner, Govil and Weems [3] proved an important inequality on lacunary polynomials, which is our next lemma.

Lemma 2.5. *If $p(z) = a_0 + \sum_{k=t}^{n} a_k z^k$, $t \geq 1$, is a polynomial of degree n having no zeros in $|z| < 1$, then*

$$\max_{|z|=1} |p'(z)| \leq n \left(\frac{1 + \frac{t|a_t|}{n(|a_0|-m)}}{2 + \frac{t|a_t|}{n(|a_0|-m)}} \right) \left(\max_{|z|=1} |p(z)| - m \right) \tag{2.3}$$

where $m = \min_{|z|=1} |p(z)|$.

There is an equality in (2.3) if $p(z) = z^n + 1$.

Now we prove a lemma which will play a crucial role in proving our theorem.

Lemma 2.6. *If* $p(z) = a_0 + \sum_{k=t}^{n} a_k z^k, \quad t \geq 1,$ *is a polynomial of degree* $n > 3$ *having no zeros in* $|z| < 1$*, then for any* $R \geq 1$*,*

$$\max_{|z|=1} |p(z)| \leq \left[1 + \frac{n}{2}\left(\frac{R^n - 1}{n} + R - 1\right)\left(\frac{1 + \frac{t|a_t|}{n(|a_0|-m)}}{2 + \frac{t|a_t|}{n(|a_0|-m)}}\right)\right] \max_{|z|=1} |p(z)|$$
$$- \left[\frac{n}{2}\left(\frac{R^n - 1}{n} + R - 1\right)\left(\frac{1 + \frac{t|a_t|}{n(|a_0|-m)}}{2 + \frac{t|a_t|}{n(|a_0|-m)}}\right)\right] m$$
$$-2\ |a_2| \left[\frac{R^n - 1}{n(n-1)} - \frac{R^{n-2} - 1}{(n-2)(n-3)} + \frac{2(R-1)}{(n-1)(n-3)}\right]$$

where $m = \min_{|z|=1} |p(z)|$.

Proof. Let the polar representation of z be $z = re^{i\theta}$. Then for every θ with $0 \leq \theta < 2\pi$, we have

$$p(Re^{i\theta}) - p(e^{i\theta}) = \int_1^R e^{i\theta} p'(re^{i\theta}) dr.$$

But then

$$|p(Re^{i\theta}) - p(e^{i\theta})| \leq \int_1^R |p'(re^{i\theta})| dr.$$

Note that by Gauss Lucas theorem, it follows that $p'(z)$ also has no zeros in $|z| < 1$. Now applying Lemma 2.4 and then Lemma 2.5 to the polynomial $p'(z)$ which is of degree greater than 2, we get

$$|p(Re^{i\theta}) - p(e^{i\theta})|$$
$$\leq \int_1^R \left[\frac{r^{n-1} + 1}{2} \max_{|z|=1} |p'(z)| - 2|a_2| \left(\frac{r^n - 1}{n-1} - \frac{r^{n-3} - 1}{n-3}\right)\right] dr$$
$$= \frac{1}{2}\left(\frac{R^n - 1}{n} + R - 1\right) \max_{|z|=1} |p'(z)|$$
$$-2|a_2| \left[\frac{R^n - 1}{n(n-1)} - \frac{R^{n-2} - 1}{(n-2)(n-3)} + \frac{2(R-1)}{(n-1)(n-3)}\right]$$
$$\leq \frac{n}{2}\left(\frac{R^n - 1}{n} + R - 1\right)\left(\frac{1 + \frac{t|a_t|}{n(|a_0|-m)}}{2 + \frac{t|a_t|}{n(|a_0|-m)}}\right) \max_{|z|=1} |p(z)|$$
$$-\frac{n}{2}\left(\frac{R^n - 1}{n} + R - 1\right)\left(\frac{1 + \frac{t|a_t|}{n(|a_0|-m)}}{2 + \frac{t|a_t|}{n(|a_0|-m)}}\right) m$$
$$-2|a_2| \left[\frac{R^n - 1}{n(n-1)} - \frac{R^{n-2} - 1}{(n-2)(n-3)} + \frac{2(R-1)}{(n-1)(n-3)}\right].$$

Applying the property of difference between moduli of two functions to the left hand side of the above inequality and then rearranging, we get the desired inequality. Hence the proof is established. □

3. Proof of the theorem

Note that the polynomial $p(z)$ is of degree $n > 3$. Since the zeros of $p(z)$ are z_k $(1 \leq k \leq n)$, the zeros of the polynomial $q(z) = p(Kz)$ are $\frac{z_k}{K}(1 \leq k \leq n)$. Also observe that, the zeros of $q(z)$ all lie in $|z| \leq 1$, since all the zeros of $p(z)$ lie in $|z| \leq K,\ \ K \geq 1$. Now by Lemma 2.2 we have

$$\max_{|z|=1} |q'(z)| \geq \sum_{k=1}^{n} \frac{1}{1+\frac{|z_k|}{K}} \max_{|z|=1} |q(z)|$$

which gives

$$\max_{|z|=K} |p'(z)| \geq \sum_{k=1}^{n} \frac{1}{K+|z_k|} \max_{|z|=K} |p(z)|. \tag{3.1}$$

Since the degree of $p(z)$ is greater than 3, the degree of $p'(z)$ is greater than 2. Also by the Gauss Lucas theorem, $p'(z)$ has all its zeros in $|z| \leq K,\ \ K \geq 1$. Therefore applying Lemma 2.4 to $p'(z)$ we get, for $K \geq 1$,

$$\max_{|z|=K} |p'(z)| \leq \frac{K^{n-1}+1}{2} \max_{|z|=1} |p'(z)| - 2\left(\frac{K^{n-1}-1}{n-1} - \frac{R^{n-3}-1}{n-3}\right)|a_2|. \tag{3.2}$$

From Equations (3.1) and (3.2) it follows that

$$\sum_{k=1}^{n} \frac{1}{K+|z_k|} \max_{|z|=K} |p(z)| \leq \frac{K^{n-1}+1}{2} \max_{|z|=1} |p'(z)|$$
$$-2\left(\frac{K^{n-1}-1}{n-1} - \frac{R^{n-3}-1}{n-3}\right)|a_2|. \tag{3.3}$$

Denote $r(z) = z^n p(1/z)$. Since the polynomial $p(z)$ has all its zeros in $|z| \leq K,\ \ K \geq 1$, the polynomial $r(z/K)$ has all its zeros in $|z| \geq 1$. Also observe that

$$\max_{|z|=1} |r(z/K)| = \frac{1}{K^n} \max_{|z|=K} |p(z)| \quad \max_{|z|=K} |r(z/K)| = \max_{|z|=1} |p(z)|$$

and

$$\min_{|z|=K} |r(z/K)| = m = \min_{|z|=1} |p(z)|.$$

Now applying Lemma 2.6 to the polynomial $r(z/K)$, we get for $K \geq 1$,

$$\max_{|z|=K} |r(z/K)|$$
$$\leq \left[1 + \frac{n}{2}\left(\frac{K^n-1}{n} + K - 1\right)\left(\frac{1+\frac{(n-t)|a_t|}{n(|a_n|-m)}}{2+\frac{(n-t)|a_t|}{n(|a_n|-m)}}\right)\right] \max_{|z|=1} |r(z/K)|$$
$$-\left[\frac{n}{2}\left(\frac{K^n-1}{n} + K - 1\right)\left(\frac{1+\frac{(n-t)|a_t|}{n(|a_n|-m)}}{2+\frac{(n-t)|a_t|}{n(|a_n|-m)}}\right)\right] m$$
$$-2|a_{n-2}|\left[\frac{K^n-1}{n(n-1)} - \frac{K^{n-2}-1}{(n-2)(n-3)} + \frac{2(K-1)}{(n-1)(n-3)}\right].$$

Hence we have

$$\frac{1}{K^n}\left[1+\frac{n}{2}\left(\frac{K^n-1}{n}+K-1\right)\left(\frac{1+\frac{(n-t)|a_t|}{n(|a_n|-m)}}{2+\frac{(n-t)|a_t|}{n(|a_n|-m)}}\right)\right]\max_{|z|=K}|p(z)|$$

$$\geq \max_{|z|=1}|p(z)|+\left[\frac{n}{2}\left(\frac{K^n-1}{n}+K-1\right)\left(\frac{1+\frac{(n-t)|a_t|}{n(|a_n|-m)}}{2+\frac{(n-t)|a_t|}{n(|a_n|-m)}}\right)\right]m$$

$$+\,2|a_{n-2}|\left[\frac{K^n-1}{n(n-1)}-\frac{K^{n-2}-1}{(n-2)(n-3)}+\frac{2(K-1)}{(n-1)(n-3)}\right]. \tag{3.4}$$

Combining the equations (3.3) and (3.4) we get

$$\frac{K^{n-1}+1}{2K^n}B\max_{|z|=1}|p'(z)|-\frac{2}{K^n}B\left(\frac{K^{n-1}-1}{n-1}-\frac{K^{n-3}-1}{n-3}\right)|a_2|$$

$$\geq \sum_{k=1}^{n}\frac{1}{1+|z_k|}\max_{|z|=1}|p(z)|$$

$$+\quad \sum_{k=1}^{n}\frac{1}{1+|z_k|}\left[\frac{n}{2}\left(\frac{K^n-1}{n}+K-1\right)\left(\frac{1+\frac{(n-t)|a_t|}{n(|a_n|-m)}}{2+\frac{(n-t)|a_t|}{n(|a_n|-m)}}\right)\right]m$$

$$+\,2\,|a_{n-2}|\sum_{k=1}^{n}\frac{1}{1+|z_k|}\left[\frac{K^n-1}{n(n-1)}-\frac{K^{n-2}-1}{(n-2)(n-3)}+\frac{2(K-1)}{(n-1)(n-3)}\right],$$

where

$$B=1+\frac{n}{2}\left(\frac{K^n-1}{n}+K-1\right)\left(\frac{1+\frac{(n-t)|a_t|}{n(|a_n|-m)}}{2+\frac{(n-t)|a_t|}{n(|a_n|-m)}}\right).$$

Simple rearrangements give the required inequality and hence the proof is complete. □

Remark 1. One can easily observe that Lemma 2.2 is a particular case of Theorem 1.4 when $K=1$ and $n>3$. Therefore Theorem 1.4 can be regarded as a generalization of Lemma 2.2.

Remark 2. The inequalities for the cases $n=2,3$ are simple and can similarly be derived using Lemma 2.3 instead of Lemma 2.4. I leave the details of it to the readers.

Remark 3. In fact the result obtained in Theorem 1.4 is very much better than that given in Theorem 1.3. To illustrate this, we consider this example. Let $p(z)$ be any lacunary polynomial of degree $n=4$, $K=\sqrt{2}$ and $t=2$, such that $\max_{|z|=1}|p(z)|=1$. Then Theorem 1.3 gives a lower bound for $|p'(z)|$ as 0.8, while Theorem 1.4 gives a value greater than 1.59 for the same lower bound. The bound for $|p'(z)|$ given in this paper is thus very much sharper than that given in Theorem 1.3.

Thus, this paper established a refined result on the inequality for the estimation of polynomials and their derivatives. The numerical example is also given to expose the quantum of improvement in the bound with its feasibility. The future work on this may be done on other special classes of polynomials with restricted zeros about which the subsequent theory to be developed. These classes are all distinguished by the location of their zeros relative to the unit circle, i.e., by the absolute values of their zeros rather than by their arguments. Readers are encouraged to work on such special classes of polynomials such as Neumann polynomials, Self inversive polynomials etc.

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