ACTA MATHEMATICA VIETNAMICA
Volume 37, Number 1, 2012, pp. 53–61

A NOTE ON WEAKLY s-PERMUTABLY EMBEDDED AND WEAKLY s-SUPPLEMENTED SUBGROUPS

CHANGWEN LI

Abstract. Suppose that G is a finite group and H is a subgroup of G. H is called weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$; H is called weakly s-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. We investigate the influence of weakly s-permutably embedded and weakly s-supplemented subgroups on the p-nilpotency of finite groups. Some recent results are generalized.

1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation. G always means a group and $|G|$ denotes the order of G. Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation, provided that (1) if $G \in \mathcal{F}$ and $H \lhd G$, then $G/H \in \mathcal{F}$, and (2) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for any normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$.

A subgroup H of a group G is said to be s-permutable in G if H permutes with all Sylow subgroups of G, i.e., $HS = SH$ for any Sylow subgroup S of G. This concept was introduced by Kegel in [1]. From Ballester-Bolinches and Pedraza-Aguilera [2], we know that H is said to be s-permutably embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. In [3], Yangming Li showed the theorem: Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If G is A_4-free and all 2-maximal subgroups of P are s-permutably embedded in G, then G is p-nilpotent.

As a generalization of the above subgroups, Yangming Li [10] introduced a new subgroup embedding property in a finite group called weakly s-permutably embedded subgroup again. A subgroup H of group G is called weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$.

Received September 27, 2010; in revised form July 7, 2011.
2000 *Mathematics Subject Classification.* Primary: 20D10, 20D20.
Key words and phrases. Weakly s-permutably embedded, weakly s-supplemented, p-nilpotent.

As another generalization of s-permutable subgroups, Skiba [11] introduced the following concept: A subgroup H of a group G is called weakly s-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. In fact, this concept is also a generalization of c-supplemented subgroups given in [4]. Skiba proposed in [11] two open questions related to weakly s-supplemented subgroups. In this paper we are concerned with another problem in this context. There are examples to show that weakly s-supplemented subgroups are not weakly s-permutably embedded subgroups and in general the converse is also false. The aim of this article is to unify and improve some earlier results using weakly s-permutably embedded and weakly s-supplemented subgroups, such as, [3, 4, 12].

2. Preliminaries

Lemma 2.1.([10], Lemma 2.5) *Let H be a weakly s-permutably embedded subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is weakly s-permutably embedded in L.*

(2) *If $N \lhd G$ and $N \leq H \leq G$, then H/N is weakly s-permutably embedded in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-permutably embedded in G/N.*

Lemma 2.2.([11], Lemma 2.10) *Let H be a weakly s-supplemented subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is weakly s-supplemented in L.*

(2) *If $N \lhd G$ and $N \leq H \leq G$, then H/N is weakly s-supplemented in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-supplemented in G/N.*

Lemma 2.3.([6], Lemma 3.12) *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. If G is A_4-free and $|P| \leqslant p^2$, then G is p-nilpotent.*

Lemma 2.4.([7], A, 1.2) *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(1) *$U \cap VW = (U \cap V)(U \cap W)$;*

(2) *$UV \cap UW = U(V \cap W)$.*

Lemma 2.5.([3], Lemma 2.3) *Suppose that H is s-permutable in G, P is a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-permutable in G.*

Lemma 2.6.([3], Lemma 2.2) *If P is an s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.7.([3], Lemma 2.4) *Suppose P is a p-subgroup of G contained in $O_p(G)$. If P is s-permutably embedded in G, then P is s-permutable in G.*

Lemma 2.8.([8], Lemma 2.6) *Let H be a solvable normal subgroup of a group G $(H \neq 1)$. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.9.([6], Lemma 3.16) *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type. Also let P be a normal p-subgroup of a group G such that $G/P \in \mathcal{F}$. If G is A_4-free and $|P| \leqslant p^2$, then $G \in \mathcal{F}$.*

Lemma 2.10. *Let p be the smallest prime dividing $|G|$. Suppose that P is a Sylow p-subgroup of G such that every maximal subgroup of P is p-nilpotent supplemented in G. Then G is p-nilpotent.*

Proof. If $p^2 \nmid |G|$, then P is cyclic and so G is p-nilpotent by Burnside's theorem. Thus, we can suppose that $p^2 \| |G|$. Let P_1 be a maximal subgroup of P. By the hypothesis, P_1 has a p-nilpotent supplement K_1 in G. Let $K_{1p'}$ be a normal Hall p'-subgroup of K_1. Then $G = P_1K_1 = P_1N_G(K_{1p'})$. Obviously, $K_{1p'}$ is a Hall p'-subgroup of G. We claim that $K_{1p'}$ is normal in G. Indeed, if $K_{1p'}$ is not normal in G, then $P \cap N_G(K_{1p'}) < P$. It follows that P has a maximal subgroup P_2 such that $P \cap N_G(K_{1p'}) \leq P_2$. It is clear $P_1 \neq P_2$. By the hypothesis, P_2 is also p-nilpotent supplemented in G. By repeating the above argument, we can find a Hall p'-subgroup $K_{2p'}$ of G such that $G = P_2K_2 = P_2N_G(K_{2p'})$. If $p = 2$, then $K_{1p'}$ and $K_{2p'}$ are conjugate in G by applying a deep result of Gross ([9], the main theorem). If $p > 2$, then G is a soluble group by Feit-Thompson's theorem and hence $K_{1p'}$ and $K_{2p'}$ are conjugate in G. Since $K_{2p'}$ is normalized by K_2, there exists an element $g \in P_2$ such that $K_{2p'}^g = K_{1p'}$. Then

$$G = (P_2N_G(K_{2p'}))^g = P_2N_G(K_{1p'}).$$

This induces that

$$P = P \cap G = P \cap P_2N_G(K_{1p'}) = P_2(P \cap N_G(K_{1p'})) = P_2.$$

This contradiction completes the proof. □

Lemma 2.11. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If G is A_4-free and every 2-maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent.*

Proof. Let P_2 be a 2-maximal subgroup of P. By our hypotheses, P_2 is weakly s-permutably embedded in G, then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_2)_{se}$ of G contained in P_2 such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. If P_2 is not s-permutably embedded in G, then G has a normal subgroup M such that $G = P_2M$ and $|G : M| = p$ by [10, Lemma 2.5(d)]. It follows that every maximal subgroup of $P \cap M$ is weakly s-permutably embedded in M by Lemma 2.1(1). Hence we have M is p-nilpotent by [10, Theorem 3.1]. It is easy to see that G is p-nilpotent. Now we may assume that every 2-maximal subgroup of P is s-permutably embedded in G. By [3, Theorem 3.3], we get G is p-nilpotent too. □

3. Main results

Theorem 3.1. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. Suppose that every 2-maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in G. If G is A_4-free, then G is p-nilpotent.*

Proof. Assume that the theorem is false and choose G to be a counterexample of minimal order. We have the following assertions.

(1) By Lemma 2.3, $|P| \geqslant p^3$ and so every 2-maximal subgroup P_2 of P is non-identity.

(2) G is not a non-abelian simple group.

Suppose that G is simple. Since G is a counterexample, P has a maximal subgroup P_1 which has no p-nilpotent supplement in G by Lemma 2.10. It follows that for any 2-maximal subgroup P_2 of P contained in P_1, P_2 has no p-nilpotent supplement in G. By the hypothesis, P_2 is either weakly s-permutably embedded or weakly s-supplemented in G. If P_2 is weakly s-permutably embedded in G, then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_2)_{se}$ of G contained in P_2 such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. Obviously, $T = G$ and so $P_2 = (P_2)_{se}$ is s-permutably embedded in G. Thus there is an s-permutable subgroup K of G such that P_2 is a Sylow p-subgroup of K. Since G is simple, we have $K_G = 1$. By Lemma 2.5, P_2 is s-permutable in G. Therefore $N_G(P_2) \geq O^p(G) = G$ by Lemma 2.6. It follows that $P_2 \trianglelefteq G$, a contradiction. If P_2 is weakly s-supplemented in G, then there is a non-p-nilpotent subgroup T of G such that $G = P_2T$ and

$$P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = 1.$$

It follows that $|T|_p = p^2$. Obviously, T is also A_4-free. Therefore T is p-nilpotent by Lemma 2.3, a contradiction.

(3) G has a unique minimal normal subgroup N. Moreover G/N is p-nilpotent, and $\Phi(G) = 1$.

By step (2), we may assume that G has a minimal normal subgroup N. We shall prove that G/N satisfies the hypothesis of our theorem. Since P is a Sylow p-subgroup of G, PN/N is a Sylow p-subgroup of G/N. If $|PN/N| \leqslant p^2$, then G/N is p-nilpotent by Lemma 2.3. Thus we can suppose $|PN/N| \geqslant p^3$. Let M_2/N be a 2-maximal subgroup of PN/N. Then $M_2 = M_2 \cap PN = N(M_2 \cap P)$. Let $P_2 = M_2 \cap P$. It follows that $P_2 \cap N = M_2 \cap P \cap N = P \cap N$ is a Sylow p-subgroup of N. Since

$$|P : P_2| = |P : M_2 \cap P| = |PN : (M_2 \cap P)N| = |PN/N : M_2/N| = p^2,$$

we have P_2 is a 2-maximal subgroup of P. If P_2 is weakly s-supplemented in G, then there is a subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{sG}$.

Therefore,

$$G/N = M/N \cdot TN/N = P_2N/N \cdot TN/N.$$

Since $(|N : P_2 \cap N|, |N : T \cap N|) = 1$, we have

$$(P_2 \cap N)(T \cap N) = NN \cap G = N \cap P_2T.$$

By Lemma 2.4, $(P_2N) \cap (TN) = (P_2 \cap T)N$. It follows that

$$(P_2N/N) \cap (TN/N)(P_2N \cap TN)/N = (P_2 \cap T)N/N \leq (P_2)_{sG}N/N \leq (P_2N/N)_{sG}.$$

Hence M/N is weakly s-supplemented in G/N. If P_2 is weakly s-permutably embedded in G, then we can prove M_2/N is weakly s-permutably embedded in G/N too. Therefore, G/N satisfies the hypothesis of the theorem. The minimal choice of G yields that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(4) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by step (3). Since

$$G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$$

and G/N is p-nilpotent, we have G is p-nilpotent, a contradiction.

(5) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (3) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, $O_p(G) \cap M \lhd G$. The uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Then $P = NP_1$. Take a 2-maximal subgroup P_2 of P such that $P_2 \leq P_1$. By the hypothesis, P_2 is either weakly s-permutably embedded or weakly s-supplemented in G. If P_2 is weakly s-permutably embedded in G, then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_2)_{se}$ of G contained in P_2 such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. So there is an s-permutable subgroup K of G such that $(P_2)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \leq (P_2)_{se} \leq P_1$, and so $P = N(P \cap M) = NP_1 = P_1$, a contradiction. If $K_G = 1$, by Lemma 2.5, $(P_2)_{se}$ is s-permutable in G. From Lemma 2.6 we have $O^p(G) \leq N_G((P_2)_{se})$. Since $(P_2)_{se}$ is subnormal in G, $P_2 \cap T \leq (P_2)_{se} \leq O_p(G) = N$. Thus, $(P_2)_{se} \leq P_1 \cap N$ and

$$(P_2)_{se} \leq ((P_2)_{se})^G = ((P_2)_{se})^{O^p(G)P} = ((P_2)_{se})^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N.$$

It follows that $((P_2)_{se})^G = 1$ or $((P_2)_{se})^G = P_1 \cap N = N$. If $((P_2)_{se})^G = 1$, then $P_2 \cap T = 1$ and so $|T|_p = p^2$. Hence T is p-nilpotent by Lemma 2.3. Let $T_{p'}$ be the normal p-complement of T, then $T_{p'}$ is a normal Hall p'-subgroups of G since $T \lhd \lhd G$, a contradiction. If $((P_2)_{se})^G = P_1 \cap N = N$, then $N \leq P_1$ and so $P = P_1$, a contradiction. Therefore we may assume that P_2 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_2T$ and

$P_2 \cap T \leq (P_2)_{sG}$. From Lemma 2.6, we have $O^p(G) \leq N_G((P_2)_{sG})$. Since $(P_2)_{sG}$ is subnormal in G, $P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = N$. Thus, $(P_2)_{sG} \leq P_1 \cap N$ and

$$(P_2)_{sG} \leq ((P_2)_{sG})^G = ((P_2)_{sG})^{O^p(G)P} = ((P_2)_{sG})^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N.$$

It follows that $((P_2)_{sG})^G = 1$ or $((P_2)_{sG})^G = P_1 \cap N = N$. If $((P_2)_{sG})^G = P_1 \cap N = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. If $((P_2)_{sG})^G = 1$, then $P_2 \cap T = 1$ and so $|T|_p = p^2$. Hence T is p-nilpotent by Lemma 2.3. Let $T_{p'}$ be the normal p-complement of T. Since M is p-nilpotent, we may suppose that M has a normal Hall p'-subgroup $M_{p'}$ and $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \lhd G$, $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence we must have $M = N_G(M_{p'})$. By applying a deep result of Gross ([9], the main theorem) and Feit-Thompson's theorem, there exists $g \in G$ such that $T_{p'}^g = M_{p'}$. Hence

$$T^g \leq N_G(T_{p'}^g) = N_G(M_{p'}) = M.$$

However, $T_{p'}$ is normalized by T, so g can be considered as an element of P_2. Thus $G = P_2T^g = P_2M$ and $P = P_2(P \cap M) = P_1$, a contradiction.

(6) G has a Hall p'-subgroup and any two Hall p'-subgroups of G are conjugate in G.

If every 2-maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent by Lemma 2.11, a contradiction. Thus there is a 2-maximal subgroup P_2 of P such that P_2 is weakly s-supplemented in G. Then there exists a subgroup T of G such that $G = P_2T$ and

$$P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = 1.$$

By Lemma 2.3, T is p-nilpotent and so T has the normal p-complement $T_{p'}$. Obviously, $T_{p'}$ is also a Hall p'-subgroup of G. A new application of the result by Gross ([9], the main theorem) and Feit-Thompson's theorem yield any two Hall p'-subgroups of G are conjugate in G.

(7) Conclusion.

If $NP < G$, then NP satisfies the hypothesis of the theorem. The minimal choice of G yields that NP is p-nilpotent. Let $N_{p'}$ be the normal p-complemented of N. It is clear that $N_{p'} \lhd G$, so that $N_{p'} = 1$ by step (4) and N is a non-trivial p-group, contrary to step (5). Therefore we must have $G = NP$. By step (6), G has a Hall p'-subgroup. Then we may suppose that N has a Hall p'-subgroup $N_{p'}$. By Frattini's argument,

$$G = NN_G(N_{p'}) = (P \cap N)N_{p'}N_G(N_{p'}) = (P \cap N)N_G(N_{p'})$$

and so

$$P = P \cap G = P \cap (P \cap N)N_G(N_{p'}) = (P \cap N)(P \cap N_G(N_{p'})).$$

Since $N_G(N_{p'}) < G$, $P \cap N_G(N_{p'}) < P$. We take a maximal subgroup P_1 of P such that $P \cap N_G(N_{p'}) \leq P_1$. Then $P = (P \cap N)P_1$. Let P_2 be a 2-maximal subgroup of P such that $P_2 \leq P_1$. By the hypothesis, P_2 is either weakly s-permutably embedded or weakly s-supplemented in G. If P_2 is weakly s-permutably embedded in G, then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_2)_{se}$ of G contained in P_2 such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. So there is a s-permutable subgroup K of G such that $(P_2)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $(P_2)_{se} \cap N$ is a Sylow p-subgroup of N. We know $(P_2)_{se} \cap N \leq P_2 \cap N \leq P \cap N$ and $P \cap N$ is a Sylow p-subgroup of N, so $(P_2)_{se} \cap N = P_2 \cap N = P \cap N$. Consequently, $P = (N \cap P)P_1 = (P_2 \cap N)P_1 = P_1$, a contradiction. Therefore $K_G = 1$. By Lemma 2.5, $(P_2)_{se}$ is s-permutable in G and so $(P_2)_{se} \lhd \lhd G$. Hence $P_2 \cap T \leq (P_2)_{se} \leq O_p(G) = 1$. Since $|T|_p = p^2$, T is p-nilpotent by Lemma 2.3. Let $T_{p'}$ be the normal p-complement of T, then $T_{p'}$ is a normal Hall p'-subgroup of G, a contradiction. Therefore we may suppose P_2 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = 1$. Since $|T|_p = p^2$, T is p-nilpotent by Lemma 2.3. Let $T_{p'}$ be the normal p-complement of T, then $T_{p'}$ is a Hall p'-subgroup of G. By step (6), $T_{p'}$ and $N_{p'}$ are conjugate in G. Since $T_{p'}$ is normalized by T, there exists $g \in P_2$ such that $T_{p'}^g = N_{p'}$. Hence

$$G = (P_2T)^g = P_2T^g = P_2N_G(T_{p'}^g) = P_2N_G(N_{p'})$$

and

$$P = P \cap G = P \cap P_2N_G(N_{p'}) = P_2(P \cap N_G(N_{p'})) \leq P_1,$$

a contradiction.

□

Corollary 3.2. *Let p be the smallest prime dividing the order of a group G and G be A_4-free. Suppose that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in G, then G is p-nilpotent.*

Proof. By Lemmas 2.1(1) and 2.2(1), every 2-maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in H. By Theorem 3.1, H is p-nilpotent. Now we may assume that $H_{p'}$ is the normal p-complement of H. It is clear that $H_{p'} \lhd G$. If $H_{p'} \neq 1$, then we consider $G/H_{p'}$. It is easy to see that $G/H_{p'}$ satisfies all the hypotheses of our corollary for the normal subgroup $H/H_{p'}$ of $G/H_{p'}$ by Lemmas 2.1 and 2.2. Now by induction, we see that $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence we assume $H_{p'} = 1$ and therefore $H = P$ is a p-group. Since G/H is p-nilpotent, let K/H be the normal p-complement of G/H. By Schur-Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. By Theorem 3.1, K is p-nilpotent and so $K = H \times K_{p'}$. Hence $K_{p'}$ is a normal p-complement of G. This completes the proof. □

Corollary 3.3. *Suppose that every 2-maximal subgroup of any Sylow subgroup of a group G is either weakly s-permutably embedded or weakly s-supplemented in G. If G is A_4-free, then G is a Sylow tower group of supersolvable type.*

Proof. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then every 2-maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in G. From Theorem 3.1, we get G is p-nilpotent. Let L be the normal p-complement of G. By Lemmas 2.1 and 2.2, every 2-maximal subgroup of any Sylow subgroup of U is either weakly s-permutably embedded or weakly s-supplemented in U. Thus U satisfies the hypothesis of the corollary. By induction, U is a Sylow tower group of supersolvable type. It follows that G is also a Sylow tower group of supersolvable type. □

Corollary 3.4. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and G be A_4-free. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every 2-maximal subgroup of any Sylow subgroup of H is either weakly s-permutably embedded or weakly s-supplemented in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. By Lemmas 2.1 and 2.2, every 2-maximal subgroup of any Sylow subgroup of H is either weakly s-permutably embedded or weakly s-supplemented in H. By Corollary 3.3, H is a Sylow tower group of supersolvable type. Let p be the maximal prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Obviously, P is normal in G. Let N be a minimal normal subgroup of G contained in P.

(1) $G/P \in \mathcal{F}$.

Let M_2/P be a 2-maximal subgroup of a Sylow q-subgroup QP/P of H/P, where $q \neq p$ and Q is a Sylow q-subgroup of H. It is clear that $M_2 = Q_2P$, where Q_2 is a 2-maximal subgroup of Q. By the hypothesis of the theorem, Q_2 is either weakly s-permutably embedded or weakly s-supplemented in G. Hence M_2/N is either weakly s-permutably embedded or weakly s-supplemented in G/N by Lemmas 2.1(3) and 2.2(3). Since $(G/P)/(H/P) \cong G/H \in \mathcal{F}$ and G/P is A_4-free, we have G/P satisfies the hypothesis of the theorem. By the minimal choice of G, we have $G/P \in \mathcal{F}$.

(2) $P = N$.

If $N < P$, then $(G/N)/(P/N) \cong G/P \in \mathcal{F}$ by step (1). We will show that $G/N \in \mathcal{F}$. If $|P/N| \leqslant p^2$, then $G/N \in \mathcal{F}$ by Lemma 2.3. If $|P/N| > p^2$, then every 2-maximal subgroup of P/N is either weakly s-permutably embedded or weakly s-supplemented in G/N by Lemmas 2.1 and 2.2. By the minimality of G, we have $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$. By Lemma 2.8, it follows that $P = F(P) = N$, a contradiction.

(3) Conclusion.

Since $N \lhd G$, we may take a 2-maximal N_2 of N such that $N_2 \lhd G_p$, where G_p is a Sylow p-subgroup of G. Then N_2 is either weakly s-permutably embedded or weakly s-supplemented in G by step (2). Let T be a supplement of N_2 in G. Then $G = N_2T = NT$ and $N = N \cap N_2T = N_2(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in G and N is minimal normal in G, we have $N \cap T = N$ and so $T = G$. If N_2 is weakly s-permutably embedded in G, then N_2 is s-permutably embedded in G. By Lemma 2.7, N_2 is s-permutable in G. By Lemma 2.6, $O^p(G) \leq N_G(N_2)$. Thus $N_2 \lhd G_pO^p(G) = G$. It follows that $|N| = p^2$ and so $G \in \mathcal{F}$ by Lemma 2.9, a contradiction. If N_2 is weakly

s-supplemented in G, then $N_2 = (N_2)_{sG}$ is s-permutable in G. We get the same contradiction. □

Corollary 3.5.([3], Corollary 3.5) *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and N a normal subgroup of a group G. Suppose that G is A_4-free. If, for every prime p dividing the order of N and $P \in Syl_p(N)$, every 2-maximal subgroup of P is s-permutably embedded in G, then G belongs to $\mathcal{F}$.*

Corollary 3.6.([12], Theorem 3.1) *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and N a normal subgroup of a group G. Suppose that G is A_4-free. If, for every prime p dividing the order of N and $P \in Syl_p(N)$, every 2-maximal subgroup of P is c-supplemented in G, then G belongs to $\mathcal{F}$.*

Acknowledgements

The author would like to thank the referees for their useful comments and suggestions which have improved the original manuscript to its present form.

References

[1] O. H. Kegel, Sylow Gruppen und subnormalteiler endlicher, *Gruppen, Math. Z.* **78** (1962), 205–221.

[2] A. Ballester-Bolinches, M. C. Pedraza-Aguilera, Sufficient conditions for supersolvability of finite groups, *J. Pure Appl. Algebra* **127** (1998), 113–118.

[3] Y. Li, Y. Wang and H. Wei, On p-nilpotency of finite groups with some subgroups π-quasinormally embedded, *Acta Math. Hungar.* **108** (2005), 283–298.

[4] Y. Wang, Finite groups with some subgroups of Sylow subgroups c-supplemented, *J. Algebra* **224** (2000), 467–478.

[5] A. Ballester-Bolinches, Y. Wang and X. Guo, c-Supplemented subgroups of finite groups, *Glasgow Math. J.* **42** (2000), 383–389.

[6] X. Guo and K. P. Shum, Cover-avoidance properties and structure of finite groups, *J. Pure Appl. Algebra* **181** (2003), 297–308.

[7] K. Doerk and T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.

[8] D. Li and X. Guo, The influence of c-normality of subgroups on the structure of finite groups, *J. Pure Appl. Algebra* **150** (2000), 53–60.

[9] F. Gross, Conjugacy of odd order Hall subgroups, *Bull London Math. Soc.* **19** (1987), 311-319.

[10] Y. Li, S. Qiao and Y. Wang, On weakly s-permutably embedded subgroups of finite groups, *Comm. Algebra* **37** (2009), 1086–1097.

[11] A. N. Skiba, On weakly s-permutable subgroups of finite groups, *J. Algebra* **315** (2007), 192–209.

[12] Y. Li, Y. Wang and H. Wei, On p-nilpotency of finite groups with some c-supplemented subgroups of prime power order, *Publ. Math. Debrecen* **68** (2006), 77-88.

School of Mathematical Science
Xuzhou Normal University, Xuzhou, 221116, China
E-mail address: lcw2000@126.com