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ON THE STRUCTURE OF SOLUTION SETS OF AN INTEGRAL EQUATION IN A FRÉCHET SPACE

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Abstract. In this paper we consider Aronszajn's-type topological characterization (or compact R_δ property) of the set of solutions to the following integral equation

$$x(t) = V\Big(t, x(\theta_1(t)), \int_0^t F\big(t, s, x(\theta_2(s)), \int_0^s r(s,\tau)x(\theta_3(\tau))d\tau\big)ds\Big) + \int_0^t K(t,s)g(s, x(\theta_4(s)))ds,$$

where $t \in [0,\infty)$; $\theta_i : [0,\infty) \to [0,\infty)$, $i = 1,2,3,4$; $K : [0,\infty) \times [0,\infty) \to L(E,E)$; $V : [0,\infty) \times E \times E \to E$; $F : [0,\infty) \times [0,\infty) \times E \times E \to E$; $r : [0,\infty) \times [0,\infty) \to R$; $g : [0,\infty) \times E \to E$; E is a real Banach space with norm $|.|$; $L(E,E)$ the Banach space of continuous linear operators with domain E and range in E.

1. Introduction

Besides the existence of solutions, some properties of the solution set for many equations have been considered by many mathematicians. For example Hoa and Ngoc [6] have proved the existence of solutions and the connectivity and compactness of the solution set of an integral equation of the form

$$x(t) = \int_0^t f(t,s,x(s))ds + \int_0^t g(t,s,x(s))ds, t \geq 0,$$

where $f, g : [0,\infty) \times [0,\infty) \times E \to E$ are supposed to be continuous and E is a real Banach space.

In [12] Long and Ngoc have proved the existence of asymptotically stable solutions to an integral equation of the form

$$x(t) = q(t) + f(t, x(t), x(\pi(t))) + \int_0^t V(t,s,x(s),x(\sigma(s)))ds + \int_0^t G(t,s,x(s),x(\chi(s)))ds,$$

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where E is a real Banach space; $t \geq 0$; $q : [0, \infty) \to E$; $\pi, \sigma, \chi : [0, \infty) \to [0, \infty)$; $f : [0, \infty) \times E \times E \to E$; $V,\ G : [0, \infty) \times [0, \infty) \times E \times E \to E$ are supposed to be continuous.

In this article, we consider the following integral equation

$$x(t) = V\left(t, x(\theta_1(t)), \int_0^t F\left(t, s, x(\theta_2(s)), \int_0^s r(s,\tau)x(\theta_3(\tau))d\tau\right)ds\right) + \int_0^t K(t,s)g(s, x(\theta_4(s)))ds, \tag{1.1}$$

where $t \in [0, \infty)$; $\theta_i : [0, \infty) \to [0, \infty)$, $i = 1, 2, 3, 4$; $K : [0, \infty) \times [0, \infty) \to L(E, E)$; $V : [0, \infty) \times E \times E \to E$; $F : [0, \infty) \times [0, \infty) \times E \times E \to E$; $r : [0, \infty) \times [0, \infty) \to R$ are supposed to be continuous; $g : [0, \infty) \times E \to E$ is a *Carathéodory* function. E is a real Banach space with norm $|.|$; $L(E, E)$ the Banach space of continuous linear operators with domain E and range in E.

We shall give sufficient conditions which guarantee that the set of solutions to equation (1.1) is a compact R_δ, i.e. it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. Note that any compact R_δ is a nonempty connected compact space which is acyclic with respect to the *Čech* homology functor, i.e. the set of solutions (1.1) may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point, in the sense that it has the same homology groups as one point space. Our results extend some results in [1-2], [8-10] and [14].

The paper is organized in two sections. Our main result will be presented in Section 2. In the argument of our results, the result of paper [7] are used. We apply the following theorems.

Theorem 1.1 ([2]). *Let K be a convex unbounded subset of a normed space, $(E, |.|)$ be a Banach space and let $C = C(K, E)$ denote the Fréchet space of all continuous locally bounded functions $K \to E$ with the topology of locally uniform convergence. Assume that $T : C \longrightarrow C$ is a continuous mapping such that*

1) *There exist $t_0 \in K, x_0 \in C$ such that $T(x)(t_0) = x_0$ for every $x \in C$;*
2) *The family $T(C)$ is locally equiuniformly continuous;*
3) *For every $\epsilon > 0$ the following implication holds*

$$x\big|_{K_\epsilon} = y\big|_{K_\epsilon} \Rightarrow T(x)\big|_{K_\epsilon} = T(y)\big|_{K_\epsilon}, \quad x, y \in C,$$

where $K_\epsilon = K \bigcap B(t_0, \epsilon)$ and $B(t_0, \epsilon)$ denotes the closed ball of center t_0 and radius ϵ;

4) *Every sequence $(x_n)_{n \in N}$, $x_n \subset C$ such that*

$$\lim_{n \to \infty} (x_n - T(x_n)) = 0$$

has a limit point.

Then the set of all fixed points of T is a compact R_δ.

Remark 1.2 ([11])**.** Specially, for a close convex set $K \subset R$ and a compact map $T : C \to C$ then 2) and 4) are fulfilled.

Condition (A) ([7]). *Let X be a locally convex topological vector space and P be a separating family of seminorms on X. Let D be a subset of X and $U : D \longrightarrow X$. For any $a \in X$, define $U_a : D \longrightarrow X$ by $U_a(x) = U(x) + a$.*

The operator $U : D \longrightarrow X$ is said to satisfy condition (A) on a subset Ω of X if:

(A.1) *For any $a \in \Omega$, $U_a(D) \subset D$;*

(A.2) *For any $a \in \Omega$ and $p \in P$ there exists $k_a \in Z_+$ with the property: for any $\epsilon > 0$, there exist $r \in N$ and $\delta > 0$ such that x, $y \in D$, $\alpha_a^p(x,y) < \epsilon + \delta$ implies $\alpha_a^p(U_a^r(x),(U_a^r(y)) < \epsilon$.*

Here $\alpha_a^p(x,y) = \max\{p(U_a^i(x) - U_a^j(y)), i,j = 0,1,2,...,k_a\}$, $N = \{1,2,3,...\}$ and $Z_+ = N \cup \{0\}$.

Theorem 1.3. [7]. *Let X be a sequentially complete locally convex space with a separating family of seminorms P. Let U and C be operators on X such that*

1) *U satisfies condition (A) on X;*
2) *For any $p \in P$, there exists $k > 0$ (depending on p) such that*
$$p(U(x) - U(y)) \le kp(x-y), \quad \forall x,y \in X;$$
3) *There exists $x_0 \in X$ with the property: for any $p \in P$, there exist $r \in N$ and*
$\lambda \in [0,1)$ (r and λ depend on p) such that
$$p(U_{x_0}^r(x) - U_{x_0}^r(y)) \le \lambda p(x-y) \ ,\forall x,y \in X;$$
4) *C is completely continuous such that $p(C(A)) < \infty$ whenever $p(A) < \infty$, for $A \subset X$;*
5)
$$\lim_{p(x)\to\infty} \frac{p(C(x))}{p(x)} = 0, \quad for\ all\ x \in X.$$

Then $U + C$ has a fixed point.

Remark 1.4. From the proof of Theorem 1.3 we obtain: There exists a bounded closed convex subset D of X, such that $0 \in D$ and $(I-U)^{-1}C(D) \subset D$.

Next we put (Ω, Σ, μ) is a measure space (where $\Omega = [0,a]$, $a \in [0,\infty)$, equipped with the Lebesgue measure) and E is a real Banach space with norm $|.|$. For any $p \in [1,\infty]$, we consider the space $L^p(\Omega; E)$ of all strongly measurable functions $u : \Omega \to E$ such that $|u(.)|^p$ is Lebesgue integrable on Ω. $L^p(\Omega; E)$ is a Banach space under the norm

$$\|u\|_p = \left(\int_\Omega |u(t)|^p dt\right)^{\frac{1}{p}} \quad \text{if} \quad p \in [1,\infty)$$

and

$$\|u\|_\infty = \text{esssup}_{t\in\Omega}|u(t)| = \inf\{c \ge 0; |u(t)| \le c\ a.e.\ t \in \Omega\}.$$

When the role of the space is important, we shall denote $\|u\|_p$ also by $\|u\|_{L^p(\Omega;E)}$. In particular, $L^1(\Omega; E)$ is the space of Bochner integrable functions on Ω. When $E = R_+$, the space $L^p(\Omega; R_+)$ is simply denoted by $L^p(\Omega)$.

Definition 1.5. A function $g : \Omega \times E \longrightarrow E$ is said to be a *Carathéodory* function, if

(a) $t \longmapsto g(t, x)$ is strongly measurable for each $x \in E$;
(b) $x \longmapsto g(t, x)$ is continuous almost every where for each $t \in \Omega$.

Definition 1.6. A function $g : \Omega \times E \longrightarrow E$ is L^p $-$ *Carathéodory* if it is a *Carathéodory* function and for each real number $C > 0$ there exists a nonnegative function $r_C \in L^p(\Omega)$, $p \in [0, \infty]$ and a compact set $K_C \subset E$ such that if $|x| \leq C$ then $g(t, x) \in r_C(t)K_C$ for a.e. $t \in \Omega$.

2. Main results

Let E be a real Banach space with the norm $|.|$ and $X = C([0, \infty), E)$ be the space of all continuous functions on $[0, \infty)$ to E which is equipped with the numerable family of seminorms

$$p_n(x) = \sup_{t\in[0,n]} \{|x(t)|\}, \qquad n \in N.$$

Then X is complete in the metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)}$$

and X is a *Fréchet* space.

Let $X_n = C([0, n], E)$ be the Banach space of all continuous functions on $[0, n]$ to E with the norm

$$\|x\|_n = \sup_{t\in[0,n]} \{|x(t)|\}.$$

This norm is equivalent to the norm

$$\|x\|_N = \sup_{t\in[0,n]} \left\{\left|e^{-Nt}x(t)\right|\right\}.$$

At first we make the following assumptions:

V2.1 $\theta_1, \theta_2, \theta_3, \theta_4 : [0, \infty) \to [0, \infty)$ are continuous such that

$$\theta_1(t) \leq t\ ,\ \theta_2(t) \leq t,\ \theta_3(t) \leq\ t,\ \theta_4(t) \leq\ t;\ \forall t \in [0, +\infty).$$

V2.2 $V : [0, \infty) \times E \times E \to E$ is a continuous mapping such that the followings hold

1) There exists a continuous function $h : [0, \infty) \to [0, \infty)$ and a constant $L \in [0, 1)$ such that

$$|V(t, x, z) - V(t, y, w)| \leq L\,|x - y| + h(t)\,|z - w|\,, \quad \forall x, y, z, w \in E,\ t \in [0, \infty);$$

2) $V(0, x(0), 0) = 0,\ \ \forall x \in E.$

V2.3 $F : [0, \infty) \times [0, \infty) \times E \times E \longrightarrow E$ such that

1) There exists $q : \triangle = \{(t,s) \in [0,\infty) \times [0,\infty), s \leq t\} \to [0,\infty)$ such that for each $n \in N$

$$q_t(s) = q(t,s) \in L^2([0,n]) \quad \text{and} \quad \sup_{t\in[0,n]} \{\|q_t\|_{L^2([0,n])}\} < +\infty;$$

2) $|F(t,s,x,z) - F(t,s,y,w)| \leq q(t,s)(|x-y| + |z-w|)$, $\forall x, y \in E$, $(t,s) \in \triangle$.

V2.4 $g : [0,\infty) \times E \longrightarrow E$ such that

1) for each $n \in N$, $g : [0,n] \times E \to E$ is a $L^p - Carath\acute{e}odory$ function;

2)

$$\lim_{|x|\to\infty} \frac{|g(t,x)|}{|x|} = 0 \quad \text{uniformly in } t \text{ in bounded subsets of } [0,\infty).$$

V2.5 $K : [0,\infty) \times [0,\infty) \longrightarrow L(E,E)$ such that the followings hold

1) The map $t \mapsto \left(\|K(t,s)\|_{L(E,E)}\right)^q$ satisfies

$$\forall \epsilon > 0,\ \exists \delta > 0 :\ \forall \Omega \subset R,\ \mu(\Omega) < \delta,\ \forall s \Longrightarrow \int_\Omega \left(\|K(t,s)\|_{L(E,E)}\right)^q d\mu < \epsilon;$$

2) $$\left\| \|K_t - K'_t\|_{L(E,E)} \right\|_{L^q([0,n])} \longrightarrow 0 \text{ uniformly as } t \to t',$$

where $q = \frac{p-1}{p}$.

Theorem 2.1. *Let (V2.1)-(V2.5) hold. Then the solution set of Equation (1.1) on $[0,\infty)$ is a compact R_δ.*

To this aim we need some further auxiliary results

Put

$$\begin{aligned} U(x)(t) &= V\left(t, x(\theta_1(t)), \int_0^t F\left(t, s, x(\theta_2(s)), \int_0^s r(s,\tau)x(\theta_3(\tau))d\tau\right)ds\right); \\ U_z(x) &= z + U(x),\ \forall z \in X_n. \end{aligned}$$

Lemma 2.2. *Let $(V2.1)-(V2.3)$ hold. Then for any $z \in X_n$, U_z is a contraction mapping (with the norm $\|.\|_N$).*

Proof. Put $Q = \sup_{t\in[0,n]} \{\|q_t\|_{L^2([0,n])}\}$; $H = \sup_{t\in[0,n]} \{|h(t)|\}$; $R = \sup_{(s,\tau)\in[0,n]\times[0,n]} \{|r(s,\tau)|\}$.

We have

$$\begin{aligned} |U_z(x)(t) - U_z(y)(t)| &\leq L\,|x(\theta_1(t) - y(\theta_1(t)| \\ &+ h(t)\int_0^t q(t,s)|x(\theta_2(s)) - y(\theta_2(s))|ds \\ &+ h(t)\int_0^t q(t,s)\left(\int_0^s r(s,\tau)\,|x(\theta_3(\tau)) - y(\theta_3(\tau))|d\tau\right)ds \end{aligned}$$

It follows that

$$
\begin{aligned}
& e^{-Nt}\left|U_z(x)(t)-U_z(y)(t)\right| \\
\leq & e^{-N\theta_1(t)} L\left|x(\theta_1(t))-y(\theta_1(t))\right| \\
& + e^{-Nt}h(t)\int_0^t q(t,s)e^{Ns}\left(e^{-N\theta_2(s)}\left|x(\theta_2(\tau))-y(\theta_2(\tau))\right|\right)ds \\
& + e^{-Nt}h(t)\int_0^t q(t,s)\left(\int_0^s r(s,\tau)e^{N\tau}e^{-N\theta_3(\tau)}\left|x(\theta_3(\tau))-y(\theta_3(\tau))\right|d\tau\right)ds \\
\leq & \left[L+e^{-Nt}h(t)\int_0^t q(t,s)e^{Ns}ds+e^{-Nt}h(t)\int_0^t q(t,s)\left(\int_0^s r(s,\tau)e^{N\tau}d\tau\right)ds\right]\|x-y\|_N \\
< & \left[L+e^{-Nt}H\int_0^t q(t,s)e^{Ns}ds+\frac{e^{-Nt}HR}{N}\int_0^t q(t,s)e^{Ns}ds\right]\|x-y\|_N \\
\leq & \left[L+\left(H+\frac{HR}{N}\right)e^{-Nt}\int_0^t q(t,s)e^{Ns}ds\right]\|x-y\|_N \\
\leq & \left[L+\left(H+\frac{HR}{N}\right)e^{-Nt}\left(\int_0^t \left|q_t(s)\right|^2 ds\right)^{\frac{1}{2}}\left(\int_0^t e^{2Ns}ds\right)^{\frac{1}{2}}\right]\|x-y\|_N \\
\leq & \left[L+\left(H+\frac{HR}{N}\right)\frac{Q}{\sqrt{2N}}\, e^{-Nt}\left(e^{2Nt}-1\right)^{\frac{1}{2}}\right]\|x-y\|_N \\
< & \left[L+\left(H+\frac{HR}{N}\right)\frac{Q}{\sqrt{2N}}\, e^{-Nt}e^{Nt}\right]\|x-y\|_N \\
< & \left[L+\left(H+\frac{HR}{N}\right)\frac{Q}{\sqrt{2N}}\right]\|x-y\|_N\,, \quad \forall t\in[0,n].
\end{aligned}
$$

Therefore

$$
\|U_z(x)-U_z(y)\|_N < \left[L+\left(H+\frac{HR}{N}\right)\frac{Q}{\sqrt{2N}}\right]\|x-y\|_N\,.
$$

Choose $N=\frac{1}{2}\left[1+\frac{HQ(2R+1)}{1-L}\right]^2$, then we deduce $\left[L+\left(H+\frac{HR}{N}\right)\frac{Q}{\sqrt{2N}}\right]<1$. Hence U_z is a contraction mapping (with the norm $\|.\|_N$). □

Lemma 2.3. *Let (V2.4.1), (V2.5) hold, then*

$$
C(x)(t)=\int_0^t K(t,s)(g(s,x(\theta_4(s))))ds
$$

is a compact mapping on $(X_n,\|.\|_N)$.

Proof. Let Δ be a bounded subset of X_n. Put $A = \{x(\theta_4(s)) : x \in \Delta, s \in [0,n]\}$. Then A is bounded in E (i.e. there exists $C > 0$ such that $|x(t)| \leq C$ $\forall x(t) \in A$). Thus for all $x(\theta_4(s)) \in A$, by (V2.4.1) there exists a nonnegative function $r_C \in L^p([0,n])$, $p \in [1,\infty)$ and a compact set $K_C \subset E$ such that $g(s, x(\theta_4(s))) \in r_C(s)K_C$ for a.e. $s \in [0,n]$. This implies that there exists a constant $M > 0$ such that $|g(s, x(\theta_4(s)))| \leq Mr_C(s)$ for a.e. $s \in [0,n]$. Put $G = M\left(\int\limits_0^n r_c^p(s)ds\right)^{\frac{1}{p}}$ (Note that G is finite).

We first show that $C(\Delta)$ is equicontinuous in X_n.
For any $t,\, t' \in [0,n],\ |t-t'| < \delta,\ x \in \Delta$, we have

$$\begin{aligned}
&\left|C(x)(t) - C(x)(t')\right| \\
=&\left|\int_0^t K(t,s)\left(g(s,x(\theta_4(s)))\right)ds - \int_0^{t'} K(t',s)\left(g(s,x(\theta_4(s)))\right)ds\right| \\
\leq&\left|\int_0^t \big(K(t,s)-K(t',s)\big)\left(g(s,x(\theta_4(s)))\right)ds\right| + \left|\int_t^{t'} K(t',s)\left(g(s,x(\theta_4(s)))\right)ds\right| \\
\leq&\left(\int_0^n |g(s,x(\theta_4(s)))|^p\,ds\right)^{\frac{1}{p}}\left(\int_0^n \left\|K(t,s)-K(t',s)\right\|^q_{L(E,E)}ds\right)^{\frac{1}{q}} \\
&+\left(\int_0^n |g(s,x(\theta_4(s)))|^p\,ds\right)^{\frac{1}{p}}\left|\int_t^{t'} \left\|K(t',s)\right\|^q_{L(E,E)}ds\right|^{\frac{1}{q}} \\
\leq& G\left[\left(\int_0^n \left\|K(t,s)-K(t',s)\right\|^q_{L(E,E)}ds\right)^{\frac{1}{q}} + \left|\int_t^{t'} \left\|K(t',s)\right\|^q_{L(E,E)}ds\right|^{\frac{1}{q}}\right].
\end{aligned}$$

From (V2.5), the above inequality shows that $C(\Omega)$ is equicontinuous on $[0,n]$.

Next we prove that $C : X_n \longrightarrow X_n$ is continuous.

Let $x_m, x_0 \in X_n$ be such that $\|x_m - x_0\|_n \to 0$ as $m \to \infty$. Put $A = \{x_m(\theta_4(s)),\ s \in [0,n],\ m \in N\}$ then A is a compact set in E, this implies A is bounded.

Put

$$\rho_m(s) = \left|g(s, x_m(\theta_4(s))) - g(s, x_0(\theta_4(s)))\right|.$$

By (V2.4.1), we have

$$\rho_m(s) \to 0 \qquad \text{and} \qquad |\rho_{\mathrm{m}}(\mathrm{s})| \leq 2\mathrm{Mr_C(s)} \qquad \text{a.e.} \qquad \text{on } [0,\mathrm{n}].$$

Put $K = \sup\limits_{t\in[0,n]} \left\{ \left\| \|K_t(s)\|_{L(E,E)} \right\|_{L^q([0,n])} \right\}$ (Note that K is finite because of (V2.5.2)). We have

$$\begin{aligned}
&|C(x_m)(t) - C(x_0)(t)| \\
=&\left| \int_0^t K(t,s)(g(s, x_m(\theta_4(s))))ds - \int_0^t K(t,s)(g(s, x_0(\theta_4(s))))ds \right| \\
\le& \int_0^t \left\| K(t,s) \right\|_{L(E,E)} \left| g(s, x_m(\theta_4(s))) - g(s, x_0(\theta_4(s))) \right| ds \\
\le& K \left(\int_0^n \left| g(s, x_m(\theta_4(s))) - g(s, x_0(\theta_4(s))) \right|^p ds \right)^{\frac{1}{p}}, \ \forall t \in [0,n] \\
\le& K \left(\int_0^n |\rho_m(s)|^p \, ds \right)^{\frac{1}{p}}, \ \forall t \in [0,n].
\end{aligned}$$

And so

$$\|C(x_m) - C(x_0)\|_n \le K \left(\int_0^n |\rho_m(s)|^p \, ds \right)^{\frac{1}{p}}.$$

The Lebesgue dominated convergence theorem implies that C is continuous on X_n.

Finally we prove that for every $t \in [0,n]$ the set

$$C(\Delta)(t) = \left\{ \int_0^t K(t,s)(g(s, x(\theta_4(s))))ds; \ x \in \Delta \right\}$$

is relatively compact in E.

Let $b^* \in E^*$ be such that K_C lies in the half-space $b^* \le r$, that is, $b \in K_C \Rightarrow b^*(b) \le r$. Then, for each $x \in \Delta$ we have

$$b^*(g(s, x(\theta_4(s)))) \le r_C(s) r \text{ a.e. } s \in [0,n].$$

Now, if $\int_0^t r_C(s)ds > 0$, then by a standard property of the Bochner integral,

$$b^* \left(\frac{\int_0^t g(s, x(\theta_4(s)))ds}{\int_0^t r_C(s)ds} \right) = \frac{\int_0^t b^*(g(s, x(\theta_4(s))))ds}{\int_0^t r_C(s)ds} \le r.$$

Since the intersection of all closed half-spaces containing K_C is its closed convex hull, it follows that

$$\int_0^t g(s, x(\theta_4(s)))ds \in \left(\int_0^t r_C(s)ds \right) \overline{co}\,(K_C).$$

If $\int\limits_0^t r_C(s)ds = 0$, then $r_C = 0$ a.e on $[0,t]$, $g(s, x(\theta_4(s))) = 0$ a.e on $[0,t]$, and the previous inclusion still holds.
As above we have

$$C(\Delta)(t) \subseteq \left(\int\limits_0^t r_C(s)ds\right) K(t,s)\overline{co}(K_C \cup \{0\}).$$

Since $K(t,s) \in L(E,E)$, it implies that $K(t,s)\overline{co}(K_C \cup \{0\})$ is compact in E. So $C(\Omega)(t)$ is relatively compact in E, consequently $C(\Omega)$ is relatively compact in X_n by the general Arzela-Ascoli theorem and C is a compact mapping on $(X_n, \|.\|_n)$. So C is a compact mapping on $(X_n, \|.\|_N)$. □

Lemma 2.4 ([12]). *A set S in X is relatively compact if and only if for each $n \in N$, S is equicontinuous in $[0,n]$ and the set $\{x(t) : x \in S, t \in [0,n]\}$ is relatively compact in E.*

Lemma 2.5. *Let (V2.4.2) hold, then*

$$\lim_{\|x\|_N \to \infty} \frac{\|C(x)\|_N}{\|x\|_N} = 0, \quad \forall x \in X.$$

Proof. For any given $\epsilon > 0$, (V2.4) implies there exists $\delta > 0$ such that

$$\frac{|g(s, x(\theta_4(s)))|}{|x(\theta_4(s))|} < \frac{\varepsilon}{2Kn},$$

for all x with $|x| \geq \delta$ and $s \in [0,n]$. Since g is $L^p - Carathéodory$ function, there exists a function r_C in $L^p([0,n])$ and a constant $M > 0$ such that $|g(s, x(\theta_4(s)))| \leq M r_C(s)$ for a.e. $s \in [0,n]$ and x with $|x| \leq \delta$.

Put $I_1 = \{s \in [0,n] : |x(\theta_4(s))| \leq \delta\}$, $I_2 = [0,n] \backslash I_1$. Choose δ_1 such that $\delta_1 > \frac{2KG}{\varepsilon}$. We have for all $x \in X_n, \|x\|_n \geq \delta_1$

$$\frac{|C(x)(t)|}{\|x\|_n} = \frac{1}{\|x\|_n}\left|\int\limits_0^t K(t,s)\left(g(s, x(\theta_4(s)))\right)ds\right|$$

$$\leq \frac{K}{\|x\|_n}\left(\int\limits_0^n |g(s, x(\theta_4(s)))|^p\, ds\right)^{\frac{1}{p}}$$

$$\leq \frac{K}{\|x\|_n}\left[\left(\int\limits_{I_1} |g(s, x(\theta_4(s)))|^p\, ds\right)^{\frac{1}{p}} + \left(\int\limits_{I_2} \frac{|g(s, x(\theta_4(s)))|^p\, |x(\theta_4(s))|^p}{|x(\theta_4(s))|^p} ds\right)^{\frac{1}{p}}\right]$$

$$\leq \frac{KM}{\|x\|_n}\left(\int_0^n r_c^p(s)ds\right)^{\frac{1}{p}} + K\frac{\|x\|_n}{\|x\|_n}\left(\int_{I_2}\frac{|g(s,x(\theta_4(s)))|^p}{|x(\theta_4(s))|^p}ds\right)^{\frac{1}{p}}$$

$$\leq \frac{KG}{\|x\|_n} + K\left(\int_{I_2}\left(\frac{\varepsilon}{2Kn}\right)^p ds\right)^{\frac{1}{p}}$$

$$\leq \frac{KG}{\delta_1} + \frac{\varepsilon}{2} \leq \frac{\varepsilon KG}{2KG} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall t \in [0,n].$$

Thus

$$\frac{\|C(x)\|_n}{\|x\|_n} \leq \epsilon,$$

which shows that

$$\lim_{\|x\|_n\to\infty}\frac{\|C(x)\|_n}{\|x\|_n} = 0, \quad \forall x \in X \quad \text{and so} \quad \lim_{\|x\|_N\to\infty}\frac{\|C(x)\|_N}{\|x\|_N} = 0, \quad \forall x \in X.$$

□

Proof of Theorem 2.1. By Lemma 2.2 we have U satisfies condition 1)-3) of Theorem 1.3.

Let $\{x_m\}_m \subset X$ be such that $\|x_m - x_0\|_n \to 0$ as $m \to \infty$ for every $n \in N$. Since $\{x_m\}_m$ converges uniformly to x_0 on $[0,n]$, for every n, then by Lemma 2.3, we have $\|C(x_m) - C(x_0)\|_n \to 0$ as $m \to \infty$ for all $n \in N$, i.e. C is continuous on X. By Lemmas 2.4 and 2.5 we have C satisfies conditions 4)-5) of Theorem 1.3, hence by that theorem, there exists a bounded, closed and convex $D \subset X$ such that $(I-U)^{-1}C(D) \subset D$.

Now we shall apply Theorem 1.1 to $C = D,\ T = (I-U)^{-1}C$, $t_0 = 0$, $x_0 = 0$. Theorem 2.1 is proved completely. □

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