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PENALTY FUNCTIONS FOR THE VECTOR VARIATIONAL INEQUALITY PROBLEM

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Abstract. In this paper we apply the penalty function method to the vector variational inequality problem, in order to transform a constrained problem, referred to as the original problem, into a sequence of simpler, unconstrained problems, referred to as the penalized problems. We show that any limit point of a sequence of solutions of the penalized problems is a solution of the original problem. Moreover, under certain assumptions on the feasible region D and the function $\mathbf{F}$, we can show that every penalized problem has a solution, and that a sequence of solutions of the penalized problems always has at least one limit point. As far as we know, this is the first study on solving a vector variational inequality problem using the penalty function method.

1. Introduction

The penalty function method is often employed to transform a constrained problem into a sequence of unconstrained problems, so that a sequence of solutions of the unconstrained problems converges to a solution of the constrained problem. There have been several studies on solving the scalar variational inequality problem via the penalty function method (see, for instance [5, 4, 6, 1, 8, 9]). However, according to our best knowledge, there has not been any study on how to apply the penalty function method to the vector variational inequality problem. In this paper we investigate the relationship between $S(t)$, the set of solutions of the penalized problem $(\mathrm{VVIP})_t$, as $t \to +\infty$, and S, the set of solutions of the original problem (VVIP). We show that under certain assumptions, the penalized problems always have solutions. Furthermore, a sequence of solutions of the penalized problems always has at least one limit point, and each limit point of this sequence is a solution of the original problem (VVIP).

The paper is organized as follows. Section 2 provides the definitions and notations that are used throughout the paper. Section 3 is devoted to the treatment of the vector variational inequality problem with penalty functions. The paper is concluded in Section 4.

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2. Definitions and notations

For $\mathbf{x} = (x_1, \dots, x_k)^T \in \mathbb{R}^k$ and $\mathbf{y} = (y_1, \dots, y_k)^T \in \mathbb{R}^k$, we adopt the following conventions

$$\mathbf{x} < \mathbf{y} \Longleftrightarrow x_i < y_i, \ \forall i = 1, \dots, k,$$
$$\mathbf{x} \not< \mathbf{y} \Longleftrightarrow \exists i: \ x_i \geq y_i.$$

Let D be a nonempty subset of $\mathbb{R}^k$. Suppose that $\mathbf{F} : \mathbb{R}^k \to \mathbb{R}^{r \times k}$ is a continuous function whose range is a set of $r \times k$ real matrices. The *D-constrained vector variational inequality problem* is defined as follows

$$\text{(VVIP)} \qquad \text{Find } \mathbf{x} \in D \text{ such that } \mathbf{F}(\mathbf{x})(\mathbf{y} - \mathbf{x}) \not< 0, \forall \mathbf{y} \in D.$$

If $D = \mathbb{R}^k$, (VVIP) is called an *unconstrained vector variational inequality problem.* Let $\mathbf{F}_i : \mathbb{R}^k \to \mathbb{R}^k$, $i = 1, \dots, r$ be the component functions of $\mathbf{F}$, i.e. $\mathbf{F}_i(\mathbf{x})$ is the ith row of the matrix $\mathbf{F}(\mathbf{x})$. Then

$$\mathbf{F}(\mathbf{x})(\mathbf{y} - \mathbf{x}) = (\langle \mathbf{F}_1(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \dots, \ \langle \mathbf{F}_r(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle)^T.$$

Therefore $\mathbf{x} \in D$ is a solution of (VVIP) if and only if for all $\mathbf{y} \in D$

$$(\langle \mathbf{F}_1(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \dots, \ \langle \mathbf{F}_r(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle)^T \not< 0,$$

or in other words, there exists i such that

$$\langle \mathbf{F}_i(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0.$$

For a vector $\mathbf{y} = (y_1, \dots, y_k)^T \in \mathbb{R}^k$, let $\|\mathbf{y}\|$ denote the Euclidean norm of $\mathbf{y}$, namely,

$$\|\mathbf{y}\| = \sqrt{\sum_{i=1}^{k} y_i^2}.$$

3. The vector variational inequality problem and the penalty functions

3.1. **A sufficient condition for the existence of solutions for the vector variation inequality problem.** We first recall a sufficient condition, established in [3], for a vector variation inequality problem to be solvable, i.e., to have a solution. Before stating the theorem, we need a few definitions.

Definition 3.1 ([3])**.** The function $\mathbf{F} : \mathbb{R}^k \to \mathbb{R}^{r \times k}$ is said to be *monotone* on $\mathbb{R}^k$ if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ we have $(\mathbf{F}(\mathbf{y}) - \mathbf{F}(\mathbf{x}))(\mathbf{y} - \mathbf{x}) \geq 0$, i.e. $\langle \mathbf{F}_i(\mathbf{y}) - \mathbf{F}_i(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0$ for all $i = 1, \dots, r$.

Definition 3.2 ([3])**.** The function $\mathbf{F} : \mathbb{R}^k \to \mathbb{R}^{r \times k}$ is said to be *weak coercive on $D \subset \mathbb{R}^k$* if there exists a vector $\mathbf{s} \in \mathbb{R}^r_+ := \{\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{R}^r : z_i \geq 0, \ i = 1, \dots, r\}$ and a vector $\mathbf{a} \in D$ such that

$$\langle \mathbf{s}^T \mathbf{F}(\mathbf{y}), \mathbf{y} - \mathbf{a} \rangle \to +\infty \text{ as } \|\mathbf{y}\| \to +\infty, \ \mathbf{y} \in D.$$

Chen and Yang [3] established the following sufficient condition for the existence of a solution of (VVIP).

Theorem 3.3 ([3]). *Let $D \neq \varnothing$ be a closed convex subset of $\mathbb{R}^k$, and $\mathbf{F} : \mathbb{R}^k \to \mathbb{R}^{r\times k}$ a continuous and monotone function on $\mathbb{R}^k$. Assume that*

(1) *D is bounded, or*
(2) *$\mathbf{F}$ is weak coercive on D.*

Then (VVIP) *is solvable.*

Note that Theorem 3.3 in its original form ([3, Theorem 2.1]) only requires that $\mathbf{F}$ is a v-hemicontinuous map instead of a continuous map. However, as the continuity of $\mathbf{F}$ is crucial for our main results presented in Section 3.3, the stronger version of Theorem 3.3 is not needed.

3.2. **The penalized vector variational inequality problems.** Let $\mathbf{P} : \mathbb{R}^k \to \mathbb{R}$ be a Fréchet differentiable function, i.e. for all $\mathbf{x} \in \mathbb{R}^k$, there exists a vector $\nabla\mathbf{P}(\mathbf{x}) \in \mathbb{R}^k$ which satisfies

$$\lim_{\mathbf{y}\to 0} |\mathbf{P}(\mathbf{x}+\mathbf{y}) - \mathbf{P}(\mathbf{y}) - \langle \nabla\mathbf{P}(\mathbf{x}), \mathbf{y}\rangle|/\|\mathbf{y}\| = 0.$$

The vector-valued function $\nabla\mathbf{P}(\mathbf{x})$ is referred to as the Fréchet derivative of $\mathbf{P}$.

Suppose that $\mathbf{P}$ is a convex, Fréchet differentiable function on $\mathbb{R}^k$. Due to the property of a convex function (see, for instance [7], Corollary 25.5.1), $\nabla\mathbf{P}(\mathbf{x})$ is continuous on $\mathbb{R}^k$. Moreover, $\mathbf{P}(\mathbf{y}) - \mathbf{P}(\mathbf{x}) \geq \langle \nabla\mathbf{P}(\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$. We furthermore assume that $\mathbf{P}$ satisfies the property of a penalty function for D, namely

$$\mathbf{P}(\mathbf{x}) \begin{cases} = 0, & \mathbf{x} \in D, \\ > 0, & \mathbf{x} \notin D. \end{cases} \tag{3.1}$$

For instance, if D is defined as

$$D = \{\mathbf{x} \in \mathbb{R}^k : g_j(\mathbf{x}) \leq 0, j = 1, \ldots, m\}, \tag{3.2}$$

where $g_j : \mathbb{R}^k \to \mathbb{R}$, $j = 1, \ldots, m$ are continuous functions, we can take

$$\mathbf{P}(\mathbf{x}) = \sum_{j=1}^{m} \big[\max\{0, g_j(\mathbf{x})\}\big]^2. \tag{3.3}$$

It is straightforward to verify that $\mathbf{P}$ defined as above not only is convex and Fréchet differentiable on $\mathbb{R}^k$, but also satisfies (3.1).

We now define the penalized problems. Let $\mathbf{Q}_i := \nabla\mathbf{P}$ and $\mathbf{Q} : \mathbb{R}^k \to \mathbb{R}^{r\times k}$ with the component functions $\mathbf{Q}_i : \mathbb{R}^k \to \mathbb{R}^k$, $i = 1, \ldots, r$. For $t > 0$ we consider the following penalized problem

$$(\text{VVIP})_t \qquad \text{Find } \mathbf{x}^{(t)} \in \mathbb{R}^k \text{ such that } (\mathbf{F}^{(t)}(\mathbf{x}^{(t)}))(\mathbf{y} - \mathbf{x}^{(t)}) \not< 0,\ \forall \mathbf{y} \in \mathbb{R}^k,$$

where $\mathbf{F}^{(t)} := \mathbf{F} + t\mathbf{Q}$.

Definition 3.4. The function $\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^k$ is said to be *D-coercive on* $\mathbb{R}^k$ if there exists a vector $\mathbf{a} \in D$ such that

$$\langle \mathbf{f}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle \to +\infty \text{ as } \|\mathbf{y}\| \to +\infty,\ \mathbf{y} \in \mathbb{R}^k.$$

Example 3.5. The function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $\mathbf{f}(\mathbf{y}) = (y_1 + y_2, y_2 + e^{y_2/4} - 1)^T$, is D-coercive on $\mathbb{R}^2$ for every subset D of $\mathbb{R}^2$ which contains the origin. Indeed,

$$\begin{aligned}\langle \mathbf{f}(\mathbf{y}), \mathbf{y} - 0\rangle &= (y_1 + y_2)y_1 + (y_2 + e^{y_2/4} - 1)y_2 \\ &= (y_1^2 + y_1 y_2 + y_2^2) + (e^{y_2/4} y_2 - y_2) \\ &\to +\infty,\end{aligned}$$

as $\|\mathbf{y}\| \to +\infty$.

Definition 3.6. The function $\mathbf{F} : \mathbb{R}^k \to \mathbb{R}^{r\times k}$ is said to be *D-coercive on* $\mathbb{R}^k$ if there exists a vector $\mathbf{s} \in \mathbb{R}^r_+$ and a vector $\mathbf{a} \in D$ such that

$$\langle \mathbf{s}^T \mathbf{F}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle \to +\infty \text{ as } \|\mathbf{y}\| \to +\infty,\ \mathbf{y} \in \mathbb{R}^k.$$

The following lemma provides a sufficient condition for both the original and the penalized problems to be solvable.

Lemma 3.7. *Let $D \neq \varnothing$ be a closed convex subset of $\mathbb{R}^k$, and $\mathbf{F} : \mathbb{R}^k \to \mathbb{R}^{r\times k}$ a continuous, monotone function which is D-coercive on $\mathbb{R}^k$. Then the (VVIP) has at least one solution. Moreover, for every $t > 0$, the penalized problem (VVIP)$_t$ also has at least one solution.*

Proof. As $\mathbf{F}$ is D-coercive on $\mathbb{R}^k$, it is also weak coercive on D if D is unbounded. Hence, by Theorem 3.3, (VVIP) has at least one solution. Now we show that (VVIP)$_t$ also has a solution. Obviously $\mathbf{F}^{(t)}$ is continuous. Moreover, $\mathbf{F}^{(t)}$ is monotone on $\mathbb{R}^k$. We want to show that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, the vector $(\mathbf{F}^{(t)}(\mathbf{y}) - \mathbf{F}^{(t)}(\mathbf{x}))(\mathbf{y} - \mathbf{x})$ has nonnegative coordinates. Indeed

$$\begin{aligned}(\mathbf{F}^{(t)}(\mathbf{y}) - \mathbf{F}^{(t)}(\mathbf{x}))(\mathbf{y} - \mathbf{x}) &= (\mathbf{F}(\mathbf{y}) + t\mathbf{Q}(\mathbf{y}) - \mathbf{F}(\mathbf{x}) - t\mathbf{Q}(\mathbf{x}))(\mathbf{y} - \mathbf{x}) \\ &= (\mathbf{F}(\mathbf{y}) - \mathbf{F}(\mathbf{x}))(\mathbf{y} - \mathbf{x}) + t(\mathbf{Q}(\mathbf{y}) - \mathbf{Q}(\mathbf{x}))(\mathbf{y} - \mathbf{x}).\end{aligned}$$

Note that the first term is always greater than or equal to 0, i.e. all of its coordinates are nonnegative. The ith coordinate of the second term is

$$\begin{aligned}t\langle \mathbf{Q}_i(\mathbf{y}) - \mathbf{Q}_i(\mathbf{x}), \mathbf{y} - \mathbf{x}\rangle &= -t\langle \nabla\mathbf{P}(\mathbf{y}), \mathbf{x} - \mathbf{y}\rangle - t\langle \nabla\mathbf{P}(\mathbf{x}), \mathbf{y} - \mathbf{x}\rangle \\ &\geq -t(\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{y})) - t(\mathbf{P}(\mathbf{y}) - \mathbf{P}(\mathbf{x})) \\ &= 0.\end{aligned}$$

Note that in the second inequality we exploit the fact that $\mathbf{P}$ is convex. Thus the second term also has nonnegative coordinates. Therefore $\mathbf{F}^{(t)}$ is monotone on $\mathbb{R}^k$.

It remains to show that $\mathbf{F}^{(t)}$ is weak coercive on $\mathbb{R}^k$. Indeed, there exist $\mathbf{s} \in \mathbb{R}^r_+$ and $\mathbf{a} \in D$ such that

$$\langle \mathbf{s}^T \mathbf{F}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle \to +\infty \text{ as } \|\mathbf{y}\| \to +\infty, \mathbf{y} \in \mathbb{R}^k.$$

We have

$$\begin{aligned}\langle \mathbf{s}^T \mathbf{F}^{(t)}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle &= \langle \mathbf{s}^T (\mathbf{F}(\mathbf{y}) + t\mathbf{Q}(\mathbf{y})), \mathbf{y} - \mathbf{a}\rangle \\ &= \langle \mathbf{s}^T \mathbf{F}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle + t\langle \mathbf{s}^T \mathbf{Q}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle.\end{aligned}$$

We examine the second term divided by t. Since

$$\mathbf{Q}(\mathbf{y})(\mathbf{y}-\mathbf{a}) = (\langle \mathbf{Q}_1(\mathbf{y}), \mathbf{y}-\mathbf{a}\rangle, \ldots, \langle \mathbf{Q}_r(\mathbf{y}), \mathbf{y}-\mathbf{a}\rangle),$$

we have

$$\begin{aligned}\langle \mathbf{s}^T\mathbf{Q}(\mathbf{y}), \mathbf{y}-\mathbf{a}\rangle &= \sum_i s_i \langle \mathbf{Q}_i(\mathbf{y}), \mathbf{y}-\mathbf{a}\rangle \\ &= \sum_i s_i \langle \nabla \mathbf{P}(\mathbf{y}), \mathbf{y}-\mathbf{a}\rangle \\ &\geq \sum_i s_i (\mathbf{P}(\mathbf{y}) - \mathbf{P}(\mathbf{a})) \\ &= \sum_i s_i \mathbf{P}(\mathbf{y}),\end{aligned}$$

which is nonnegative, as $s_i \geq 0$ and $\mathbf{P}(\mathbf{y}) \geq 0$. As $\mathbf{F}$ is D-coercive on $\mathbb{R}^k$, we have

$$\langle \mathbf{s}^T\mathbf{F}(\mathbf{y}), \mathbf{y}-\mathbf{a}\rangle \to +\infty \text{ as } \|\mathbf{y}\| \to +\infty, \mathbf{y} \in \mathbb{R}^k.$$

Hence we deduce that

$$\langle \mathbf{s}^T\mathbf{F}^{(t)}(\mathbf{y}), \mathbf{y}-\mathbf{a}\rangle \to +\infty \text{ as } \|\mathbf{y}\| \to +\infty, \mathbf{y} \in \mathbb{R}^k.$$

Therefore $\mathbf{F}^{(t)}$ is weak coercive on $\mathbb{R}^k$ for every $t > 0$. By Theorem 3.3, the penalized problem $(\text{VVIP})_t$ has at least one solution. □

Example 3.8. Let

$$D = \{\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 - x_1 - 1 \leq 0, -x_2 + x_1^2 - 1 \leq 0\}.$$

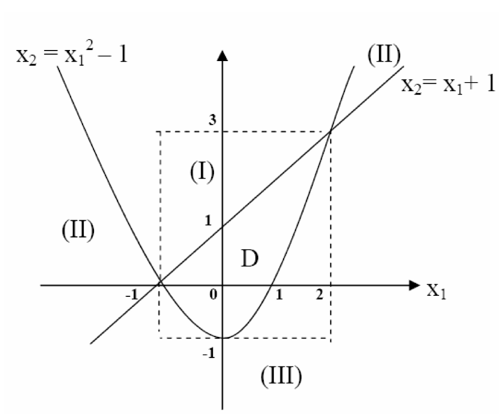

FIGURE 1. The feasible region D of (VVIP)

Take $\mathbf{P}$ as in (3.3).

$$\begin{aligned} P(x_1,x_2) &= [\max\{0, x_2 - x_1 - 1\}]^2 + [\max\{0, -x_2 + x_1^2 - 1\}]^2 \\ &= \begin{cases} 0, & (x_1,x_2) \in D, \\ (x_2 - x_1 - 1)^2, & (x_1,x_2) \in (I), \\ (x_2 - x_1 - 1)^2 + (-x_2 + x_1^2 - 1)^2, & (x_1,x_2) \in (II), \\ (-x_2 + x_1^2 - 1)^2, & (x_1,x_2) \in (III). \end{cases} \end{aligned}$$

We have

$$\nabla P(x_1,x_2) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (x_1,x_2) \in D, \\ \begin{pmatrix} -2(x_2 - x_1 - 1) \\ 2(x_2 - x_1 - 1) \end{pmatrix}, & (x_1,x_2) \in (I), \\ \begin{pmatrix} -2(x_2 - x_1 - 1) + 4x_1(-x_2 + x_1^2 - 1) \\ 2(x_2 - x_1 - 1) - 2(-x_2 + x_1^2 - 1) \end{pmatrix}, & (x_1,x_2) \in (II), \\ \begin{pmatrix} 4x_1(-x_2 + x_1^2 - 1) \\ -2(-x_2 + x_1^2 - 1) \end{pmatrix}, & (x_1,x_2) \in (III). \end{cases}$$

Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ be a function defined as

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 & x_2 + e^{x_2/2} - 2 \\ x_1 + 2x_2 & x_1 + 3x_2 + e^{x_2} - 1 \end{pmatrix}.$$

Then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ we have

$$\begin{aligned} \langle \mathbf{F}_1(\mathbf{y}) - \mathbf{F}_1(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle &= (y_1 - x_1)^2 + (y_2 - x_2)(y_1 - x_1) + (y_2 - x_2)^2 \\ &\quad + (e^{y_2/2} - e^{x_2/2})(y_2 - x_2) \\ &\geq 0, \end{aligned}$$

and

$$\begin{aligned} \langle \mathbf{F}_2(\mathbf{y}) - \mathbf{F}_2(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle &= (y_1 - x_1)^2 + 3(y_2 - x_2)(y_1 - x_1) + 3(y_2 - x_2)^2 \\ &\quad + (e^{y_2} - e^{x_2/2})(y_2 - x_2) \\ &\geq 0. \end{aligned}$$

Hence, $\mathbf{F}$ is monotone on $\mathbb{R}^2$. Moreover, $\mathbf{F}$ is D-coercive on $\mathbb{R}^2$. Indeed, choosing $\mathbf{a} = 0 \in D$ and $\mathbf{s} = (1,1)^T$, we have

$$\begin{aligned} \langle \mathbf{s}^T \mathbf{F}(\mathbf{y}), \mathbf{y} - \mathbf{a} \rangle &= \langle \mathbf{F}_1(\mathbf{y}) + \mathbf{F}_2(\mathbf{y}), \mathbf{y} \rangle \\ &= 2y_1^2 + 4y_1y_2 + 4y_2^2 + e^{y_2/2}y_2 + e^{y_2}y_2 - 3y_2 \\ &\to +\infty, \end{aligned}$$

as $\|\mathbf{y}\| \to +\infty$. Thus, the region D and the function $\mathbf{F}$ defined as above satisfy all the requirements in Lemma 3.7. Therefore, both (VVIP) and $(\text{VVIP})_t$ $(t > 0)$ corresponding to these D and $\mathbf{F}$ are solvable.

3.3. **The convergence theorems.** Let S and $S(t)$ denote the set of solutions of (VVIP) and (VVIP)$_t$, respectively. Let $\{t_n\}_n$ be a sequence of positive real numbers which monotonically tends to $+\infty$ as $n \to +\infty$.

Lemma 3.9. *Assume that* $\mathbf{x}_n \in S(t_n)$ *for all* $n \in \mathbb{N}$. *Suppose that* $\{\mathbf{x}_{n_m}\}_m$ *is a convergent subsequence of* $\{\mathbf{x}_n\}_n$ *and that* $\lim_{m\to+\infty} \mathbf{x}_{n_m} = \mathbf{x}$. *Then* $\mathbf{x} \in D$.

Proof. We prove this lemma by contradiction. Suppose that $\mathbf{x} \notin D$. Then $\mathbf{P}(\mathbf{x}) > 0$ and hence $\mathbf{P}(\mathbf{x}) > \epsilon$ for some $\epsilon > 0$. Take $\mathbf{y} \in D$. Since $\mathbf{x}_{n_m}$ solves (VVIP)$_{t_{n_m}}$, there exists an index i_{n_m} such that

$$\langle \mathbf{F}_{i_{n_m}}^{(t_{n_m})}(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle \geq 0.$$

Since $i_{n_m} \in \{1, 2, \ldots, r\}$, there exists an infinite sequence $\{i_{n_{m_k}}\}_k$ of indices, all of which have the same value, say $i_{n_{m_k}} = 1$, for all $k \in \mathbb{N}$. To simplify the notation, we assume that $i_{n_m} = 1$ for all $m \in \mathbb{N}$. Therefore for all $m \in \mathbb{N}$ we have

$$\langle \mathbf{F}_1^{(t_{n_m})}(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle \geq 0. \tag{3.4}$$

On the other hand, since $\mathbf{P}(\mathbf{x}_{n_m}) \to \mathbf{P}(\mathbf{x}) > \epsilon$, for all m sufficiently large we have $\mathbf{P}(\mathbf{x}_{n_m}) > \epsilon$. Therefore

$$\begin{aligned}
\langle \mathbf{F}_1^{(t_{n_m})}(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle &= \langle \mathbf{F}_1(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle + t_{n_m} \langle \nabla \mathbf{P}(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle \\
&\leq \langle \mathbf{F}_1(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle + t_{n_m} (\mathbf{P}(\mathbf{y}) - \mathbf{P}(\mathbf{x}_{n_m})) \\
&= \langle \mathbf{F}_1(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle - t_{n_m} \mathbf{P}(\mathbf{x}_{n_m}) \\
&< \langle \mathbf{F}_1(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle - \epsilon t_{n_m} \\
&\to \langle \mathbf{F}_1(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle - \infty = -\infty,
\end{aligned}$$

as $m \to +\infty$. This contradicts (3.4). □

The following theorem shows that if a sequence of solutions of the penalized problems converges to a point $\mathbf{x}$ then $\mathbf{x}$ solves the original problem (VVIP). Note that if there exists some $n \in \mathbb{N}$ such that $\mathbf{x}_n \in S(t_n) \cap D$, then it is easy to verify that $\mathbf{x}_n$ is also a solution of (VVIP).

Theorem 3.10. *Assume that* $\mathbf{x}_n \in S(t_n)$ *for all* $n \in \mathbb{N}$. *Then any limit point of the sequence* $\{\mathbf{x}_n\}_n$ *is a solution of (VVIP).*

Proof. We assume that $\mathbf{x}$ is a limit point of the sequence $\{\mathbf{x}_n\}_n$. Let $\{\mathbf{x}_{n_m}\}_m$ be a subsequence of $\{\mathbf{x}_n\}_n$ which converges to $\mathbf{x}$. By Lemma 3.9, we already have $\mathbf{x} \in D$. Suppose for contradiction that $\mathbf{x}$ is not a solution of (VVIP). Then there exists $\mathbf{y} \in D$ satisfying $\langle \mathbf{F}_i(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle < 0$ for all $i = 1, \ldots, r$. Since $\mathbf{x}_{n_m}$ is a solution of (VVIP)$_{t_{n_m}}$, for $\mathbf{y}$ above, there exists an index i_{n_m} such that

$$\langle \mathbf{F}_{i_{n_m}}^{(t_{n_m})}(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle \geq 0.$$

Since $i_{n_m} \in \{1, 2, \ldots, r\}$, there exists an infinite sequence $\{i_{n_{m_k}}\}_k$ of indices, all of which have the same value, say $i_{n_{m_k}} = 1$, for all $k \in \mathbb{N}$. Again, to simplify the notation, we assume that the sequence $\{i_{n_m}\}_m$ itself satisfies this property, namely, $i_{n_m} = 1$ for all $m \in \mathbb{N}$. Therefore for all $m \in \mathbb{N}$ we have

$$\langle \mathbf{F}_1^{(t_{n_m})}(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle \geq 0. \tag{3.5}$$

Since

$$\langle \mathbf{F}_1(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle < 0,$$

we deduce that

$$\langle \mathbf{F}_1(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle < -\epsilon$$

for some $\epsilon > 0$. Taking into account the continuity of $\mathbf{F}$ and the assumption that $\mathbf{x}_{n_m} \to \mathbf{x}$, it follows that for all sufficiently large m we have

$$\langle \mathbf{F}_1(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle < -\epsilon.$$

Hence, for sufficiently large m,

$$\begin{aligned}
\langle \mathbf{F}_1^{(t_{n_m})}(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle &= \langle \mathbf{F}_1(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle + t_{n_m} \langle \nabla \mathbf{P}(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle \\
&\leq \langle \mathbf{F}_1(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle + t_{n_m}(\mathbf{P}(\mathbf{y}) - \mathbf{P}(\mathbf{x}_{n_m})) \\
&= \langle \mathbf{F}_1(\mathbf{x}_{n_m}), \mathbf{y} - \mathbf{x}_{n_m} \rangle - t_{n_m}\mathbf{P}(\mathbf{x}_{n_m}) \\
&< -\epsilon.
\end{aligned}$$

This contradicts (3.5). Here we employ the fact that $\mathbf{P}(\mathbf{x}_{n_m}) \geq 0$ and $t_{n_m} > 0$ for all $m \in \mathbb{N}$. ☐

Definition 3.11. A function $\mathbf{F} : \mathbb{R}^k \to \mathbb{R}^{r \times k}$ is said to be *strong D-coercive on $\mathbb{R}^k$* if all of its component functions are D-coercive on $\mathbb{R}^k$ with the same $\mathbf{a}$, namely, there exists a vector $\mathbf{a} \in D$ such that for all $j = 1, \ldots, r$,

$$\langle \mathbf{F}_j(\mathbf{y}), \mathbf{y} - \mathbf{a} \rangle \to +\infty, \text{ as } \|\mathbf{y}\| \to +\infty, \mathbf{y} \in \mathbb{R}^k.$$

It is easy to verify that the function $\mathbf{F}$ and the region D defined in Example 3.8 satisfy that $\mathbf{F}$ is strong D-coercive on $\mathbb{R}^k$. It is also clear that if $\mathbf{F}$ is strong D-coercive on $\mathbb{R}^k$, then $\mathbf{F}$ is also D-coercive on $\mathbb{R}^k$. However, the converse is not always true. For example, consider $D = \{0\}$ and

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 & x_2/4 \\ x_1 & x_2 \end{pmatrix}.$$

Take $\mathbf{s} = (1,1)^T$. Since

$$\begin{aligned}
\langle \mathbf{s}^T \mathbf{F}(\mathbf{y}), \mathbf{y} - 0 \rangle &= \langle \mathbf{F}_1(\mathbf{y}) + \mathbf{F}_2(\mathbf{y}), \mathbf{y} \rangle \\
&= 2y_1^2 + y_1 y_2 + \frac{5}{4} y_2^2 \\
&\to +\infty
\end{aligned}$$

as $\|\mathbf{y}\| \to +\infty$, we conclude that $\mathbf{F}$ is D-coercive on $\mathbb{R}^2$. On the other hand, as

$$\langle \mathbf{F}_1(\mathbf{y}), \mathbf{y} - 0 \rangle = (y_1 + y_2/2)^2 = 0$$

for all $\mathbf{y} = (y_1, -2y_1)^T$, with $y_1 \to +\infty$, we deduce that $\mathbf{F}_1$ is not even D-coercive on $\mathbb{R}^2$. Therefore, $\mathbf{F}$ is D-coercive on $\mathbb{R}^2$, but not strong D-coercive on $\mathbb{R}^2$.

Theorem 3.12. *Let $D \neq \varnothing$ be a closed convex subset of $\mathbb{R}^k$, and $\mathbf{F} : \mathbb{R}^k \to \mathbb{R}^{r\times k}$ a continuous, monotone function which is strong D-coercive on $\mathbb{R}^k$. Suppose that $\mathbf{x}_n \in S(t_n)$ for all $n \in \mathbb{N}$. Then the sequence $\{\mathbf{x}_n\}_n$ has at least one limit point, and every limit point of this sequence is a solution of (VVIP).*

Proof. Note that since $\mathbf{F}$ is also D-coercive on $\mathbb{R}^k$, by Lemma 3.7, $S(t_n) \neq \varnothing$ for all $n \in \mathbb{N}$, hence the sequence $\{\mathbf{x}_n\}_n$ stated in the theorem is well-defined. We show that $\{\mathbf{x}_n\}_n$ is bounded, namely, there exists some ball B of finite radius, centered at the origin, such that $\mathbf{x}_n \in B$ for all $n \in \mathbb{N}$.

Let $\mathbf{a}$ be the constant vector stated in Definition 3.11. For each $t > 0$, we define $B(t)$ as the smallest closed ball centered at the origin, such that for all $\mathbf{y} \notin B(t)$ and for all $j = 1, \dots, r$, we have

$$\langle \mathbf{F}_j^{(t)}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle > 0,$$

or in other words,

$$\langle \mathbf{F}_j(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle + t\langle \nabla\mathbf{P}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle > 0.$$

As each $\mathbf{F}_j$ is D-coercive on $\mathbb{R}^k$ and

$$\langle \nabla\mathbf{P}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle \geq \mathbf{P}(\mathbf{y}) - \mathbf{P}(\mathbf{a}) = \mathbf{P}(\mathbf{y}) \geq 0,$$

we deduce that for every $t > 0$, $B(t)$ has a finite radius.

We claim that $S(t) \subseteq B(t)$ for every $t > 0$. For suppose $\mathbf{y} \in S(t)\backslash B(t)$, then by the definition of $B(t)$, for all $j = 1, \dots, r$ we have

$$\langle \mathbf{F}_j^{(t)}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle > 0,$$

which results in $\langle \mathbf{F}_j^{(t)}(\mathbf{y}), \mathbf{a} - \mathbf{y}\rangle < 0$ for all $j = 1, \dots, r$. This contradicts our assumption that $\mathbf{y} \in S(t)$.

On the other hand, we claim that if $t' > t$, then $B(t') \subseteq B(t)$. Indeed, for all $\mathbf{y} \notin B(t)$, we have

$$\langle \mathbf{F}_j(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle + t\langle \nabla\mathbf{P}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle > 0,$$

which implies

$$\langle \mathbf{F}_j(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle + t'\langle \nabla\mathbf{P}(\mathbf{y}), \mathbf{y} - \mathbf{a}\rangle > 0 \tag{3.6}$$

for all $j = 1, \dots, r$. By definition, $B(t')$ is the smallest ball such that for all $\mathbf{y} \notin B(t')$, the inequality (3.6) is satisfied. Therefore, $B(t')$ is contained in $B(t)$.

Finally, as $\{t_n\}_n$ is monotonically increasing, we have

$$B(t_1) \supseteq B(t_2) \supseteq \cdots \supseteq B(t_n) \supseteq \cdots$$

Therefore, for all $n \in \mathbb{N}$, $\mathbf{x}_n \in S(t_n) \subseteq B(t_n) \subseteq B(t_1)$. Since the radius of $B(t_1)$ is finite, we conclude that the sequence $\{\mathbf{x}_n\}_n$ is bounded. Thus, it has at least one limit point. The assertion that every limit point of $\{\mathbf{x}_n\}_n$ is a solution of (VVIP) comes directly from Theorem 3.10. □

4. Conclusion

This paper studies the vector variation inequality problem using the penalty function method. Some basic convergence theorems are established. Note that instead of transforming the D-constrained problem into a sequence of unconstrained problems, namely $\mathbb{R}^k$-constrained problems, we can transform it into a sequence of Z-constrained problems, for some subset Z of $\mathbb{R}^k$, $Z \supseteq D$. All the results presented in this paper hold almost verbatim for this case, with $\mathbb{R}^k$ replaced by Z. Moreover, if Z is bounded, then all the results in this paper still hold without any assumption on the coercivity of $\mathbf{F}$.

Under certain conditions, a weak convex vector optimization problem is equivalent to a vector variational inequality problem (see, for instance [2] for more details). Therefore the results obtained in this paper have direct applications to the vector optimization problem. Note that, however, not every vector variational inequality problem is equivalent to a weak vector optimization problem.

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