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CONVERGENCE THEOREMS FOR PSEUDO-COMPLETE LOCALLY CONVEX ALGEBRAS

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Abstract. The set of bounded elements of a locally convex algebra is characterized as the union of certain naturally defined normed subalgebras. Pseudo-complete locally convex algebras are characterized in terms of the completeness of these subalgebras. In this paper it is proved, among other results, that a convergent sequence of elements in a bounded pseudo-complete locally convex strict inductive limit algebra is locally convergent. Along the way, a theorem on i-bounded sets is obtained, which identifies non-normable locally convex algebras in which every bounded subset is i-bounded.

1. Introduction

The following definition is from Webb [10].

Definition 1.1. A sequence $\{x_n\}$ of points of a locally convex space (E, τ) is said to be Mackey-convergent to x_0 (we write $x_n \rightarrow_M x_0$) if there exists a sequence of positive numbers $\{\alpha_n\}$ tending to ∞ such that $\{\alpha_n(x_n - x_0)\}$ is τ-null.

Köthe ([7, Section 28, 3.(1)]) has shown that: a sequence $\{x_n\}$ is Mackey-convergent to x_0 if and only if there exists a bounded, absolutely convex, closed subset B of $E := (E, \tau)$ such that $\{x_n\}$ and x_0 lie in $E_B := \bigcup_{n=1}^{\infty} nB$, and $p_B(x_n - x_0) \to 0$, where p_B is the norm on E_B which has B as its unit ball. It may be remarked that Köthe names this concept 'local convergence'.

Mackey-convergence implies convergence in the topology τ (Webb [10]).

Conversely, in a metrizable locally convex space every convergent sequence is Mackey-convergent (Köthe [7, Section 28, 1.(5)]). In Section 4 of the present paper, it is proved that a similar result holds good for every convergent sequence of points in a pseudo-complete locally convex strict inductive limit algebra $E := (E, \tau)$ whose every element is bounded. It is also shown that in a metrizable pseudo-complete locally convex algebra, in which the idempotent envelope of each bounded subset is bounded, the normed space E_B may be embedded in

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a Banach algebra. An example of a locally convex algebra, in which sequential convergence does not imply Mackey-convegence is given. Section 3 is concerned with i-bounded sets (Warner [9]). In Section 3, we identify pseudo-complete locally convex algebras not necessarily normable in which each bounded subset is i-bounded. Section 2 contains some preliminaries concerning locally convex algebras, particularly pseudo-complete algebras, which are required for the remainder of this paper.

2. Preliminaries

In what follows, an algebra stands for a linear associative algebra over the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. A locally convex algebra is an algebra E endowed with a Hausdorff topology τ such that (E, τ) is a locally convex space and such that multiplication in the algebra is separately continuous. A locally convex algebra E is said to be locally m-convex if the topology τ of E is defined by a family $\{p_\alpha : \alpha \in \mathbf{A}\}$ of semi-norms satisfying the sub-multiplicative condition:

$$p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$$

for all $x, y \in E$ and $\alpha \in \mathbf{A}$. Every normed algebra is a locally m-convex algebra.

Definitions 2.1 and 2.2 below are found in Allan [1].

Definition 2.1. Let E be a locally convex algebra and let $\mathcal{B}_1$ denote the collection of all subsets B of E satisfying

(i) B is absolutely convex and $B^2 \subset B$,
(ii) B is bounded and closed.

If E has an identity $\mathbf{1}$, we take $\mathbf{1} \in B$. For every $B \in \mathcal{B}_1$, $E(B) = \text{span}(B)$, the linear span of B, forms an algebra normed by the Minkowski functional $p_B(.)$ of B. Note that

$$p_B(x) := \inf\{\lambda > 0 : x \in \lambda B\} \quad (x \in E(B)).$$

If for every $B \in \mathcal{B}_1$, $E(B)$ is a Banach algebra with respect to p_B, then E is said to be pseudo-complete. By Proposition 2.6 (Allan [1]) if E is sequentially complete then E is pseudo-complete. Pseudo-complete normed algebras are Banach.

Note. Unless otherwise stated, it is assumed throughout that $E(B)$ carries the topology induced by the norm p_B; since B is bounded in (E, τ), the norm topology on $E(B)$ is at least as strong as its topology as a subspace of (E, τ).

Definition 2.2. An element x of a locally convex algebra $E := (E, \tau)$ is bounded if, and only if, for some non-zero complex number λ, the set

$$\{(\lambda x)^n : n = 1, 2, \dots\}$$

is bounded in E. The set of all bounded elements of E is denoted by E_0.

We note that an identity $\mathbf{1}$ in a locally convex algebra is necessarily a bounded element, for the set $\{\mathbf{1}^n : n = 1, 2, \dots\} = \{\mathbf{1}\}$ is certainly bounded.

Every element of a normed algebra is bounded (Allan [1], p. 400). Also, in a pseudo-complete locally m-convex algebra which is a Q-algebra every element is bounded.

Let $E := (E, \tau)$ be a locally convex algebra in which each element is bounded (i.e., $E = E_0$, in the notation above). Let $\mathcal{B} = \{B\}$ denote a family of bounded, absolutely convex, closed subsets B of E such that $B^2 \subset B$ and that each such subset of $\mathcal{B}$ is contained in some B in $\mathcal{B}$. By Propositions (2.2)-(2.4) (Allan [1]),

$$E = E_0 = \cup\{E(B) : B \in \mathcal{B}\}.$$

Then E is called a bounded locally convex algebra. A bounded pseudo-complete algebra is, in fact, algebraically, the inductive limit of the inductive system given by the (sort of increasing) net $\{E(B)\}_{B\in\mathcal{B}}$ and the continuous algebra isomorphism $I_{B'B}$ (the injection of $E(B)$ into $E(B')$ for $B \leq B'$. Here $B \leq B' \Leftrightarrow \exists\, \lambda > 0$ such that $B \subset \lambda B'$).

Conversely, an inductive limit of a system of complex Banach algebras and continuous (unital) isomorphisms equipped with pointwise multiplication, is a bounded pseudo-complete algebra. We refer the reader to Allan *et al.* [2], and Beckenstein *et al.* [3] for details.

In Allan *et al.* [2], and Husain and Warsi [6] several examples of bounded pseudo-complete locally convex algebras are listed.

3. A theorem concerning i-bounded sets

The concept of idempotently bounded (hereafter abbreviated i-bounded) sets is due to Warner ([9], p. 197).

If A is a subset of an algebra E, there exists a smallest subset containing A which is idempotent, namely, the intersection of all idempotent subsets of E containing A. This set is also the same as $\bigcup_{n=1}^{\infty} A^n$.

Definition 3.1. If A is any subset of an algebra E, the idempotent envelope of A is the smallest idempotent subset containing A.

Definition 3.2 (cf. Warner [9], p. 197)**.** Let E be a locally convex algebra and let $A \subseteq E$. A is said to be i-bounded if for some $\lambda > 0$, λA is contained in a bounded, idempotent set, or equivalently, if for some $\lambda > 0$, the idempotent envelope of λA is bounded.

Every bounded subset of a normed algebra is i-bounded. We identify below some non-normable locally convex algebras whose every bounded subset is i-bounded.

Let $\mathbf{A}$ denote a countable index set, let $\mathcal{B}_1 := \{B_\alpha : \alpha \in \mathbf{A}\}$, a collection of subsets satisfying the conditions (i) and (ii) of Definition 2.1, and let $E_\alpha := E(B_\alpha)$. For $\alpha' > \alpha$ and some $\lambda > 0$, we have $B_\alpha \subset \lambda B_{\alpha'}$ $(\alpha, \alpha' \in \mathbf{A})$ so that $E_\alpha \subset E_{\alpha'}$. We prove the following theorem.

Theorem 3.3. *Let $E := (E, \tau)$ be a bounded pseudo-complete locally convex algebra with respect to the bound structure $\mathcal{B}$. Suppose E is the inductive limit of* *the Banach algebras $\{E_\alpha : \alpha \in \mathbf{A}\}$ such that a subset B of E is bounded if and only if $B \subset E_{\alpha_0}$ for some $\alpha_0 \in \mathbf{A}$ and is bounded in E_{α_0}. Then every bounded subset B of E is i-bounded.*

Proof. Let B be a bounded subset of E. Then for each balanced o-neighborhood V in E there exists $\lambda_V > 0$ such that $B \subset \lambda_V V$ (or what is the same as $\lambda_V^{-1} B \subset V$). Select $\alpha_0 \in \mathbf{A}$ such that $B \subset E_{\alpha_0}$ and let W_{α_0} denote the closed unit ball of E_{α_0}. By the hypothesis, there exists $\lambda > 0$ such that $\lambda B \subset W_{\alpha_0}$. W_{α_0} is an idempotent set so that $W_{\alpha_0} W_{\alpha_0} \subset W_{\alpha_0}$ implies

$$(\lambda B)(\lambda B) \subset W_{\alpha_0} W_{\alpha_0} \subset W_{\alpha_0}.$$

Therefore,

$$B_1 := \bigcup_{k=1}^{\infty} (\lambda B)^k \subset W_{\alpha_0}.$$

The set B_1 so defined is an idempotent, i.e., $B_1 B_1 \subset B_1$. □

Let us give an example.

Example 3.4. Consider the algebra φ of complex sequences with only finitely many non-zero coordinates. Then its algebraic dual is the space (in fact, algebra) ω of all complex sequences under the duality given by $\langle x, f\rangle = \sum_{n=1}^{\infty} f_n \zeta_n$ for $x = (\zeta_n) \in \varphi$ and $f = (f_n) \in \omega$. So the Mackey topology $\tau(\varphi, \omega)$ is the finest locally convex topology on φ. A base of o-neighborhoods for $\tau(\varphi, \omega)$ is given by:

$$\left\{ U_f = \left\{ x = (\zeta_n) \in \varphi : \sum_{n=1}^{\infty} |f_n \zeta_n| \leq 1 \right\},\ \ f = (f_n) \in \omega \right\}.$$

With this system of o-neighborhoods, $(\varphi, \tau(\varphi, \omega))$ is a locally m-convex algebra whose underlying locally convex space is a bornological (DF)-space (Chilana and Sharma [4], Example 4.5, p. 149).

For each fixed $n \in \mathbb{N}$, consider $l_n^\infty(\mathbb{C})$ as a finite dimensional normed subalgebra of $(\varphi, \tau(\varphi, \omega))$. Note that $l_n^\infty(\mathbb{C})$ is viewed as the direct sum algebra $\bigoplus_{i=1}^{n} \mathbb{C}_{(i)}$, with $\mathbb{C}_{(i)} = \mathbb{C}$ for $i = 1, 2, \ldots, n$ endowed with the norm $\|\cdot\|_{n,\infty}$ defined as

$$\|x\|_{n,\infty} := \sup_{1 \leq i \leq n} |\zeta_i|,\ \ x = (\zeta_1, \ldots, \zeta_n) \in \bigoplus_{i=1}^{n} \mathbb{C}_{(i)}.$$

The element $x \in \bigoplus_{i=1}^{n} \mathbb{C}_{(i)}$ is identified with the sequence $(\zeta_1, \zeta_2, \ldots, \zeta_n, 0, 0, 0, \ldots) \in \varphi$.

For a fixed $n \in \mathbb{N}$, let B_n be the closed unit ball of $l_n^\infty(\mathbb{C})$, let $E(B_n)$ denote the span(B_n) and let $p_{B_n}(\cdot)$ denote the Minkowski functional of B_n. Then $p_{B_n}(\cdot)$ is a submultiplicative norm so that $(E(B_n), p_{B_n}(\cdot))$ is a normed subalgebra of φ.

Note that $B_n \subset B_{n'}$ for $n' > n$ which implies that $p_{B_{n'}}(\cdot) \leq p_{B_n}(\cdot)$. Thus the following inclusion is strict:

$$E(B_1) \subset E(B_2) \subset E(B_3) \subset \cdots, \tag{1}$$

all embeddings in (1) are continuous.

The Minkowski functional $p_{B_n}(\cdot)$ of each B_n is a norm, which is equivalent to the original norm $\|\cdot\|_{n,\infty}$, since $l_n^\infty(\mathbb{C})$ is a finite-dimensional space. It is not hard to see, via the standard basis of $l_n^\infty(\mathbb{C})$, that $\dim E(B_n) = \dim l_n^\infty(\mathbb{C})$ (in fact, the standard basis of each $l_n^\infty(\mathbb{C})$ lie on the boundary of the corresponding B_n and span $E(B_n)$). Therefore $(E(B_n), p_{B_n}(\cdot)) \cong l_n^\infty(\mathbb{C})$ for every n. Since $(\varphi, \tau(\varphi, \omega))$ is a bornological (DF)-space then by Section 28, 2.(2); Section 29, 6.(5) Köthe [7], we have

$$\varphi = s - \underrightarrow{\lim}\{E(B_n) : n \in \mathbb{N}\}$$

with the injection $i_{E(B_n)}$ from each $E(B_n)$ into φ. Hence φ can be expressed as

$$\varphi = \bigcup_{n \in \mathbb{N}} E(B_n) \qquad \text{(cf. Köthe [7], p. 222).}$$

By Section 30, 5.(9) Köthe [7], $(\varphi, \tau(\varphi, \omega))$ is a complete Hausdorff space. It is now clear that $(\varphi, \tau(\varphi, \omega))$ is a bounded pseudo-complete locally convex algebra with respect to the bound structure $\mathcal{B}_n = \{B_n\}_{n \in \mathbb{N}}$. By Köthe ([7], Section 18, 5.(4)) any bounded subset of $(\varphi, \tau(\varphi, \omega))$ is contained in some $E(B_n)$ and is bounded there. Therefore, by Theorem 3.3 above, each bounded subset of $(\varphi, \tau(\varphi, \omega))$ is i-bounded.

Remark 3.5. The result of Theorem 3.3 covers any locally convex algebra $E = \bigcup_{\alpha \in \mathbf{A}} E_\alpha$, which is an inductive limit of the Banach algebras E_α such that each bounded subset B of E is contained in some E_{α_0} and is bounded there.

For example, let G be a real Lie group, $\mathcal{G}$ the complex Lie algebra of G and $E(\mathcal{G})$ the complex universal enveloping algebra of $\mathcal{G}$. For any integer $n \geq 0$, let $E_n(\mathcal{G})$ denote the finite-dimensional subalgebra of elements of $E(\mathcal{G})$ of order $\leq n$. The collection $(E_n(\mathcal{G}))_{n \geq 0}$ is such that

$$E_0(\mathcal{G}) \subset E_1(\mathcal{G}) \subset E_2(\mathcal{G}) \subset \cdots, E(\mathcal{G}) = \bigcup_{n=0}^{\infty} E_n(\mathcal{G})$$

(see Dixmier [5], p.75 for the details). Equipped with the strict inductive limit topology τ with respect to the $E_n(\mathcal{G})$, $E(\mathcal{G})$ is a complete locally convex unital algebra. Multiplication in $E(\mathcal{G})$ is jointly continuous, because bilinear maps on $E(\mathcal{G}) \times E(\mathcal{G})$ are continuous. $(E(\mathcal{G}), \tau)$ satisfies the result of Theorem 3.3.

4. The convergence theorems

We can now prove the following theorems.

Theorem 4.1. *Let (E, τ) be a metrizable pseudo-complete locally convex algebra in which the idempotent envelope of each bounded set is bounded and let $\{x_n\}$ be a sequence of elements of E which converges to x_0. Then there exists a bounded,* *absolutely convex, closed and idempotent subset B^* of E (with $\mathbf{1} \in B^*$ if E has an identity $\mathbf{1}$) such that both $\{x_n\}$ and x_0 all lie in a Banach algebra $E(B^*)$ and $p_{B^*}(x_n - x_0) \to 0$ as $n \to \infty$.*

Proof. Let $E := (E, \tau)$ be a metrizable pseudo-complete locally convex algebra satisfying the specified condition. As (E, τ) is a metrizable locally space then by Section 28, 3.(1)(c) (Köthe [7]) $\exists$ a bounded, closed absolutely convex subset B of (E, τ) such that $\{x_n\}$ and x_0 lie in the normed space E_B and $p_B(x_n - x_0) \to 0$ as $n \to \infty$. Note that B is not assumed to be an idempotent subset of (E, τ). Therefore we consider

$$B_1 \;:=\; \cup\{B^n : n = 1, 2, \dots\} \text{ or,}$$
$$B_1 \;:=\; \left(\bigcup_{n\in\mathbb{N}} B^n\right) \cup \{\mathbf{1}\}$$

if (E, τ) has an identity $\mathbf{1}$ (p. 86, Waelbroeck [8]). Warner [9] has shown that B_1 is the smallest idempotent set containing B. B_1 is bounded by the hypothesis. Let B^* denote the closed, absolutely convex hull of B_1. Then B^* belongs to $\mathcal{B}_1$ or $\mathcal{B} = \{B \in \mathcal{B}_1 : \mathbf{1} \in B\}$ as the case may be and $B \subset B^*$. Thus B^* is a bounded, absolutely convex, closed subset of $E := (E, \tau)$ such that $(B^*)^2 \subset B^*$ (with $\mathbf{1} \in B^*$ as the case may be) and $B \subset B^*$. As $B \subset B^*$, it is obvious that $E_B \subset E(B^*)$ and in this case

$$p_{B^*}(x) \le p_B(x) \qquad \forall\, x \in E_B$$

so the injection of E_B into $E(B^*)$ is continuous. It follows from the inclusion relation $E_B \subset E(B^*)$ that both $\{x_n\}$ and x_0 all lie in $E(B^*)$ while the last inequality shows that $p_{B^*}(x_n - x_0) \to 0$ as $n \to \infty$. □

The following lemma is required for the proof of Theorem 4.3 below.

Lemma 4.2. *A convergent sequence of points in a Hausdorff topological vector space is bounded.*

Proof. Let $E := (E, \tau)$ be a Hausdorff topological vector space and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E such that $x_n \longrightarrow_\tau x_0$ as $n \longrightarrow \infty$. For each $n \in \mathbb{N}$, put $y_n = x_n - x_0$. Then $\lim\limits_{n\to\infty} y_n = 0$. Let U be a 0-neighborhood in E. There exists a balanced 0-neighborhood V in E such that $V \subset U$. Then $V \subset \nu V$ for all ν with $|\nu| \ge 1$. As $y_n \longrightarrow_\tau 0$ as $n \longrightarrow \infty$, there exists $n_0 \in \mathbb{N}$ such that $y_n \in V$ whenever $n \ge n_0$. Hence $y_n \in V \subset \mu V \subset \mu U$ whenever $n \ge n_0$ and $\mu \ge 1$. Let $P = \{y_1, \cdots, y_{n_0}\}$ and $Q = \{y_n : n > n_0\}$. Since P is a finite set, it is bounded so that $P \subset \mu U$ for all sufficiently large μ. Thus $P \cup Q \subset \mu U$ for sufficiently large μ, that is, $\{y_n\}_{n\in\mathbb{N}}$ is bounded. But $\{x_n\}_{n\in\mathbb{N}} = x_0 + P \cup Q$ and so it is bounded. □

Theorem 4.3. *Let $E := (E, \tau)$ be a bounded pseudo-complete locally convex algebra (with respect to the completant bound structure $\mathcal{B} = \{B\}$) in which every τ-bounded subset is i-bounded. Assume that the relative $E-$ topology on each $E(B)$ coincides with its norm topology. Then for any sequence of elements $x_n \in E$*

with $x_n \longrightarrow_\tau x_0 \in E$ there exists a Banach algebra $E(B^*)$ such that both $(x_n), x_0$ all lie in $E(B^*)$ and $p_{B^*}(x_n - x_0) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. Let (x_n) be a sequence of elements $x_n \in E$ and let $\lim_{n\to\infty} x_n = x_0 \in E$. Let $\epsilon > 0$ be given. Then for each 0-neighborhood U in E there exists $n_0 = n_0(\epsilon, U) \in \mathbb{N}$ such that $x_n - x_0 \in U$ whenever $n \geq n_0$. Since, by Lemma 4.2, (x_n) is bounded, let B be a bounded subset of E with $(x_n) \subset B$. Since $\lambda B \subset B_0$ for some $\lambda > 0$ and some bounded indempotent subset B_0 in E, then $(x_n)_{n\in\mathbb{N}} \subset \lambda^{-1}B_0$ as well. Let B^* denote the closed absolutely convex hull of B_0. Then B^* is a closed bounded absolutely convex idempotent subset of E with $B_0 \subset B^*$. Thus $(x_n)_{n\in\mathbb{N}} \subset \lambda^{-1}B^*$, i.e., $x_n \in \lambda^{-1}B^*$ for every $n \in \mathbb{N}$. But $\lambda^{-1}B^*$ is a 0-neighborhood in $E(B^*)$. Thus $(x_n)_{n\in\mathbb{N}} \subset E(B^*)$.

Let $\{p_\gamma : \gamma \in \Gamma\}$ be a system of (continuous) seminorms defining the Hausdorff topology τ on E, and let $\tau|_{E(B^*)}$ be the topology induced by τ on $E(B^*)$. The identity map

$$i_{E(B^*)} : (E(B^*), p_{B^*}) \longrightarrow \left(E(B^*), \tau|_{E(B^*)}\right)$$

is bi-continuous. This is because the norm topology τ_{B^*} defined by p_{B^*} on $E(B^*)$ coincides with the topology $\tau|_{E(B^*)}$ induced by E. Therefore, by continuity of the inverse map $i^{-1}_{E(B^*)}$, there exists $\beta := \beta(\gamma, B^*) > 0$ such that

$$p_{B^*}(x) \leq \beta p_\gamma(x), \quad \forall\ x \in E(B^*), \quad \gamma \in \Gamma.$$

Hence, $p_\gamma(x_n - x_0) \longrightarrow 0$ as $n \longrightarrow \infty$ implies $p_{B^*}(x_n - x_0) \longrightarrow 0$ as $n \longrightarrow \infty$. Now $x_0 \in E(B^*)$ because $(E(B^*), p_{B^*}(\cdot))$ being a Banach space, each Cauchy sequence in $E(B^*)$ converges to a point of $E(B^*)$. □

Corollary 4.4. *Let $E := (E, \tau)$ be a bounded pseudo-complete locally convex strict inductive limit algebra with respect to a sequence of closed, bounded, absolutely convex, idempotent subsets $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$, and suppose $(x_n)_{n\in\mathbb{N}} \subset E$ is a sequence such that $\lim_{n\to\infty} x_n = x_0$ for some $x_0 \in E$. Then there exists some $B_{n_0} \in \mathcal{B}$ such that both $(x_n)_{n\in\mathbb{N}}$ and x_0 all lie in the Banach algebra $E_{n_0} := \big(E(B_{n_0}), p_{B_{n_0}}(\cdot)\big)$ with $p_{B_{n_0}}(x_n - x_0) \to 0$ as $n \to \infty$.*

Corollary 4.5. *A convergent sequence of elements in a bounded pseudo-complete locally convex strict inductive limit algebra is Mackey convergent.*

Here is an example of a locally convex algebra in which τ-sequential convergence is not Mackey-convergence.

Example 4.6. Let $I = [0, 1]$, an uncountable index set and let $\mathbb{R}^I$ denote the Cartesian product $\prod_{i\in I} \mathbb{R}_i$ (without topology), where for all i, $\mathbb{R}_i = \mathbb{R}$. Now let $\mathbb{R}^I$ be endowed with the product topology τ and consider $E := (\mathbb{R}^I, \tau)$ as an algebra, with jointly continuous multiplication defined pointwise. Then E is a bornological space.

Now the cardinality of $I = [0, 1]$ is $2^{\aleph_0}$. Also the set of all sequences of positive real numbers which tend to ∞ has cardinality $2^{\aleph_0}$, so may be put into one-to-one correspondence with the points of the interval $I = [0, 1]$. Denote this correspondence by $q : q(i) = \{\alpha_n\}$, where $i \in I$ and $0 < \alpha_n \to \infty$. Define the sequence $\{x_n\}$ in E by $x_n(i) = \frac{1}{\alpha_n}$, where $q(i) = \{\alpha_n\}$. Then $\{x_n\}$ is a τ-null sequence in E. But there does not exist a sequence $\{\alpha_n\}$ of positive numbers tending to ∞ such that $\{\alpha_n\, x_n\}$ is τ-null. In other words, $\{x_n\}$ is τ-convergent, but not Mackey-convergent, to 0. So E is a locally convex algebra in which τ-sequential convergence is not Mackey.

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