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GENERALIZED GROWTH OF SOLUTIONS TO A CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In the present paper, we study the generalized growth of solutions to a class of elliptic partial differential equations. The characterizations of generalized order and generalized type of solutions to a class of elliptic partial differential equations have been obtained in terms of its Taylor's series coefficients.

1. Introduction

Following McCoy [4], we first give some definitions. A Helmholtz type equation is given by

$$L[H] := [\partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + F(r^2)]H(r,\theta) = 0. \tag{1.1}$$

Here (r,θ) are polar coordinates and F is an entire function of r^2. Let $H(r,\theta) = H(r, e^{i\theta})$ be a regular solution of (1.1) in some sufficiently small star-shaped neighborhood Ω about origin. Let R be the radius of convergence of this regular solution. Following Bergman [1], we have

$$\begin{aligned} H(r,\theta) = B[f(z)] \ &= \ \textstyle\int_{-1}^{+1} E(r^2,t)\, f(\sigma)\, d\mu(t)\ , \\ \sigma = z(1-t^2)/2\ , \qquad & d\mu(t) = (1-t^2)^{-1/2} dt\ \ , \end{aligned}$$

where $z = re^{i\theta} \in \Omega$ and the associated f is analytic for $2z \in \Omega$. The Taylor's series expansion of kernel $E(r^2,t)$ is given as

$$E(r^2,t) = 1 + \sum_{n=1}^{\infty} t^{2n} Q^{2n}(r^2),$$

analytic for $t \in [-1,+1]$ and entire for $r \geq 0$. The Taylor coefficients $Q^{2n}(r^2)$ are entire function solutions of the system

$$\frac{d}{dr^2}\left(Q^2(r^2)\right) + 2F(r^2) = 0, \qquad Q^0(r^2) = 1,$$

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$$(2n+1)\frac{d}{dr^2}\left(Q^{2n+2}(r^2)\right)+2\frac{d}{dr^2}\left(r^2Q^{2n}(r^2)\right)+F(r^2)Q^{2n}(r^2)-n\frac{d}{dr^2}\left(Q^{2n}(r^2)\right)=0\ ,$$

$$Q^{2n+2}(r^2)|_{r=0}=0,\quad n=1,2,3...\ .$$

McCoy [4] defined the basic set of particular solutions

$$\Phi_n(r,e^{i\theta})=[r^nG_n(r^2)/R^nG_n(R^2)]e^{in\theta}$$

normalized by

$$\Phi_n(R,e^{i\theta})=e^{in\theta}\ ,\quad n=0,1,2,3...\ .$$

Here

$$G_n(r^2)=\int_{-1}^{+1}E(r^2,t)(1-t^2)^n d\mu(t).$$

This basic set is complete relative to compact convergence on a disk $D_R : |z| < R$. Let $J(D_R)$ be the space of regular solutions of (1.1) on D_R. Then $H \in J(D_R)$ has the uniformly convergent expansion

$$H(r,e^{i\theta})=\sum_{n=0}^{\infty}a_n\Phi_n(r,e^{i\theta})\ ,$$

where a_n are real numbers. If $A(D_R)$ is the space of analytic functions on D_R, then $f \in A(D_R)$ has the Taylor's series expansion

$$f(z)=\sum_{n=0}^{\infty}a_nz^n\ ,\qquad z\in D_R.$$

McCoy [4] associated H with f by defining an integral operator as given below

$$H(r,e^{i\theta})=T_\varepsilon[f(z)]=\frac{1}{2\pi i}\int_{|\zeta|=1-\varepsilon}K_R(\zeta)f(z/\zeta)d\zeta/\zeta,\qquad z=re^{i\theta}/R,$$

where $\varepsilon > 0$ is arbitrarily small. The kernel for this integral operator defined over the basis $\{\Phi_n\}$ is given by

$$K_R(\zeta)=\sum_{n=0}^{\infty}\zeta^n[G_n(r^2)/G_n(R^2)].$$

For $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, we have

$$(1-\varepsilon)\ \leq\ |G_n(r^2)/G_n(R^2)|\ \leq\ (1+\varepsilon).$$

Hence we can say that the kernel of this operator has a uniformly convergent series expansion. The above integral operator maps function $f \in A(D_{R(1-\varepsilon)})$ onto regular solution $H \in J(D_{R(1-\varepsilon)})$ and the disk of regularity of H coincides with the disk of analyticity of f.

McCoy obtained the characterizations of order and type of function H in terms of its Taylor's series coefficients (see [4], Theorem 2, pp. 209-210). In the present paper we have obtained the characterizations of generalized order and generalized type of function H in terms of its Taylor's series coefficients.
The maximum modulus of H on D_r is given by

$$M(r,H) = \max\{|H(s,e^{i\theta})| : s \leq r\}.$$

The definition of order and type for regular solution H are the same as those of the associated analytic function f (see [4], pp. 209). So the order ρ of regular solution H on D_R is given by

$$\rho = \lim_{r\to R}\sup \frac{\log^+ \log^+ M(r,H)}{\log[R/(R-r)]} \ ,$$

where

$$\log^+ x = \begin{cases} \log x \ , & x > 1 \\ 0 \ , & x \leq 1 \end{cases} \quad .$$

Further, for $0 < \rho < \infty$ the type σ of regular solution H on D_R is given by

$$\sigma = \lim_{r\to R}\sup \frac{\log^+ M(r,H)}{[R/(R-r)]^{\rho}}.$$

Now following Janik [3] and Seremeta [5] we define the generalized order and generalized type of function H. Hence, let L^0 denote the class of functions $h(x)$ satisfying the following conditions:

(i) $h(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii) $\lim_{x\to\infty} \frac{h\{(1+1/\psi(x))x\}}{h(x)} = 1$,
for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions $h(x)$ satisfying conditions (i) and

(iii) $\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$,
for every $c>0$, that is $h(x)$ is slowly increasing.
For $\alpha \in \Lambda$ and $\beta \in L^0$ we define the generalized order of H on D_R as

$$\rho(\alpha,\beta,H) = \lim_{r\to R}\sup \frac{\alpha\ [\log^+ M(r,H)]}{\beta\ [R/(R-r)]}. \tag{1.2}$$

Further for $\alpha,\ \beta,\ \gamma \in \Lambda$ and $0 < \rho < \infty$, we define the generalized type of H on D_R as

$$\sigma(\alpha,\beta,\gamma,\rho,H) = \lim_{r\to R}\sup \frac{\alpha\ [\log^+ M(r,H)]}{\beta\ \{[\gamma\{R/(R-r)\}]^{\rho}\}}. \tag{1.3}$$

2. Main results

We now prove

Theorem 2.1. *Let H be a regular solution of* (1.1) *on D_R $(R > 1)$ and have the series expansion $H(r, e^{i\theta}) = \sum_{n=0}^{\infty} a_n \Phi_n(r, e^{i\theta})$. For positive x and μ_1 set $U(x, \mu_1) = \beta^{-1}\{\mu_1 \alpha(x)\}$. Assume that $\alpha(x/U(x, \mu_1)) = [1 + o(x)]\,\alpha(x)$, when $x \to \infty$. Then for $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$ the generalized order ρ $(0 < \rho < \infty)$ of H is given by*

$$\rho = \rho(\alpha, \beta, H) = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\beta\left\{n/\log^+(|a_n|R^n)\right\}}. \tag{2.1}$$

Proof. Write

$$\eta_1 = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\beta\left\{n/\log^+(|a_n|R^n)\right\}}.$$

Now first we prove that $\eta_1 \le \rho$. From (1.2), for $\mu_1 > \rho$ and r sufficiently close to R, we have

$$\log^+ M(r, H) \le \alpha^{-1}[\mu_1 \beta\, \{R/(R - r)\}].$$

Now following Janik (see [2], Lemma 3.4), we have

$$|a_n| \le \frac{M(r, H)}{(r - 1)\, r^n} \;, \quad 1 < r < R.$$

So for every r sufficiently close to R, we get

$$\log^+(|a_n|R^n) \le -\log(r - 1) - n\log(r/R) + \alpha^{-1}[\mu_1 \beta\, \{R/(R - r)\}].$$

Putting

$$r = r_n = R\left[1 - 1/U\left(n/U(n, \mu_1^{-1}), \mu_1^{-1}\right)\right] \;,$$

we get

$$\log^+(|a_n|R^n) \le -\log(r_n - 1) - n\log\left[1 - 1/U\left(n/U(n, \mu_1^{-1}), \mu_1^{-1}\right)\right] + n/U(n, \mu_1^{-1}).$$

Now using the properties of logarithm and assumptions of the theorem, we get for sufficiently large value of n

$$\log^+(|a_n|R^n) \le C_1\, n\, /\beta^{-1}\left\{\mu_1^{-1}\, \alpha(n)\right\},$$

where C_1 is a positive constant. Hence by using the properties of β, we get

$$\frac{\alpha(n)}{\beta\left\{n/\log^+(|a_n|R^n)\right\}} \le \mu_1.$$

Now proceeding to limits and taking *sup* on both sides, we get

$$\eta_1 \le \mu_1.$$

Since $\mu_1 > \rho$ is arbitrarily, therefore finally we get

$$\eta_1 \le \rho. \tag{2.2}$$

Now we will prove that $\rho \le \eta_1$. If $\eta_1 = \infty$, then there is nothing to prove. So let us assume that $0 \le \eta_1 < \infty$. Therefore for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n > n_0$, we have

$$0 \le \frac{\alpha(n)}{\beta\left\{n/\log^+\left(|a_n|R^n\right)\right\}} \le \eta_1 + \varepsilon = \overline{\eta_1}$$

or

$$|a_n|R^n \;\le\; \exp\left[n\;/\beta^{-1}\left\{\left(\overline{\eta_1}\right)^{-1}\;\alpha(n)\right\}\right]$$

or

$$|a_n|r^n \;\le\; r^n R^{-n}\exp\left[n\;/\beta^{-1}\left\{\left(\overline{\eta_1}\right)^{-1}\;\alpha(n)\right\}\right]\;.$$

Now from the property of maximum modulus, we have

$$M(r,H) \;\le\; \sum_{n=0}^{\infty}|a_n|\,r^n$$

or

$$M(r,H) \le \sum_{n=0}^{n_0}|a_n|\,r^n + \sum_{n=n_0+1}^{\infty} r^n R^{-n}\exp\left[n\;/\beta^{-1}\left\{\left(\overline{\eta_1}\right)^{-1}\;\alpha(n)\right\}\right]$$

or

$$M(r,H) \le A_1 r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n R^{-n}\exp\left[n\;/\beta^{-1}\left\{\left(\overline{\eta_1}\right)^{-1}\;\alpha(n)\right\}\right]\;, \tag{2.3}$$

where A_1 is a positive real constant. We take

$$N(r) = \left[\alpha^{-1}\left(\overline{\eta_1}\beta\left\{[\log\{R/(N+1)r\}]^{-1}\right\}\right)\right],$$

where $[x]$ denotes the integer part of $x \ge 0$. Since $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$, the integer $N(r)$ is well defined. Now if r is sufficiently large, then from (2.3) we have

$$M(r,H) \le A_1 r^{n_0} + r^{N(r)}\textstyle\sum_{n_0+1\le n\le N(r)} R^{-n}\exp\left[n\;/\beta^{-1}\left\{\left(\overline{\eta_1}\right)^{-1}\;\alpha(n)\right\}\right]$$
$$+\textstyle\sum_{n>N(r)} r^n R^{-n}\exp\left[n\;/\beta^{-1}\left\{\left(\overline{\eta_1}\right)^{-1}\;\alpha(n)\right\}\right]$$

or

$$(2.4)\qquad \begin{aligned} M(r,H) \le A_1 r^{n_0} + r^{N(r)} \textstyle\sum_{n=1}^{\infty} R^{-n} \exp\left[n \,/\beta^{-1}\left\{(\overline{\eta_1})^{-1}\ \alpha(n)\right\}\right] \\ + \textstyle\sum_{n>N(r)} r^n R^{-n} \exp\left[n \,/\beta^{-1}\left\{(\overline{\eta_1})^{-1}\ \alpha(n)\right\}\right]\ . \end{aligned}$$

Now we have

$$\lim_{n\to\infty} \sup \left(R^{-n} \exp\left[n \,/\beta^{-1}\left\{(\overline{\eta_1})^{-1}\ \alpha(n)\right\}\right]\right)^{1/n} = \frac{1}{R} < 1.$$

Hence the first series in (2.4) converges to a positive real constant A_2. So from (2.4) we get

$$M(r,H) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} r^n R^{-n} \exp\left[n \,/\beta^{-1}\left\{(\overline{\eta_1})^{-1}\ \alpha(n)\right\}\right]$$

or

$$M(r,H) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} r^n R^{-n} \exp[n \log\{R/(N+1)r\}\,]$$

or

$$M(r,H) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} \left(\frac{1}{N+1}\right)^n$$

or

$$(2.5)\qquad M(r,H) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n=1}^{\infty} \left(\frac{1}{N+1}\right)^n .$$

It can be easily seen that the series in (2.5) converges to a positive real constant A_3. Therefore from (2.5), we get

$$M(r,H) \le A_1 r^{n_0} + A_2 r^{N(r)} + A_3$$

or

$$M(r,H) \ \le\ A_2 r^{N(r)}[1+o(1)]$$

or

$$\log M(r,H) \ \le\ [1+o(1)]\, N(r) \log r$$

or

$$\log M(r,H) \ \le\ [1+o(1)]\ \left[\alpha^{-1}\left(\overline{\eta_1}\beta\left\{[\log\{R/(N+1)r\}]^{-1}\right\}\right)\right] \log r$$

or

$$\log M(r,H) \ \le\ [1+o(1)]\ \alpha^{-1}\left[\{\overline{\eta_1}+\delta_1\}\,\beta\left\{[\log\{R/(N+1)r\}]^{-1}\right\}\right],$$

where $\delta_1 > 0$ is suitably small. Hence

$$\alpha[\log M(r,H)] \quad \leq \quad \{\overline{\eta_1} + \delta_1\}\, \beta\left\{[1+o(1)]^{-1}[\log(R/r)]^{-1}\right\}.$$

Thus for r sufficiently close to R, we get

$$\frac{\alpha[\log M(r,H)]}{\beta\left\{[1+o(1)]^{-1}[R/(R-r)]\right\}} \leq \quad \overline{\eta_1} + \delta_1.$$

Proceeding to limits as $r \to R$ and using property of β, we get

$$\limsup_{r\to R} \frac{\alpha[\log M(r,H)]}{\beta\left\{R/(R-r)\right\}} \leq \quad \overline{\eta_1} + \delta_1.$$

Since ε and δ_1 are arbitrarily small, therefore finally we get $\rho \leq \eta_1$. Combining this with the inequality (2.2), we get (2.1). Hence Theorem 2.1 is proved. □

Next we prove

Theorem 2.2. *Let H be a regular solution of* (1.1) *on D_R $(R > 1)$ and have the series expansion $H(r, e^{i\theta}) = \sum_{n=0}^{\infty} a_n \Phi_n(r, e^{i\theta})$. For positive x, μ_2 and ρ set*

$$V(x,\mu_2,\rho) = \gamma^{-1}\{[\beta^{-1}\left(\mu_2\, \alpha(x)\right)]^{1/\rho}\}.$$

Assume that

$$V\left(\frac{x\,(\rho+1)}{\rho\; V(x/\rho, 1/\mu_2, \rho+1)}, \frac{1}{\mu_2}, \rho\right) = [1+o(x)]\;\; V\left(x/\rho, 1/\mu_2, \rho+1\right),$$

when $x \to \infty$. Then for $\alpha(x),\ \beta(x),\ \gamma(x) \in \Lambda$ the generalized type σ $(0 < \sigma < \infty)$ of H is given by

(2.6)

$$\sigma = \sigma(\alpha,\beta,\gamma,\rho,H) = \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\;\left[\rho\log^{+}\left(|a_n|R^n\right)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}}.$$

Proof. Write

$$\eta_2 = \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\;\left[\rho\log^{+}\left(|a_n|R^n\right)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}}.$$

Now first we prove that $\eta_2 \leq \sigma$. From (1.3), for $\mu_2 > \sigma$ and r sufficiently close to R, we have

$$\log^{+} M(r,H) \;\leq\; \alpha^{-1}[\mu_2 \beta\{[\gamma\{R/(R-r)\}]^{\rho}\}].$$

Thus as in Theorem 2.1, here we have

$$\log^{+}\left(|a_n|R^n\right) \;\leq\; -\log(r-1) - n\log(r/R) + \alpha^{-1}[\mu_2\beta\{[\gamma\{R/(R-r)\}]^{\rho}\}].$$

Putting

$$r = r_n = R\left[1 - \left\{V\left(\frac{n\,(\rho+1)}{\rho\,V(n/\rho, 1/\mu_2, \rho+1)}, \frac{1}{\mu_2}, \rho\right)\right\}^{-1}\right],$$

we get

$$\log^+(|a_n|R^n) \le -\log(r_n - 1) - n\log\left[1 - \left\{V\left(\frac{n\,(\rho+1)}{\rho\,V(n/\rho, 1/\mu_2, \rho+1)}, \frac{1}{\mu_2}, \rho\right)\right\}^{-1}\right]$$
$$+ n\,\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{\mu_2^{-1}\,\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}.$$

Now using the properties of logarithm and assumptions of theorem, we get for sufficiently large value of n

$$\log^+(|a_n|R^n) \;\le\; C_2\, n\,\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{\mu_2^{-1}\,\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1},$$

where C_2 is a positive constant. Hence by using the properties of α, β and γ, we get

$$\frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\log^+(|a_n|R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}} \;\le\; \mu_2\,.$$

Now proceeding to limits and taking *sup* on both sides, we get

$$\eta_2 \le \mu_2.$$

Since $\mu_2 > \sigma$ is arbitrarily, therefore finally we get

$$\eta_2 \le \sigma. \tag{2.7}$$

Now we will prove that $\sigma \le \eta_2$. If $\eta_2 = \infty$, then there is nothing to prove. So let us assume that $0 \le \eta_2 < \infty$. Therefore for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n > n_0$, we have

$$0 \le \frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\log^+(|a_n|R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}} \le \eta_2 + \varepsilon = \overline{\eta_2}$$

or

$$|a_n|R^n \;\le\; \exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1}\,\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}$$

or

$$|a_n|r^n \leq r^n R^{-n} \exp\left\{ n\frac{\rho+1}{\rho} \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}.$$

Now from the property of maximum modulus, we have

$$M(r,H) \leq \sum_{n=0}^{\infty} |a_n|\, r^n$$

or

$$M(r,H) \leq \sum_{n=0}^{n_0} |a_n|\, r^n + \\ + \sum_{n=n_0+1}^{\infty} r^n R^{-n} \exp\left\{ n\tfrac{\rho+1}{\rho} \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}$$

or

(2.8)

$$M(r,H) \leq B_1 r^{n_0} + \\ + \sum_{n=n_0+1}^{\infty} r^n R^{-n} \exp\left\{ n\tfrac{\rho+1}{\rho} \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\},$$

where B_1 is a positive real constant. We take

$$N(r) = \left[\rho\alpha^{-1}\left\{\overline{\eta_2}\beta\left([\gamma\{(\rho+1)[\rho\log\{R/(N+1)r\}]^{-1}\}]^{(\rho+1)}\right)\right\}\right],$$

where $[x]$ denotes the integer part of $x \geq 0$. Since $\alpha(x)$, $\beta(x)$, $\gamma(x) \in \Lambda$, the integer $N(r)$ is well defined. Now if r is sufficiently large, then from (2.8) we have

$$M(r,H) \leq B_1 r^{n_0} + r^{N(r)} \times \\ \times \sum_{n_0+1\leq n\leq N(r)} R^{-n} \exp\left\{ n\tfrac{\rho+1}{\rho} \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\} \\ + \sum_{n>N(r)} r^n R^{-n} \exp\left\{ n\tfrac{\rho+1}{\rho} \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}$$

or

(2.9)

$$M(r,H) \leq B_1 r^{n_0} + r^{N(r)} \times \\ \times \sum_{n=1}^{\infty} R^{-n} \exp\left\{ n\tfrac{\rho+1}{\rho} \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\} \\ + \sum_{n>N(r)} r^n R^{-n} \exp\left\{ n\tfrac{\rho+1}{\rho} \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}.$$

Now we have

$$\lim_{n\to\infty}\sup\left(R^{-n}\exp\left\{\ n\tfrac{\rho+1}{\rho}\ \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2}\,)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}\right)^{1/n}$$
$$=\lim_{n\to\infty}\sup\left(R^{-1}\exp\left\{\ \tfrac{\rho+1}{\rho}\ \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2}\,)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}\right)$$
$$=\tfrac{1}{R}<1.$$

Hence the first series in (2.9) converges to a positive real constant B_2. So from (2.9), we get

$$M(r,H)\le B_1r^{n_0}+B_2r^{N(r)}+$$
$$+\textstyle\sum_{n>N(r)}r^nR^{-n}\exp\left\{\ n\tfrac{\rho+1}{\rho}\ \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\overline{\eta_2}\,)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}$$

or

$$M(r,H)\le B_1r^{n_0}+B_2r^{N(r)}+\sum_{n>N(r)}r^nR^{-n}\exp[n\log\{R/(N+1)r\}\,]$$

or

$$M(r,H)\le B_1r^{n_0}+B_2r^{N(r)}+\sum_{n>N(r)}\left(\frac{1}{N+1}\right)^n$$

or

$$M(r,H)\le B_1r^{n_0}+B_2r^{N(r)}+\sum_{n=1}^{\infty}\left(\frac{1}{N+1}\right)^n. \tag{2.10}$$

It can be easily seen that the series in (2.10) converges to a positive real constant B_3. Therefore from (2.10), we get

$$M(r,H)\le B_1r^{n_0}+B_2r^{N(r)}+B_3$$

or

$$M(r,H)\ \le\ B_2r^{N(r)}[1+o(1)]$$

or

$$\log M(r,H)\ \le\ [1+o(1)]\,N(r)\log r$$

or

$$\log M(r,H)\ \le\ [1+o(1)]\times$$
$$\times\ \left[\rho\alpha^{-1}\left\{\overline{\eta_2}\beta\left([\gamma\{(\rho+1)[\rho\log\{R/(N+1)r\}]^{-1}\}]^{(\rho+1)}\right)\right\}\right]\log r$$

or

$$\log M(r,H) \le [1+o(1)]\times$$
$$\times \left[\alpha^{-1}\left\{(\overline{\eta_2}+\delta_2)\,\beta\left([\gamma\{(\rho+1)[\rho\log\{R/(N+1)r\}]^{-1}\}]^{(\rho+1)}\right)\right\}\right],$$

where $\delta_2 > 0$ is suitably small. Hence

$$\alpha[\log M(r,H)] \le (\overline{\eta_2}+\delta_2)\,\beta\left([\gamma\{(\rho+1)[\rho\log\{R/(N+1)r\}]^{-1}\}]^{(\rho+1)}\right).$$

When r is sufficiently close to R, then by using properties of β and γ, we get

$$\frac{\alpha[\log M(r,H)]}{\beta\left\{[\gamma\{R/(R-r)\}]^{\rho}\right\}} \le \overline{\eta_2}+\delta_2.$$

Since ε and δ_2 are arbitrarily small, proceeding to limits as $r \to R$, we get $\sigma \le \eta_2$. Combining this with the inequality (2.7), we get (2.6). Hence Theorem 2.2 is proved. □

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