ACTA MATHEMATICA VIETNAMICA
Volume 37, Number 1, 2012, pp. 137–147

ON EXISTENCE OF WEAK SOLUTIONS OF NEUMANN PROBLEM FOR A SYSTEM OF SEMILINEAR ELLIPTIC EQUATIONS IN AN UNBOUNDED DOMAIN

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Abstract. The goal of this paper is to study the existence of non-trivial weak solution for the following system of nonlinear elliptic equations:

$$-\mathrm{div}(h_1(x)\nabla u) + a(x)u = f(x,u,v) \text{ in } \Omega$$

$$-\mathrm{div}(h_2(x)\nabla v) + b(x)v = g(x,u,v) \text{ in } \Omega$$

with Neumann condition:

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0$$

$$u(x) \longrightarrow 0, \quad v(x) \longrightarrow 0 \text{ as } |x| \longrightarrow +\infty$$

where $\Omega \subset R^N, N \geq 3$ is an unbounded domain with smooth bounded boundary $\partial\Omega$, and $h_i(x) \in L^1_{loc}(\overline{\Omega}), i = 1, 2, \quad \overline{\Omega} = \Omega \cup \partial\Omega$.
The solutions will be obtained in a subspace of the space $H^1(\Omega)$ and the proofs rely essentially on a variation of the Mountain Pass Theorem in [7].

1. Introduction and preliminaries results

We are concerned with the study of a Neumann problem for systems of semilinear elliptic equations:

$$(1.1) \quad \begin{cases} -\mathrm{div}(h_1(x)\nabla u) + a(x)u = f(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(h_2(x)\nabla v) + b(x)v = g(x,u,v) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0, \dfrac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega, & u(x) \longrightarrow 0, \quad v(x) \longrightarrow 0 \text{ as } |x| \longrightarrow +\infty \end{cases}$$

where $\Omega \subset R^N, N \geq 3$, is an unbounded domain with smooth bounded boundary $\partial\Omega, \overline{\Omega} = \Omega \cup \partial\Omega$, n is the outward unit normal to $\partial\Omega$, $f, g : \Omega \times R^2 \longrightarrow R$ are functions which will be specified later, $h_i(x), i = 1, 2$ and $a(x), b(x)$ satisfy the following conditions:

(H): $h_i(x) \in L^1_{\mathrm{loc}}(\overline{\Omega}), i = 1, 2, h_i(x) \geq 1$ for all $x \in \overline{\Omega}$.
(A-B): $a(x), b(x) \in C(\overline{\Omega}), a(x) \geq a_0 > 0, \quad b(x) \geq b_0 > 0$ for all $x \in \overline{\Omega}$.

Received January 21, 2011.
2000 *Mathematics Subject Classification.* 35J20, 35J65.
Key words and phrases. Neumann problem, Mountain Pass Theorem, weakly continuously differentiable functional.
Research supported by the National Fondation for Science and Technology Development of Vietnam (NAFOSTED).

We first make some comments on the problem (1.1). In the case when Ω is a bounded domain in R^N and $h(x) = 1$ there were extensive studies in the last decades dealing with the Neumann problems for the nonlinear elliptic equations involving the p-laplacian ($p \geq 2$). We just remember the papers [1-4, 14-16], where different techniques of finding solutions are illustrated. We also find that in the case $h_i(x) \in L^1_{loc}(\Omega)$, systems of semilinear elliptic equations of type (1.1), with Dirichlet boundary condition, have been studied by Hoang Quoc Toan, Nguyen Thanh Chung (see [11, 13]). The goal of this work we study is the existence of weak soluions of the Neumann problem for a system of semilinear elliptic equations with singular coefficients of type (1.1) in an unbounded domain $\Omega \subset R^N$ with smooth bounded boundary $\partial\Omega$.
In order to state our main results let us introduce the following hypotheses:

(F1) We assume that the functions $F, f, g : \Omega \times R^2 \longrightarrow R$ are of C^1 class such that $\frac{\partial F}{\partial u} = f(x,u,v)$, $\frac{\partial F}{\partial v} = g(x,u,v)$, $\nabla F(x,w) = (\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v})$ for all $x \in \Omega$ and $w = (u,v) \in R^2$. In addition, the following hypotheses are satisfied:
$f(x,0,0) = g(x,0,0) = 0$ for all $x \in \Omega$.

(F2) There exist functions $\tau_1, \tau_2 : \Omega \longrightarrow R$, $\tau_1(x) \geq 0, \tau_2(x) \geq 0$ for $x \in \Omega$ and constants $r, s \in (1, \frac{N+2}{N-2})$ such that

$$|\nabla f(x,w)| + |\nabla g(x,w)| \leq \tau_1(x)|w|^{r-1} + \tau_2(x)|w|^{s-1} \text{ for a.e } x \in \Omega, w = (u,v) \in R^2.$$

$$\tau_1(x) \in L^\infty(\Omega) \cap L^{r_0}(\Omega), \quad \tau_2(x) \in L^\infty \cap L^{s_0}(\Omega)),$$

$$r_0 = \frac{2N}{2N - (r+1)(N-2)}, \quad s_0 = \frac{2N}{2N - (s+1)(N-2)}.$$

(F3) There exists $\mu > 2$ such that

$$0 < \mu F(x,w) \leq w.\nabla F(x,w), x \in \Omega, w \neq (0,0).$$

Denote

$$C_0^\infty(\overline{\Omega}) = \{\varphi \in C^\infty(\overline{\Omega}) : \text{supp } \varphi \text{ compactly embedded} \subset \overline{\Omega}\}$$

and $H^1(\Omega)$ is the usual Sobolev space which can be defined as the completion of $C_0^\infty(\overline{\Omega})$ under the norm

$$||\varphi|| = \left(\int_\Omega (|\nabla \varphi|^2 + |\varphi|^2) dx \right)^{\frac{1}{2}}.$$

We now consider the following subspaces E and H of $H^1(\Omega, R^2) = H^1(\Omega) \times H^1(\Omega)$,

$$E = \{w = (u,v) \in H^1(\Omega, R^2) : \int_\Omega (|\nabla u|^2 + |\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2) dx < +\infty\}$$

and

$$H = \{w = (u, v) \in E : \int_\Omega (h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2)dx < +\infty\}.$$

By similar arguments as those used in the proof of Proposition 1.2 in [11], we can deduce that E and H are Hilbert spaces with respect to the inner product:

$$\langle w_1, w_2 \rangle_E = \int_\Omega (\nabla u_1 \nabla u_2 + \nabla v_1 \nabla v_2 + a(x)u_1u_2 + b(x)v_1v_2)dx$$

for $w_1 = (u_1, v_1), w_2 = (u_2, v_2) \in E$ and

$$\langle w_1, w_2 \rangle_H = \int_\Omega (h_1(x)\nabla u_1 \nabla u_2 + h_2(x)\nabla v_1 \nabla v_2 + a(x)u_1u_2 + b(x)v_1v_2)dx$$

for $w_1, w_2 \in H$. Moreover, by the conditions (H) and (A-B), the embeddings $H \hookrightarrow E \hookrightarrow L^q(\Omega, R^2), 2 \leq q \leq 2^* = \dfrac{2N}{N-2}$ are continuous, with $L^q(\Omega, R^2) = L^q(\Omega) \times L^q(\Omega)$.

Definition 1.1. We say that $w = (u, v) \in H$ is a weak solution of the prolem (1.1) if

$$\int_\Omega (h_1(x)\nabla u \nabla \varphi_1 + h_2(x)\nabla v \nabla \varphi_2 + a(x)u\varphi_1 + b(x)v\varphi_2)dx - \int_\Omega (f(x, u, v)\varphi_1 + g(x, u, v)\varphi_2)dx = 0 \tag{1.2}$$

for all $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(\overline{\Omega}, R^2)$.

Remark 1.1. If $w_0 = (u_0, v_0) \in C_0^\infty(\overline{\Omega}, R^2)$ satisfies the condition (1.2), then w_0 is a classical solution of the problem (1.1). Indeed, since $w_0 \in C_0^\infty(\overline{\Omega}, R^2)$, supp u_0, supp v_0 are compact then there exists $R > 0$ large enough such that $\partial\Omega \subset B_R(0)$ and supp $u_0 \cup$ supp $v_0 \subset \overline{\Omega} \cap B_R(0)$ where $B_R(0)$ is the ball of radius R. By denoting $\Omega_R = \overline{\Omega} \cap B_R(0)$, from (F1) we have

$$\int_{\Omega_R} (h_1(x)\nabla u_0 \nabla \varphi_1 + h_2(x)\nabla v_0 \nabla \varphi_2 + a(x)u_0\varphi_1 + b(x)v_0\varphi_2)dx - \int_{\Omega_R} (f(x, u_0, v_0)\varphi_1 + g(x, u_0, v_0)\varphi_2)dx = 0$$

for all $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(\overline{\Omega}, R^2)$.

Integrate by parts and remark that $u_0 = 0, v_0 = 0$ on $\partial B_R(0)$ we get

$$\int\limits_{\Omega_R} (-\text{div}(h_1(x)\nabla u_0)\varphi_1 - \text{div}(h_2(x)\nabla v_0)\varphi_2 + a(x)u_0\varphi_1 + b(x)v_0\varphi_2)dx$$

$$+\int\limits_{\partial\Omega} (h_1(x)\frac{\partial u_0}{\partial n}\varphi_1 + h_2(x)\frac{\partial v_0}{\partial n}\varphi_2)dx - \int\limits_{\Omega_R} (f(x, u_0, v_0)\varphi_1 + g(x, u_0, v_0)\varphi_2)dx = 0$$

for all $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(\overline{\Omega}, R^2)$. It follows that

$$\int\limits_{\Omega_R} (-\text{div}(h_1(x)\nabla u_0)\varphi_1 - \text{div}(h_2(x)\nabla v_0)\varphi_2 + a(x)u_0\varphi_1 + b(x)v_0\varphi_2)dx$$

$$- \int\limits_{\Omega_R} (f(x, u_0, v_0)\varphi_1 + g(x, u_0, v_0)\varphi_2)dx = 0$$

for all $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(\Omega_R, R^2)$. From this we obtain

$$\begin{gathered}
-\text{div}(h_1(x)\nabla u_0) + a(x)u_0 = f(x, u_0, v_0) \text{ in } \Omega \\
-\text{div}(h_2(x)\nabla v_0) + b(x)v_0 = g(x, u_0, v_0) \text{ in } \Omega \\
\frac{\partial u_0}{\partial n} = 0, \quad \frac{\partial v_0}{\partial n} = 0 \text{ on } \partial\Omega.
\end{gathered}$$

Our main result is given in the following theorem.

Theorem 1.1. *Assuming hypotheses* (H), (A-B), (F1)-(F3) *are fulfilled. Then the problem* (1.1) *has at least one nontrivial solution in* H.

Note that in hypothesis (A-B) we do not require the convexity for the functions $a(x), b(x)$ as in [11].
Theorem 1.1 will be proved by using a variation of the Mountain Pass Theorem in [7].

2. Existence of a weak solution

We define the functional $J : H \longrightarrow R$ by

$$J(w) = \frac{1}{2}\int\limits_{\Omega} [h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2 + a(x)|u|^2 + b(x)|v^2|]dx - \int\limits_{\Omega} F(x, w)dx \tag{2.3}$$

$$= T(w) - P(w)$$

where

$$T(w) = \frac{1}{2}\int\limits_{\Omega} (h_1(x)|\nabla u|^2 + h_2(x)|\nabla v|^2)dx + \frac{1}{2}\int\limits_{\Omega} (a(x)|u|^2 + b(x)|v|^2)dx$$

and $P(w) = \int\limits_{\Omega} F(x, w)dx, \quad w = (u, v) \in H.$

Firstly we remark that, due to the presence of $h_i(x) \in L^1_{loc}(\overline{\Omega})$, in general, the functional T (and thus J) does not belong to $C^1(H)$. This means that we cannot apply the classical Mountain Pass Theorem by Ambrossetti-Rabinowitz. In order to overcome this difficulty, we shall apply a weak version of the Mountain Pass Theorem introduced by D. M. Duc [7]. But we first recall the following useful concept of weak continuous differentiablity:

Definition 2.1. Let J be a functional from a Banach space Y into R. We say that J is weakly continuously differentiable on Y if and only if three following conditions are satisfied :

(i) J is continuous on Y.

(ii) For any $u \in Y$ there exists a linear map $DJ(u)$ from Y into R such that

$$\lim_{t \longrightarrow 0} \frac{J(u+t\varphi) - J(u)}{t} = \big\langle DJ(u), \varphi \big\rangle, \forall \varphi \in Y.$$

(iii) For any $\varphi \in Y$, the map $u \mapsto \big\langle DJ(u), \varphi \big\rangle$ is continuous on Y.

We denote by $C^1_w(Y)$ the set of weakly continuously differentiable functionals on Y. It is clear that $C^1(Y) \subset C^1_w(Y)$, where $C^1(Y)$ is the set of all continuously Fréchet differentiable functionals on Y. With similar arguments as those used in the proof of Proposition 2.2 in [11], we conclude the following proposition which concerns the smoothness of the functional J.

Proposition 2.1. *The functional J given by* (2.3) *is weakly continuously differentiable on H and we have*

$$\begin{aligned}\big\langle J'(w), \varphi \big\rangle = &\int_\Omega (h_1(x)\nabla u \nabla \varphi_1 + h_2(x)\nabla v \nabla \varphi_2 + a(x)u\varphi_1 + b(x)v\varphi_2)dx \\ &- \int_\Omega (f(x,u,v)\varphi_1 + g(x,u,v)\varphi_2)dx\end{aligned}$$

for all $w = (u,v) \in H, \varphi = (\varphi_1, \varphi_2) \in H$.

By Proposition 2.1, weak solutions of Problem (1.1) correspond to the critical points of the functional J.

Proposition 2.2. *The functional T given by* (2.3) *is weakly lower semicontinuous on the space H.*

Proof. By the convexity of the functional T, in order to prove the weak lower semicontinuity of T on H we shall prove that for any $w_0 \in H$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$T(w) \geq T(w_0) - \varepsilon, \quad \forall w \in H : ||w - w_0||_H < \delta.$$

Since T is convex, for all $w \in H$ we have

$$\begin{aligned}
T(w) &\geq T(w_0) + \langle DT(w_0), w - w_0 \rangle \\
&\geq T(w_0) - \int_\Omega (h_1(x)|\nabla u_0||\nabla u - \nabla u_0| + h_2(x)|\nabla v_0||\nabla v - \nabla v_0|)dx \\
&\quad - \int_\Omega (a(x)|u_0||u - u_0| + b(x)|v_0||v - v_0|)dx \\
&\geq T(w_0) - \left(\int_\Omega h_1(x)|\nabla u_0|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega h_1(x)|\nabla u - \nabla u_0|^2 dx\right)^{\frac{1}{2}} \\
&\quad - \left(\int_\Omega h_2(x)|\nabla v_0|^2 dx\right)^{\frac{1}{2}} . \left(\int_\Omega h_2(x)|\nabla v - \nabla v_0|^2 dx\right)^{\frac{1}{2}} \\
&\quad - \left(\int_\Omega a(x)|u_0|^2 dx\right)^{\frac{1}{2}} . \left(\int_\Omega a(x)|u - u_0|^2 dx\right)^{\frac{1}{2}} \\
&\quad - \left(\int_\Omega b(x)|v_0|^2 dx\right)^{\frac{1}{2}} . \left(\int_\Omega b(x)|v - v_0|^2 dx\right)^{\frac{1}{2}} \\
&\geq T(w_0) - c||w - w_0||_H, \text{ where } c = 4||w_0||_H.
\end{aligned}$$

Taking $\delta = \dfrac{\varepsilon}{c}$ we obtain

$$T(w) \geq T(w_0) - \varepsilon, \quad \forall w \in H : ||w - w_0||_H < \delta.$$

Thus we have proved that T is strongly lower semicontinuous on H. Since T is convex, by Corollary III.8 in [4] we conclude that T is weakly lower semicontinuous on H. □

Proposition 2.3. *The functional* $J : H \longrightarrow R$ *defined by* (2.3) *satisfies the Palais-Smale condition on* H*.*

Proof. Let $\{w_m\} = \{u_m, v_m\}$ be a sequence in H such that

$$\lim_{m \longrightarrow \infty} J(w_m) = C, \quad \lim_{m \longrightarrow +\infty} ||DJ(w_m)||_{H^*} = 0.$$

First, we shall prove that $\{w_m\}$ is bounded in H. We suppose by contradiction that $\{w_m\}$ is not bounded in H. Then there exists a subsequence $\{w_{m_k}\}$ of $\{w_m\}$ such that $||w_{m_k}||_H \longrightarrow +\infty$ as $k \longrightarrow +\infty$.

Observe further that

$$
\begin{aligned}
& w_{m_k}) - \frac{1}{\mu}\langle DJ(w_{m_k}), w_{m_k}\rangle \\
&= T(w_{m_k}) - \frac{1}{\mu}\langle DT(w_{m_k}), w_{m_k}\rangle + \frac{1}{\mu}\langle DP(w_{m_k}), w_{m_k}\rangle - P(w_{m_k}) \\
&\geq (\frac{1}{2} - \frac{1}{\mu})||w_{m_k}||_H^2
\end{aligned}
$$

yields

$$
\begin{aligned}
J(w_{m_k}) &\geq (\frac{1}{2} - \frac{1}{\mu})||w_{m_k}||_H^2 + \frac{1}{\mu}\langle DJ(w_{m_k}), w_{m_k}\rangle \\
&\geq (\frac{1}{2} - \frac{1}{\mu})||w_{m_k}||_H^2 - \frac{1}{\mu}||DJ(w_{m_k})||_{H^*}||w_{m_k}||_H \\
&\geq ||w_{m_k}||_H(\gamma_0||w_{m_k}||_H - \frac{1}{\mu}||DJ(w_{m_k})||_{H^*})
\end{aligned}
$$

where $\gamma_0 = \frac{1}{2} - \frac{1}{\mu} > 0$.
From this, letting $k \longrightarrow +\infty$, since $||w_{m_k}||_H \longrightarrow +\infty$, $||DJ(w_{m_k})||_{H^*} \longrightarrow 0$, we deduce $J(w_{m_k}) \longrightarrow +\infty$, a contradicion. Hence $\{w_m\}$ is bounded in H.
Next we prove that $\{w_m\}$ has a subsequence converging in H.
Since $\{w_m\}$ is bounded in H, H is a Hilbert space, there exists a subsequence $\{w_{m_k}\}$ such that it converges weakly to some $w = (u, v)$ in H. Then by Proposition 2.2 we find that

$$T(w) \leq \lim_{k \longrightarrow \infty} \inf T(w_{m_k}).$$

Furthermore, since the embedding $H \hookrightarrow L^{2^*}(\Omega, R^2)$ is continuous, $\{w_{m_k}\}$ is weakly converges to $w = (u, v)$ in $L^{2^*}(\Omega, R^2)$ and $w_{m_k}(x) \longrightarrow w(x)$ a.e. $x \in \Omega$. Then it is clear that $|w_{m_k}|^{r-1}w_{m_k}$ converges weakly to $|w|^{r-1}w$ in $L^{\frac{2^*}{r}}(\Omega, R^2)$. With similar arguments as those in [11], we define the map

$K_1(w) : L^{\frac{2^*}{r}}(\Omega, R^2) \longrightarrow R$ given by

$$\langle K_1(w), \varphi\rangle = \int_\Omega \tau_1(x) w\varphi dx \text{ for } \varphi = (\varphi_1, \varphi_2) \in L^{\frac{2^*}{r}}(\Omega, R^2).$$

We remark that $K_1(w)$ is linear and continuous provided that $\tau_1(x) \in L^{r_0}(\Omega)$, $w \in L^{2^*}(\Omega, R^2)$ and $\frac{1}{r_0} + \frac{1}{2^*} + \frac{r}{2^*} = 1$. Hence

$$\langle K_1(w), |w_{m_k}|^{r-1}w_{m_k}\rangle \longrightarrow \langle K_1(w), |w|^{r-1}w\rangle \text{ as } k \longrightarrow +\infty$$

i.e.

$$\lim_{k \longrightarrow +\infty} \int_\Omega \tau_1(x)|w_{m_k}|^{r-1}w_{m_k} w dx = \int_\Omega \tau_1(x)|w|^{r+1}dx. \tag{2.4}$$

With the same arguments we can show that

$$\lim_{k\longrightarrow+\infty}\int_\Omega \tau_2(x)|w_{m_k}|^{s-1}w_{m_k}wdx=\int_\Omega \tau_2(x)|w|^{s+1}dx. \tag{2.5}$$

$$\lim_{k\longrightarrow+\infty}\int_\Omega \tau_1(x)|w_{m_k}|^{r+1}dx=\int_\Omega \tau_1(x)|w|^{r+1}dx. \tag{2.6}$$

$$\lim_{k\longrightarrow+\infty}\int_\Omega \tau_2(x)|w_{m_k}|^{s+1}dx=\int_\Omega \tau_2(x)|w|^{s+1}dx. \tag{2.7}$$

Combinning (2.4) and (2.6) we get

$$\lim_{k\longrightarrow+\infty}\int_\Omega \tau_1(x)|w_{m_k}|^{r-1}w_{m_k}(w_{m_k}-w)dx=0. \tag{2.8}$$

Similarly

$$\lim_{k\longrightarrow+\infty}\int_\Omega \tau_2(x)|w_{m_k}|^{s-1}w_{m_k}(w_{m_k}-w)dx=0. \tag{2.9}$$

By (2.8), (2.9), (F1) and (F2) we obtain

$$\lim_{k\longrightarrow+\infty}\big\langle DP(w_{m_k})(w_{m_k}-w)\big\rangle=\lim_{k\longrightarrow+\infty}\int_\Omega \nabla F(x,w_{m_k})(w_{m_k}-w).$$

$$\lim_{k\longrightarrow+\infty}\big\langle DP(w_{m_k}),w_{m_k}-w\big\rangle=0. \tag{2.10}$$

It follows from (2.10) that

$$\begin{aligned}&\lim_{k\longrightarrow+\infty}\big\langle DT(w_{m_k}),w_{m_k}-w\big\rangle\\ &=\lim_{k\longrightarrow+\infty}\big\langle DJ(w_{m_k}),(w_{m_k}-w)\big\rangle+\lim_{k\longrightarrow+\infty}\big\langle DP(w_{m_k}),(w_{m_k}-w)\big\rangle=0.\end{aligned}$$

On the other hand, since T is convex the following inequality holds true

$$T(w)-T(w_{m_k})\geq\big\langle DT(w_{m_k}),w-w_{m_k}\big\rangle.$$

Letting $k\longrightarrow+\infty$ we have

$$\begin{aligned}T(w)-\lim_{k\longrightarrow+\infty}T(w_{m_k})&=\lim_{k\longrightarrow+\infty}[T(w)-T(w_{m_k})]\\ &\geq\lim_{k\longrightarrow+\infty}\big\langle DT(w_{m_k}),w-w_{m_k}\big\rangle=0.\end{aligned} \tag{2.11}$$

This implies that

$$T(w)\geq\lim_{k\longrightarrow+\infty}T(w_{m_k}).$$

Moreover we again have

$$T(w)\leq\lim_{k\longrightarrow+\infty}\inf T(w_{m_k}). \tag{2.12}$$

From (2.11) and (2.12) we get $\lim_{k \longrightarrow +\infty} T(w_{m_k}) = T(w)$.

Now we prove that the sequence $\{w_{m_k}\}$ converges strongly to w in H. Indeed, we suppose by contradiction that $\{w_{m_k}\}$ does not converge strongly to w in H. Then there exist a constant $\epsilon_0 > 0$ and a subsequence $\{w_{m_{k_j}}\}$ of $\{w_{m_k}\}$ such that $||w_{m_{k_j}} - w||_H \geq \epsilon_0$ for $j = 1, 2...$
Recalling the inequality

$$|\frac{\alpha + \beta}{2}|^2 + |\frac{\alpha - \beta}{2}|^2 = \frac{1}{2}(|\alpha|^2 + |\beta|^2), \quad \forall \alpha, \beta \in R,$$

we deduce that for any $j = 1, 2, ...$

$$\frac{1}{2}T(w_{m_{k_j}}) + \frac{1}{2}T(w) - T(\frac{w_{m_{k_j}} + w}{2}) \geq \frac{1}{4}||w_{m_{k_j}} - w||_H^2 = (\frac{\epsilon_0}{2})^2. \tag{2.13}$$

Again instead of the remark that $\{\frac{w_{m_{k_j}} + w}{2}\}$ converges weakly to w in H, applying Proposition 2.2 we have

$$T(w) \leq \lim_{j \longrightarrow +\infty} \inf T(\frac{w_{m_{k_j}} + w}{2}).$$

Then from (2.13), letting $j \longrightarrow \infty$ we obtain

$$T(w) - \lim_{j \longrightarrow +\infty} \inf T(\frac{w_{m_{k_j}} + u}{2}) \geq (\frac{\epsilon_0}{2})^2 > 0$$

which is a contradiction. Therefore, $\{w_{m_k}\}$ converges strongly to w in H. Thus the functional J satisfies the Palais-Smale condition on H. The proof of Proposition 2.3 is complete. □

We remark that the critical points of the functional J correspond to the weak solutions of the problem (1.1). Thus our idea is to apply a variation of the Mountain Pass Theorem (see [7]) in order to obtain at least one non-trivial weak solution of the Problem (1.1).
In what follows, we will prove a proposition which shows that the functional J has the Mountain Pass geometry.

Proposition 2.4. i) *There exist $\alpha > 0$ and $\rho > 0$ such that $J(w) \geq \alpha > 0$ for all $w \in H$, $||w||_H = \rho$.*

ii) *There exists $w_0 \in H$, $||w_0||_H > \rho$ and $J(w_0) < 0$.*

Proof. i) We follow the method used in proof of Theorem 1.2 in [6]. From condition (F3), it is easy to see that

$$F(x, z) \geq \min_{|s|=1} F(x, s)|z|^\mu, \quad \forall x \in \Omega, \text{ and } z = (z_1, z_2) \in R^2, |z| \geq 1 \tag{2.14}$$

$$0 < F(x, z) \leq \max_{|s|=1} F(x, s)|z|^\mu, \quad \forall x \in \Omega, \quad z = (z_1, z_2) \in R^2, |z| \leq 1 \tag{2.15}$$

where $\max\limits_{|s|=1} F(x,s) \leq c$ in view of (H2).
Since $\mu > 2$ it follows from (2.15) that

$$\lim_{|z|\longrightarrow 0} \frac{F(x,z)}{|z|^2} = 0 \text{ uniformly for } x \in \Omega. \tag{2.16}$$

From (2.16) we deduce that for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that

$$0 < F(x,z) < \varepsilon |z|^2 \tag{2.17}$$

for all z with $|z| < \delta$. Therefore, by using the continuous embedding $H \hookrightarrow E \hookrightarrow L^2(\Omega, R^2)$, with a simple calculation (2.17) we obtain $\inf\limits_{||w||_H=\rho} J(w) = \alpha > 0$ for all $\rho > 0$ small enough.

ii) Besides, by (2.14), for any given compact set $B \subset \Omega$ there exists $\overline{c} = \overline{c}(B)$ such that

$$F(x,z) \geq \overline{c}|z|^\mu \text{ for all } x \in B, |z| \geq 1. \tag{2.18}$$

Let $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(\overline{\Omega}, R^2), \varphi \neq 0$, for $t > 0$ large enough, from(2.15) we have

$$\begin{aligned} J(t\varphi) =& \frac{1}{2} t^2 ||\varphi||_H^2 - \int\limits_\Omega F(x,t\varphi)dx \\ \leq& \frac{1}{2} t^2 ||\varphi||_H^2 - t^\mu \overline{c} \int\limits_\Omega |\varphi|^\mu dx. \end{aligned} \tag{2.19}$$

Where $\overline{c} = \overline{c}(B)$, with $B = \text{supp } \varphi_1 \cup \text{supp } \varphi_2$. Since $\mu > 2$ the right hand-side of the (2.19) approaches to $-\infty$ as $t \longrightarrow +\infty$. This help us to conclude ii). The proof of Proposition 2.4 is complete. □

Proposition 2.5. i) *J(0)= 0.*
ii) *The acceptable set* $G = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = w_0\}$ *is not empty (with* w_0 *in Proposition 2.4).*

Proof. It is clear that: i) follows from (F1) and the definition of J.
ii) Let $\gamma(t) = tw_0$, then $\gamma(t) \in G$. □

Proof of Theorem 1.1. By Propositions 2.1-2.5, all assumptions of the variations of the Mountain Pass Theorem introduce in [7] are satisfied. Therefore there exists $\hat{\mathrm{w}} \in H$ such that

$$0 < \alpha \leq J(\hat{\mathrm{w}}) = \inf\{\max J(\gamma([0,1])) : \gamma \in G\}$$

and $\langle DJ(\hat{\mathrm{w}}), v\rangle = 0$ for all $v \in H$, i.e $\hat{\mathrm{w}}$ is a weak solution of problem (1.1). Moreover since $J(\hat{\mathrm{w}}) > 0 = J(0)$, $\hat{\mathrm{w}}$ is a nontrivial weak solution of Problem (1.1). □

References

[1] M. Alif and P. Omari, On a p-Laplace Neumann problem with asymptotically asymetric perturbations, *Nonlinear Anal.* **51** (2002), 369-389.

[2] G. Anello, Existence of infinitely many weak solutions for a Neumann problem, *Nonlinear Analysis* **57** (2004), 199-209.

[3] G. Bonanno and P. Candito, Three solutions to a Neumann problem for the elliptic equations involving the p-Laplacian, *Arch. Math.* **80** (2003), 424-429.

[4] H. Bresis, *Analyse fonctionelle théorie et applicaions*, Masson, 1992.

[5] P. A. Binding, P. Drábek, and Y. X. Huang, Existence of multiple solutions of critical quasilinear elliptic Neumann problems, *Nonlinear Anal.* **42** (2000), 613-629.

[6] D. G. Costa, On a class of elliptic systems in R^N, *EJDE* **07** (1994), 1-14.

[7] D. M. Duc, Nonlinear singular elliptic equation, *J. Lond. Math. Soc.* **40**(2) (1989), 420-440.

[8] D. M. Duc and N. T. Vu, Nonuniformly elliptic equations of p-Laplacian type, *Nonlinear Analysis* **61** (2005), 1483-1495.

[9] E. Giusti, Direct methods in the Calculus of variation World scientific, *New Jersey*, 2003.

[10] Hoang Quoc Toan and Ngo Quoc Anh, Multiplicity of weak solutions for a class of nonuniformly nonlinear elliptic equation of p-Laplacian type, *Nonlinear Anal.* **70** (2009), 1536-1546.

[11] Hoang Quoc Toan and Nguyen Thanh Chung, Existence of weak solutions for a class of nonuniformly nonlinear elliptic equations in unbounded domains, *Nonlinear Anal.* **70** (2009), 3987-3996.

[12] M. Mihăilescu, Existence and multiplicity of weak solution for a class of degenerate nonlinear elliptic equations, *Boundary Value Problems* (2006), Art ID 41295, 17pp.

[13] Nguyen Thanh Chung and Hoang Quoc Toan, Existence results for uniformly degenerate semilinear elliptic systems in R^N, *Glassgow Math. J.* **51** (2009), 561-570.

[14] B. Ricceri, Infinity many solutions of the Neumann problem for elliptic equations involving the p-Laplacian, *Bull. London Math. Soc.* **33** (2001), 331-340.

[15] C. L. Tang, Solvability of Neumann problem for elliptic equations at resonance, *Nonlinear Anal.* **44** (2001), 325-335.

[16] C. L. Tang, Some existence theorems for the sublinear Neumann boundary value problem *Nonlinear Anal.* **48** (2002), 1003-1011.

[17] M. Struwe, *Variational Methods*, Second Edition, Springer-Verlag, 2000.

[18] Xian Wu and Kok-Keong Tan, On existence and multiplicity of solutions of Neumann boundary value problems for quasi-linear elliptic equations, *Nonlinear Anal.* **65** (2006), 1334-1347.

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