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# ON MEROMORPHIC MAPPINGS IN SEVERAL COMPLEX VARIABLES WITH MAXIMAL DEFICIENCY SUM FOR MOVING TARGETS

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Abstract. In this paper, a theorem on the deficiency of meromorphic mappings in several complex variables with maximal deficiency sum for moving hyperplane targets is given.

## 1. Introduction

First of all, we recall the standard notations of the Nevanlinna theory. Let $f$ be a holomorphic mapping from $\mathbb{C}$ into the $n$-dimensional complex projective space $P^n\mathbb{C}$ with a reduced representation $f = (f_0, f_1, \cdots, f_n)$. Set $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$.
The characteristic function of $f$ is defined by

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log ||f(re^{i\theta})||d\theta - \log ||f(0)||.$$

Let $H = \{a_0w_0 + \cdots + a_nw_n = 0\}$ be a hyperplane of $P^n\mathbb{C}$.

We define $(H, f) = a_0f_0 + \cdots + a_nf_n$ and $(H, f(z)) = a_0f_0(z) + \cdots + a_nf_n(z)$.

Let $\nu(c)$ be the order of zero of $(H, f(z))$ at $z = c$ and for a positive integer $k$ let

$$n^{[k]}_{(H,f)}(r) = \sum_{|c|\leqslant r} \min\{\nu(c), k\},$$

$$N^{[k]}_{(H,f)}(r) = \int_0^r \frac{n^{[k]}_{(H,f)}(t) - n^{[k]}_{(H,f)}(0)}{t} dt + n^{[k]}_{(H,f)}(0) \log r \ (r > 0).$$

We call the quantity

$$\delta^{[k]}(H, f) = 1 - \limsup_{r\to\infty} \frac{N^{[k]}_{(H,f)}(r)}{T(r, f)}$$

the truncated deficiency (or defect) of $H$ with respect to $f$.

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Let $\{H_j\}_{j=1}^q$ be hyperplanes of $P^n\mathbb{C}$. Let $N \geq n$ and $N+1 \leqslant q \leqslant \infty$. Set the index set $Q = \{1, \cdots, q\}$ if $q < \infty$ and $Q = \{1, 2, \cdots\}$ if $q = \infty$. For a subset $R \subset Q$, $|R|$ denotes its cardinality. We say that the hyperplanes $\{H_j\}_{j=1}^q$ are in *$N$-subgeneral position* if for every subset $R \subset Q$ with $|R| = N+1$

$$\bigcap_{j\in R} H_j = \emptyset.$$

If they are in $n$-subgeneral position, we simply say that they are in *general position.*

Let $f : \mathbb{C} \to P^n\mathbb{C}$ be a linearly nondegenerate holomorphic mapping and $\{H_j\}_{j=1}^q$ be hyperplanes in $N$-subgeneral position in $P^n\mathbb{C}$, where $2N - n + 1 \leqslant q \leqslant \infty$. Then the Cartan-Nochka's theorem (see [1], [2], [3]) states that

$$\sum_{i=1}^{q} \delta^{[n]}(H_i, f) \leqslant 2N - n + 1.$$

A natural problem at this point is to study the case when the equality occur. Recent results of N. Toda [8]-[13] gave the interesting answers to this problem. We recall his results which are the best results available at present.

**Theorem of Toda** (see [9], Theorems 5.1 and 6.1] and [11], Theorems 3.1 and 4.1]). *Let $f : \mathbb{C}^m \to P^n\mathbb{C}$ be a linearly nondegenerate meromorphic mapping and $\{H_j\}_{j=1}^q$ be hyperplanes in $N$-subgeneral position in $P^n\mathbb{C}$, where $1 \leqslant n < N$ and $2N - n + 1 < q \leqslant \infty$. Suppose that $f$ has nonzero deficiency value at $H_i$ for each $1 \leqslant i \leqslant q$ and*

$$\sum_{j=1}^{q} \delta^{[n]}(H_j, f) = 2N - n + 1.$$

*Then one of the following two statements holds.*

(I) *There are at least* $\left[\dfrac{2N - n + 1}{n+1}\right] + 1$ *of the hyperplanes $H_j$ at which $f$ has truncated deficiecncy value 1, i.e $\delta^{[n]}(H_j, f) = 1$ ,*

(II) *The family $\{H_j\}_{j=1}^q$ has a Borel distribution.*

Recently, motivated by the establishment of the second main theorem of value distribution theory for moving targets (e.g., Ru and Stoll [4], [5] and Thai-Quang [7]), the study of the defect relation of meromorphic mappings of $\mathbb{C}^m$ into $P^n\mathbb{C}$ intersecting a finite set of moving hyperplanes (i.e, moving targets) has started. In [5], M. Ru and W. Stoll proved the Cartan-Nochka's theorem for moving targets. To state their result, we recall the following.

Let $\mathcal{M}_m$ be the field of all meromorphic functions on $\mathbb{C}^m$. Let $a_0, \ldots, a_{q-1}$ be $q$ meromorphic mappings of $\mathbb{C}^m$ into $P^n\mathbb{C}$ with reduced representations $a_j = (a_{j0} : \cdots : a_{jn})$ $(0 \leqslant j \leqslant q-1)$. Set the index set $Q = \{0, 1, \cdots, q-1\}$. For a subset $R \subset Q$, $|R|$ denotes its cardinality. Let $N \geq n$ and $q \geq N+1$. We say that the family $\{a_j\}_{j=0}^{q-1}$ is in $N$-subgeneral position if for every subset $R \subset Q$

with $|R| = N + 1$

$$\operatorname{rank}_{\mathcal{M}_m} (a_{jk})_{j\in R, 0\leqslant k\leqslant n} = n + 1.$$

If they are in $n$-subgeneral position, we simply say that they are in *general position.*

Denote by $\mathcal{R}\Big(\{a_j\}_{j=0}^{q-1}\Big) \subset \mathcal{M}_m$ the smallest subfield which contains $\mathbb{C}$ and all $\dfrac{a_{jk}}{a_{jl}}$ with $a_{jl} \not\equiv 0$. Let $f$ be a meromorphic mapping of $\mathbb{C}^m$ into $P^n\mathbb{C}$ with reduced representation $f = (f_0 : \cdots : f_n)$. We say that $f$ is linearly nondegenerate over $\mathcal{R}\Big(\{a_j\}_{j=0}^{q-1}\Big)$ if $f_0, \ldots, f_n$ are linearly independent over $\mathcal{R}\Big(\{a_j\}_{j=0}^{q-1}\Big)$.

Let $f$, $a$ be two meromorphic mappings of $\mathbb{C}^m$ into $P^n\mathbb{C}$ with reduced representations $f = (f_0 : \cdots : f_n)$, $a = (a_0 : \cdots : a_n)$ respectively. We say that $a$ is "small" with respect to $f$ if $T(r, a) = o(T(r, f))$ as $r \to \infty$. In [5], M. Ru and W. Stoll proved the following.

**Theorem of Stoll and Ru** (see [5]). *Let $f : \mathbb{C}^m \longrightarrow P^n\mathbb{C}$ be a nonconstant meromorphic mapping, and let $\{a_i\}_{i=0}^{q-1}$ be "small" (with respect to $f$) meromorphic mappings of $\mathbb{C}^m$ into $P^n\mathbb{C}$ in $N$-subgeneral position such that $f$ is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=0}^{q-1})$. Then*

$$\sum_{j=0}^{q-1} \delta\left(a_j, f\right) \leqslant 2N - n + 1.$$

The main aim of this paper is to show the theorem on the deficiency of meromorphic mappings in several complex variables with maximal deficiency sum for moving targets. In other words, we will show a theorem similar to the above-mentioned theorem of N. Toda for moving targets in the case $q < \infty$. Namely, we will prove the following.

**Main Theorem.** *Let $f : \mathbb{C}^m \longrightarrow P^n\mathbb{C}$ be a nonconstant meromorphic mapping, and let $\{a_i\}_{i=0}^{q-1}$ be "small" (with respect to $f$) meromorphic mappings of $\mathbb{C}^m$ into $P^n\mathbb{C}$ in $N$-subgeneral position such that $f$ is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=0}^{q-1})$. Suppose further that $f$ has nonzero deficiency value at $a_i$ for each $0 \leqslant i \leqslant q-1$ and*

$$\sum_{j=0}^{q-1} \delta\left(a_j, f\right) = 2N - n + 1.$$

*Then one of the following two statements holds.*

(I) *There are at least* $\left[\dfrac{2N - n + 1}{n + 1}\right] + 1$ *of the moving targets $a_j$ at which $f$ has deficiency value 1, i.e $\delta(a_j, f) = 1$ ,*

(II) *The family $\{a_j\}_{j=0}^{q-1}$ has a Borel distribution.*

## 2. Basic notions from Nevanlinna theory

**2.1.** We set $||z|| = \big(|z_1|^2 + \cdots + |z_m|^2\big)^{1/2}$ for $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ and define

$$B(r) := \{z \in \mathbb{C}^m : ||z|| < r\}, \quad S(r) := \{z \in \mathbb{C}^m : ||z|| = r\} \ (0 < r < \infty).$$

Define

$$v_{m-1}(z) := \big(dd^c||z||^2\big)^{m-1} \qquad \text{and}$$

$$\sigma_m(z) := d^c\log||z||^2 \wedge \big(dd^c\log||z||^2\big)^{m-1}\text{on} \quad \mathbb{C}^m \setminus \{0\}.$$

**2.2.** Let $F$ be a nonzero holomorphic function on a domain $\Omega$ in $\mathbb{C}^m$. For a set $\alpha = (\alpha_1, ..., \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + ... + \alpha_m$ and $\mathcal{D}^\alpha F = \dfrac{\partial^{|\alpha|}F}{\partial^{\alpha_1}z_1...\partial^{\alpha_m}z_m}$. We define the map $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max\ \{t : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < t\} \ (z \in \Omega).$$

We mean by a divisor on a domain $\Omega$ in $\mathbb{C}^m$ a map $\nu : \Omega \to \mathbb{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions $F$ and $G$ on a connected neighborhood $U \subset \Omega$ of $a$ such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leqslant m-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leqslant m-2$. For a divisor $\nu$ on $\Omega$ we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(m-1)$-dimensional analytic subset of $\Omega$ or empty.

Take a nonzero meromorphic function $\varphi$ on a domain $\Omega$ in $\mathbb{C}^m$. For each $a \in \Omega$, we choose nonzero holomorphic functions $F$ and $G$ on a neighborhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on $U$ and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leqslant m-2$, and we define the divisors $\nu_\varphi,\ \nu_\varphi^\infty$ by $\nu_\varphi := \nu_F,\ \nu_\varphi^\infty := \nu_G$, which are independent of choices of $F$ and $G$ and so globally well-defined on $\Omega$.

**2.3.** For a divisor $\nu$ on $\mathbb{C}^m$ and for positive integers $k, M$ or $M = \infty$, we define the counting function of $\nu$ by

$$\nu^{[M]}(z) = \min\ \{M, \nu(z)\},$$

$$n(t) = \begin{cases} \int\limits_{|\nu|\,\cap B(t)} \nu(z) v_{m-1} & \text{if } m \geq 2, \\ \sum\limits_{|z| \leqslant t} \nu(z) & \text{if } m = 1. \end{cases}$$

Similarly, we define $n^{[M]}(t)$.

Define

$$N(r, \nu) = \int\limits_1^r \frac{n(t)}{t^{2m-1}} dt \quad (1 < r < \infty).$$

Similarly, we define $N(r, \nu^{[M]})$ and denote it by $N^{[M]}(r, \nu)$.

Let $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r, \nu_\varphi),\ N_\varphi^{[M]}(r) = N^{[M]}(r, \nu_\varphi).$$

For brevity we will omit the character $^{[M]}$ if $M = \infty$.

**2.4.** Let $f : \mathbb{C}^m \longrightarrow P^n\mathbb{C}$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $P^n\mathbb{C}$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each $f_i$ is a holomorphic function on $\mathbb{C}^m$ and $f(z) = \big(f_0(z) : \cdots : f_n(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension $\geq 2$. Set $\|f\| = \big(|f_0|^2 + \cdots + |f_n|^2\big)^{1/2}$.

The characteristic function of $f$ is defined by

$$T(r,f) = \int\limits_{S(r)} \log\|f\|\sigma_m - \int\limits_{S(1)} \log\|f\|\sigma_m.$$

Let $a$ be a meromorphic mapping of $\mathbb{C}^m$ into $P^n\mathbb{C}$ with reduced representation $a = (a_0 : \cdots : a_n)$. We define $\quad (f,a) = \sum\limits_{i=0}^{n} a_i f_i$ and

$$m_{f,a}(r) = \int\limits_{S(r)} \log\frac{||f|| \cdot ||a||}{|(f,a)|}\sigma_m - \int\limits_{S(1)} \log\frac{||f|| \cdot ||a||}{|(f,a)|}\sigma_m,$$

where $\|a\| = \big(|a_0|^2 + \cdots + |a_n|^2\big)^{1/2}$. Define

$$\delta(a,f) = \varliminf_{r\to\infty}\left(1 - \frac{N_{(a,f)}(r)}{T(r,f)}\right) \text{ and } \delta^{[n]}(a,f) = \varliminf_{r\to\infty}\left(1 - \frac{N^{[n]}_{(a,f)}(r)}{T(r,f)}\right).$$

If $f, a : \mathbb{C}^m \to P^n\mathbb{C}$ are meromorphic mappings such that $(f,a) \not\equiv 0$, then the first main theorem for moving targets in value distribution theory (see [4]) states

$$T(r,f) + T(r,a) = m_{f,a}(r) + N_{(f,a)}(r) \ (r > 1).$$

**2.5.** Let $f$, $a$ be two meromorphic mappings of $\mathbb{C}^m$ into $P^n\mathbb{C}$ with reduced representations $f = (f_0 : \cdots : f_n)$, $a = (a_0 : \cdots : a_n)$ respectively. We say that $a$ is "small" with respect to $f$ if $T(r,a) = o(T(r,f))$ as $r \to \infty$.

**2.6.** Let $\phi$ be any non-negative function on $\mathbb{C}^m$. We say that the function $\phi$ is a $S(z,f)$-function if

$$\int_{S(r)} \log^+ |\phi|\sigma_m = o(T(r,f))$$

**2.7.** As usual, by the notation $''\| \ P''$ we mean the assertion $P$ holds for all $r \in [0,\infty)$ excluding a Borel subset $E$ of the interval $[0,\infty)$ with $\int_E dr < \infty$.

**2.8.** Let $N > n$. Let $q$ be any integer satisfying $2N - n + 1 < q < \infty$. Put $Q = \{0, 1 \ldots, q-1\}$. Let $\{a_j : j \in Q\}$ be the set of meromorphic mappings from $\mathbb{C}^m$ into $P^n\mathbb{C}$ in $N$-subgeneral position. Let $G(j_1, \ldots, j_k)(z)$ be the Grammian of $(a_{j_1}, \ldots, a_{j_k})$, where $0 \leqslant j_1 \leqslant \ldots \leqslant j_k \leqslant q-1$ and $2 \leqslant k \leqslant n+1$. We put

$$I = \{(j_1, \ldots, j_k) |\ G(j_1, \ldots, j_k) \not\equiv 0\}$$

and

$$S = \{z |\ G(j_1, \ldots, j_k)(z) = 0 \text{ for some } (j_1, \ldots, j_k) \in I\}.$$

Then $S$ is an analytic subset of codimension $\geq 1$ of $\mathbb{C}^m$ and is negligible when we integrate a function over an $S(r)$.

For $\emptyset \neq P \subset Q$ and $z \in \mathbb{C}^m$, let $V(z,P)$ be the linear space spanned by $\{a_j(z)|\ j \in P\}$ and put

$$d(z,P) = \dim V(z,p).$$

Then $d(z,P)$ is constant for $z \in \mathbb{C}^m - S$, so we put for $z \in \mathbb{C}^m - S$

$$d(P) = d(z,P).$$

If $P \subset Q$ and $N+1 \leqslant |P|$ then obviously $d(P) = n+1$.

**2.9. Definition.** Let $N > n$. Let $q$ be any integer satisfying $2N - n + 1 < q < \infty$. Put $Q = \{0, 1 \dots, q-1\}$. Let $\{a_j\}_{j \in Q}$ be the set of "small" (with respect to $f$) meromorphic mappings from $\mathbb{C}^m$ into $P^n\mathbb{C}$ in $N$-subgeneral position. The family $\{a_j\}_{j \in Q}$ is said to have a Borel distribution if $n = 2m-1$ is odd and there exist $P_1, P_2, \dots, P_p$ as subsets of $Q$ satisfying the following conditions:
i) $|P_v| = N - m + 1$ and $d(P_v) = m$ for each $1 \leqslant v \leqslant p$,
ii) $Q = \cup_{v=1}^{p} P_v$ as a disjoint union.
**2.10.** Let $f : \mathbb{C}^m \longrightarrow P^n\mathbb{C}$ be a linearly nondegenerate meromorphic mapping. Fix a homogeneous coordinate system $(w_0 : \cdots : w_n)$ of $P^n\mathbb{C}$ and let $f = (f^0 : \cdots : f^n)$ be a reduced representation. We define the Wronskian $W(f) = W(f^0, \cdots, f^n) \not\equiv 0$ and the logarithmic Wronskian $\Delta(f^0, \cdots, f^n)$ as follows

$$W(f^0, \cdots, f^n) = \begin{vmatrix} f^0 & \cdots & f^n \\ D^{(1)} f^0 & \cdots & D^{(1)} f^n \\ \vdots & \vdots & \vdots \\ D^{(n)} f^0 & \cdots & D^{(n)} f^n \end{vmatrix}$$

$$\Delta(f^0, \cdots, f^n) = \begin{vmatrix} 1 & \cdots & 1 \\ \dfrac{D^{(1)} f^0}{f^0} & \cdots & \dfrac{D^{(1)} f^n}{f^n} \\ \vdots & \vdots & \vdots \\ \dfrac{D^{(n)} f^0}{f^0} & \cdots & \dfrac{D^{(n)} f^n}{f^n} \end{vmatrix}$$

Here $D^{(j)} = \left(\frac{\partial}{\partial z^1}\right)^{\alpha_1(j)} \cdots \left(\frac{\partial}{\partial z^m}\right)^{\alpha_m(j)}$ are some partial differentiations of order atmost $j$. Because of the choice of $D^{(j)}$ we have the following functional equations for a meromorphic function $g$ on $\mathbb{C}^m$ and $A \in GL(n+1, \mathbb{C})$ :

$$\begin{aligned} W(gf^0, \cdots, gf^n) &= g^{n+1} W(f^0, \cdots, f^n), \\ W((f^0, \cdots, f^n)A) &= W(f^0, \cdots, f^n) \times (\det A), \\ \Delta(gf^0, \cdots, gf^n) &= \Delta(f^0, \cdots, f^n), \\ \Delta\left(1, \frac{f^1}{f^0}, \cdots, \frac{f^n}{f^0}\right) &= \Delta(f^0, \cdots, f^n). \end{aligned}$$

## 3. Auxiliary lemmas

First of all, by [3, Lemmas 3.3 and 3.4], we have the following.

**Lemma 1.** *Let $\{a_i\}_{i\in Q}$ be $q$ moving targets in $P^n\mathbb{C}$ in $N$-subgeneral position, and assume that $q > 2N - n + 1$. Then there are positive rational constants $\omega_j, j \in Q$ satisfying the following:*

(i) $0 < \omega_j \leqslant 1,\ \forall j \in Q$,

(ii) *setting* $\tilde{\omega} = \max_{j\in Q}\omega_j$, *one gets*

$$\sum_{j=1}^{q}\omega_j = \tilde{\omega}(q - 2N + n - 1) + n + 1,$$

(iii) $\dfrac{n+1}{2N-n+1} \leqslant \tilde{\omega} \leqslant \dfrac{n}{N}$, *and*

(iv) *For* $R \subset Q$ *with* $0 < |R| \leqslant N + 1$, $\sum_{j\in R}\omega_j \leqslant rank\{a_i\}_{i\in R}$.

The above $\omega_j$ are called *Nochka weights*, and $\tilde{\omega}$ the *Nochka constant*. We will denote $\theta = \tilde{\omega}^{-1}$ for later convenience.

**Lemma 2.** *Let $q > 2N - n + 1$, and let $\{a_i\}_{i\in Q}$ be $q$ moving targets in $P^n\mathbb{C}$ in $N$-subgeneral position. Let $\{\omega_j\}_{j\in Q}$ be its Nochka weights. Let $E_j \geq 1, j \in Q$ be arbitrarily given numbers. Then for every subset $R \subset Q$ with $0 < |R| \leqslant N + 1$, there is a subset $R^o \subset R$ such that $|R^o| =$ $rank\{a_i\}_{i\in R}$ and*

$$\prod_{i\in R} E_i^{\omega_i} \leqslant \prod_{i\in R^o} E_i.$$

**Remark 1.** Lemma 2 holds for any family $\{\omega_j\}_{j\in Q}$ satisfying the conditions (i) and (iv) in Lemma 1.

For every $1 \in \Phi \subset \mathcal{M}$ and $|\Phi| < \infty$, denote by $\mathcal{L}(\Phi)$ the $\mathbb{C}$-vector space spanned by $\Phi$. For each positive integer $k$, set $\Phi_k = \{\varphi_1\varphi_2\cdots\varphi_k|\varphi_j \in \Phi \text{ for } j = 1, ..., k\}$. Then $1 \in \Phi_k$ and $\Phi_k$ is finite.

**Lemma 3.** *Let $\epsilon$ be a real number with $0 < \epsilon < 1$. Let $1 \in \Phi \subset \mathcal{M}_m$ and $q = |\Phi|$. Let $P(\epsilon) = \binom{P_\epsilon+q+1}{q+1}$, where $P_\epsilon$ is the smallest positive integer such that $\binom{P_\epsilon+q+1}{q+1} \leqslant (1+\epsilon)^{P_\epsilon}$. Then there exists an integer $P'_\epsilon \leqslant P_\epsilon$ such that $\dfrac{\dim\mathcal{L}(\Phi_{P'_\epsilon+1})}{\dim\mathcal{L}(\Phi_{P'_\epsilon})} \leqslant (1+\epsilon)$ and $\dim\mathcal{L}(\Phi_{P'_\epsilon+1}) \leqslant P(\epsilon)$.*

*Proof.* Suppose that $\dfrac{\dim\mathcal{L}(\Phi_{p+1})}{\dim\mathcal{L}(\phi_p)} > (1+\epsilon)$ for all $1 \leqslant p \leqslant P_\epsilon$. Then

$$\dim\mathcal{L}(\Phi_{P_\epsilon+1}) \geq \prod_{i=1}^{P_\epsilon}\frac{\dim\mathcal{L}(\Phi_{i+1})}{\dim\mathcal{L}(\Phi_i)} > (1+\epsilon)^{P_\epsilon}.$$

Hence

$$\binom{P_\epsilon+q+1}{q+1} > (1+\epsilon)^{P_\epsilon}.$$

This is a contradiction. Thus, there exists a positive integer $P'_\epsilon \leqslant P_\epsilon$ such that

$$\frac{\dim \mathcal{L}(\Phi_{P'_\epsilon+1})}{\dim \mathcal{L}(\Phi_{P'_\epsilon})} \leqslant 1+\epsilon.$$

Moreover, we also have $\dim \mathcal{L}(\Phi_{P'_\epsilon+1}) \leqslant \dim \mathcal{L}(\Phi_{P_\epsilon+1}) \leqslant \binom{P_\epsilon+q+1}{q+1} \leqslant (1+\epsilon)^{P_\epsilon}$. □

**Remark 2.** For each real number $x$, denote by $[x]$ the integer part of $x$.
Let $P^*_\epsilon = \left[\frac{(q+1)^2}{\log^2(1+\epsilon)}\right]$ . Then $\binom{P^*_\epsilon+q+1}{q+1} \leqslant (1+\epsilon)^{P^*_\epsilon}$ and hence

$$P(\epsilon) \leqslant \left[(1+\epsilon)^{P^*_\epsilon}\right] \leqslant \left[(1+\epsilon)^{\frac{(q+1)^2}{\log^2(1+\epsilon)}}\right].$$

Indeed, it is easy to see that $P^*_\epsilon + 1 \geq \dfrac{(q+1)^2}{\log^2(1+\epsilon)}$, and hence $(P^*_\epsilon+1)^{\frac{1}{2}} \geq \dfrac{(q+1)}{\log(1+\epsilon)}$. This implies that $(q+1)(P^*_\epsilon+1)^{-\frac{1}{2}} \leqslant \log(1+\epsilon)$, and hence $(q+1)\frac{\log P^*_\epsilon}{P^*_\epsilon} \leqslant \log(1+\epsilon)$. This yields that $(P^*_\epsilon)^{q+1} \leqslant (1+\epsilon)^{P^*_\epsilon}$, and we have $\binom{P^*_\epsilon+q+1}{q+1} \leqslant (1+\epsilon)^{P^*}$.

The following lemma is a version for moving targets of Lemma 3.9 in [3] and Theorems 3.1, 4.1 in [6].

**Lemma 4.** *Let $f : \mathbb{C}^m \to P^n\mathbb{C}$ be a meromorphic mapping with a reduced representation $f = (f_0 : \cdots : f_n)$. Let $N > n$. Let $q$ be any integer satisfying $2N - n + 1 < q < \infty$. Put $Q = \{0, 1 \ldots, q-1\}$. Let $X = \{a_j : j \in Q\}$ be the set of "small" (with respect to $f$) meromorphic mappings from $\mathbb{C}^m$ into $P^n\mathbb{C}$ in $N$-subgeneral position. Assume that $f$ is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i\in Q})$. Let $\omega : Q \to \mathbb{R}_+$ be a function satisfying the conditions (i) and (iv) in Lemma 1. Then, we have*

$$\sum_{j=0}^{q-1} \omega(j) \cdot \delta(a_j, f) \leqslant n+1.$$

*Proof.* Assume that $a_i$ has a reduced representation $a_i = (a_{i0} : \ldots : a_{in})$ $(0 \leqslant i \leqslant q-1)$. By changing the homogeneous coordinate system of $P^n\mathbb{C}$ if necessary, without loss of generality, we may assume that $a_{i0}$ is not identically zero for all $i$. For each $0 \leqslant i \leqslant n$, set $g_{ij} = \frac{a_{ij}}{a_{i0}}$, $\tilde{a}_i = (g_{i0}, \ldots, g_{in})$, $F_i = (a_i, f)$, $\tilde{F}_i = (\tilde{a}_i, f)$ and $E_j = \dfrac{\|\tilde{a}_j\|\|f\|}{|\tilde{F}_j|} = \dfrac{\|a_j\|\|f\|}{|F_j|}$.

For any nonnegative integer $s$, let $V(s)$ be the $\mathbb{C}$-vector space spanned by

$$\{\prod_{k=0}^{n}\prod_{j=0}^{q-1} g_{jk}^{s(j,k)} | \sum_{k=0}^{n}\sum_{j=0}^{q-1} s(j,k) \leqslant s, s(j,k) \text{ are nonnegative integers}\}.$$

Put $d(s) = \dim V(s)$. Then $V(s)$ is a subspace of $V(s+1)$ and by Lemma 3, we have

$$\varliminf_{s\to\infty} \frac{d(s+1)}{d(s)} = 1.$$

Let $\{b_1, \ldots, b_{d(s)}, b_{d(s)+1}, \ldots, b_{d(s+1)}\}$ be a basis of $V(s+1)$ such that the family $\{b_1, \ldots, b_{d(s)}\}$ is a basis of $V(s)$. Then, the family $\{b_t f_k\}_{1\leqslant t\leqslant d(s+1),\ 0\leqslant k\leqslant n+1}$ is linearly independent over $\mathbb{C}$. We put

$$W = W(b_1 f_0, b_2 f_0, \ldots, b_{d(s+1)} f_n).$$

Then

$$N_{1/W}(r) = S(r, f).$$

Let $z$ be a point of $\mathbb{C}^m \setminus \{0\}$ such that $z$ is not a zero or a pole of the functions $\{\tilde{F}_j\}_{j=0}^{q-1}$ and $\{g_{jn}\}_{j=0}^{q-1}$. We rearrange $\{\tilde{F}_j(z)\}_{j=0}^{q-1}$ as follows :

$$|\tilde{F}_{j_0}(z)| \leqslant |\tilde{F}_{j_1}(z)| \leqslant \ldots \leqslant |\tilde{F}_{j_N}(z)| \leqslant \ldots \leqslant |\tilde{F}_{j_{q-1}}(z)|.$$

Then

$$\begin{aligned} \|f(z)\| &\leqslant S(z, f)|\tilde{F}_{j_k}(z)| \ (N \leqslant k \leqslant q-1) \text{ and} \\ |\tilde{F}_{j_k}(z)| &\leqslant S(z, f)\|f(z)\| \ (0 \leqslant k \leqslant q-1). \end{aligned}$$

On the other hand, at the point $z$ we have

$$\left(\prod_{j=0}^{q-1} E_j^{\omega(j)}\right)^{d(s)} \leqslant S(z, f)\left(\prod_{\nu=0}^{N} E_{j_\nu}^{\omega(j_\nu)}\right)^{d(s)} \equiv K.$$

By Lemma 2, it implies that

$$K \leqslant S(z, f)\left(\prod_{i=0}^{n} \frac{\|f\|}{|\tilde{H}_i|}\right)^{d(s)},$$

where $\{\tilde{H}_0, \ldots, \tilde{H}_n\} \subset \{\tilde{F}_{j_0}(z), \ldots, \tilde{F}_{j_N}(z)\}$ and the family $\{\tilde{H}_0, \ldots, \tilde{H}_n\}$ is linearly independent over $\mathcal{R}\Big(\{a_j\}_{j=0}^{q-1}\Big)$.

We put

$$\tilde{H}_\mu = (\tilde{a}_{j_\mu}, f) \ (0 \leqslant \mu \leqslant n) \text{ and } K' = \left(\frac{\|f\|^{n+1}}{|\tilde{H}_0 \cdots \tilde{H}_n|}\right)^{d(s)}.$$

Since $\{\tilde{H}_0, \ldots, \tilde{H}_n\}$ is linearly independent over $\mathcal{R}\Big(\{a_j\}_{j=0}^{q-1}\Big)$, it implies that the family $\{b_1\tilde{H}_0, b_2\tilde{H}_0, \ldots, b_{d(s)}\tilde{H}_n\}$ is linearly independent over $\mathbb{C}$. Since $\tilde{F}_j = (\tilde{a}_j, f)$, it implies that these $(n+1)d(s)$ functions can be represented as a linear combination with constant coefficents of $\{b_t f_k | 1 \leqslant t \leqslant d(s+1), 1 \leqslant k \leqslant n+1\}$. This means that

$$(b_1\tilde{H}_0, b_2\tilde{H}_0, \ldots, b_{d(s)}\tilde{H}_n) = (b_1 f_0, b_2 f_0, \ldots, b_{d(s+1)} f_n) D_1,$$

where $D_1$ is an $(n+1)d(s+1)\times(n+1)d(s)$-matrix whose elements are constants and whose rank is equal to $(n+1)d(s)$.

Let $D_2$ be an $(n+1)d(s+1)\times(n+1)(d(s+1)-d(s))$-matrix with constant elements such that the matrix $D=(D_1D_2)$ is invertible. Put $L=(n+1)(d(s+1)-d(s))$ and $(K_1,\dots,K_L)=(b_1f_0,b_2f_0,\dots,b_{d(s+1)}f_n)D_2$. Then

$$(b_1\tilde{H}_0,b_2\tilde{H}_0,\dots,b_{d(s)}\tilde{H}_n,K_1,\dots K_L)=(b_1f_0,b_2f_0,\dots,b_{d(s+1)}f_n)D.$$

Hence $W(b_1\tilde{H}_0,b_2\tilde{H}_0,\dots,b_{d(s)}\tilde{H}_n,K_1,\dots K_L) = (\det D)W$ with $\det D\neq 0$, where $W=W(b_1f_0,b_2f_0,\dots,b_{d(s+1)}f_n)$.

Since $|\tilde{H}_k(z)|\leqslant S(z,f)\|f(z)\|,\ K_j(z)\leqslant S(z,f)\|f(z)\|$, we have

$$\begin{aligned}\frac{1}{\prod_{k=0}^n|\tilde{H}_k|^{d(s)}} &= \frac{|W(b_1\tilde{H}_0,\dots,b_{d(s)}\tilde{H}_n,K_1,\dots,K_L)|}{|W||\det D|}\cdot\frac{1}{\prod_{k=0}^n|\tilde{H}_k|^{d(s)}}\\ &= \frac{1}{|\det D||W|}\cdot\frac{|W(b_1\tilde{H}_0,\dots,b_{d(s)}\tilde{H}_n,K_1,\dots,K_L)|}{\prod_{k=0}^n|\tilde{H}_k|^{d(s)}}\\ &\leqslant S(z,f)\frac{\|f(z)\|^L}{W}\cdot\frac{|W(b_1\tilde{H}_0,\dots,b_{d(s)}\tilde{H}_n,K_1,\dots,K_L)|}{|b_1\tilde{H}_0\cdots b_{d(s)}\tilde{H}_n\cdot K_1\cdots K_L|}.\end{aligned}$$

Hence

$$\begin{aligned}K' &= \left(\frac{\|f\|^{n+1}}{|\tilde{H}_0\cdots\tilde{H}_n|}\right)^{d(s)}\\ &\leqslant S(z,f)\cdot\frac{\|f(z)\|^{(n+1)d(s+1)}}{|W|}\cdot\frac{|W(b_1\tilde{H}_0,\dots,b_{d(s)}\tilde{H}_n,K_1,\dots,K_L)|}{|b_1\tilde{H}_0\cdots b_{d(s)}\tilde{H}_n\cdot K_1\cdots K_L|}.\end{aligned}$$

Since $K\leqslant S(z,f)\cdot K'$, we also have

$$K\leqslant S(z,f)\cdot\frac{\|f(z)\|^{(n+1)d(s+1)}}{|W|}\cdot\frac{|W(b_1\tilde{H}_0,\dots,b_{d(s)}\tilde{H}_n,K_1,\dots,K_L)|}{|b_1\tilde{H}_0\cdots b_{d(s)}\tilde{H}_n\cdot K_1\cdots K_L|}.$$

Hence

$$\Big(\prod_{j=0}^{q-1}E_j^{\omega(j)}\Big)^{d(s)}\leqslant K\leqslant S(z,f)\cdot K'$$

$$\leqslant S(z,f)\cdot\frac{\|f(z)\|^{(n+1)d(s+1)}}{|W|}\cdot\frac{|W(b_1\tilde{H}_0,\dots,b_{d(s)}\tilde{H}_n,K_1,\dots,K_L)|}{|b_1\tilde{H}_0\cdots b_{d(s)}\tilde{H}_n\cdot K_1\cdots K_L|}.$$

Taking the "log $^+$" both sides, we get the following inequality.

$$\begin{aligned}d(s)\sum_{j=0}^{q-1}\omega(j)\quad \log\tfrac{\|a_j(z)\|\|f(z)\|}{|F_j(z)|} &\leqslant \log^+\tfrac{\|f(z)\|^{(n+1)d(s+1)}}{W}+\log^+S(z,f)\\ &+\textstyle\sum_{H_0,\dots,H_n}\log^+\tfrac{|W(b_1\tilde{H}_0,\dots,b_{d(s)}\tilde{H}_n,K_1,\dots,K_L)|}{|b_1\tilde{H}_0\cdots b_{d(s)}\tilde{H}_n\cdot K_1\cdots K_L|},\end{aligned}$$

where the sum $\sum_{H_0,\dots,H_n}$ is taken over all linearly independent over $\mathcal{R}\Big(\big\{a_j\big\}_{j=0}^{q-1}\Big)$ subsets $\{H_0,\dots H_n\}$ of the family $\{F_0,\dots,F_{q-1},f_0,\dots,f_n\}$.

It is easy to see that the above inequality is independent of choosing $z \in \mathbb{C}^m \setminus \{0\}$ such that $z$ is not a zero nor a pole of the functions $\{\tilde{F}_j\}_{j=0}^{q-1}$ and $\{g_{jn}\}_{j=0}^{q-1}$.

On the other hand, by Lemma on logarithmic derivatives, we have

$$\int\limits_{S(r)} \log^+ \frac{|W(b_1\tilde{H}_0, \dots, b_{d(s)}\tilde{H}_n, K_1, \dots, K_L)|}{|b_1\tilde{H}_0 \cdots b_{d(s)}\tilde{H}_n \cdot K_1 \cdots K_L|}\sigma_m = S(r, f).$$

Integrating the above inequality over the sphere $S(r)$ with respect to $\sigma_m$, we obtain

$$d(s)\sum_{j=0}^{q-1}\omega(j)m_{a_j,f}(r) \leqslant \int\limits_{S(r)} \log^+ \frac{\|f(z)\|^{(n+1)d(s+1)}}{|W|}\sigma_m + S(r, f).$$

Hence

$$\begin{aligned} d(s)\sum_{j=0}^{q-1}\omega(j)m_{a_j,f}(r) &\leqslant \int\limits_{S(r)} \log^+ \frac{\|f(z)\|^{(n+1)d(s+1)}}{|W|}\sigma_m + S(r, f) \\ &\leqslant (n+1)d(s+1)T(r, f) - N_{1/W}(r) + S(r, f). \end{aligned}$$

This implies that

$$d(s)\sum_{j=0}^{q-1}\omega(j)\delta_(a_j, f) \leqslant (n+1)d(s+1).$$

Since $\underline{\lim}_{s\to\infty} \dfrac{d(s+1)}{d(s)} = 1$ and letting $s \to \infty$, we have

$$\sum_{j=0}^{q-1}\omega(j)\delta_(a_j, f) \leqslant n+1.$$

□

## 4. The proof of Main Theorem

First of all, having those above results, what we are going to prove is in fact a theorem in linear algebra. What is really relevant is $\dim V(z, P)$ which was proved to be independent from $z \in \mathbb{C}^m - S$. Hence by choosing a fixed $z \in \mathbb{C}^m - S$ and by identifying $a_j$ with $a_j(z)$, from now on, we can consider each $a_j$ as a fixed vector in $\mathbb{C}^{n+1}$.

Let $\vartheta = \{P \subset Q||P| \leqslant N+1\}$ and put

$$\lambda = \min_{P\in\vartheta} \frac{d(P)}{|P|}.$$

Since $\lambda$ satisfies the conditions of Lemma 4 and by Lemma 4, we have

$$\lambda\sum_{j=0}^{q-1}\delta\left(a_j,f\right)\leqslant n+1.$$

Since

$$\sum_{j=0}^{q-1}\delta\left(a_j,f\right)=2N-n+1,$$

this implies that

$$\lambda\leqslant\frac{n+1}{2N-n+1}.$$

Now we consider two cases.

**Case A.**

$$\lambda<\frac{n+1}{2N-n+1}.$$

Repeating the argument as [8, Remark 1 and Theorem 2], the assertion (I) is proved.

**Case B.**

$$\lambda=\frac{n+1}{2N-n+1}.$$

Let $\vartheta_1$ be the set of all the elements $P\in\vartheta$ such that

$$\frac{d(P)}{|P|}=\lambda.$$

**Claim 1.** If $P_1,P_2\in\vartheta_1$, then $P_1\cup P_2\in\vartheta_1$ except for the case that
i) $n$ is odd and
ii) $d(P_1)=d(P_2)=\frac{n+1}{2}$ and $V(P_1)\cap V(P_2)=\{\emptyset\}$.

Indeed, assume that $P_1\cup P_2\notin\vartheta_1$. Then $|P|-d(P)\leqslant N-n$ (*) for any $P\in\vartheta$. Actually, if we add $N+1-|P|$ vectors into the set $\{a_i : i\in P\}$ then the dimension of the set is now $n+1$ by the $N$-subgenerality. Since this dimension add at most $N+1-|P|$, we have $d(P)+N+1-|P|\geq n+1$. This implies (*) immediately.

On the other hand, since

$$\frac{d(P_1)}{|P_1|}=\frac{d(P_2)}{|P_2|}=\lambda=\frac{n+1}{2N-n+1},$$

we have

$$d(P_1)+d(P_2)=\lambda\cdot(|P_1|+|P_2|)\leqslant\lambda\cdot(d(P_1)+d(P_2)+2(N-n)).$$

This yields that

$$d(P_1)+d(P_2)\leqslant n+1. \tag{4.1}$$

Since

$$d(P_1\cup P_2)+d(P_1\cap P_2)\leqslant d(P_1)+d(P_2), \tag{4.2}$$

it implies that $d(P_1\cup P_2)\leqslant n+1$. Consider two sub-cases.

**Subcase $\mathbf{B_1}$.** $d(P_1\cup P_2)\leqslant n$.

Then $|(P_1 \cup P_2)| \leqslant N$, and hence, it belongs to $\vartheta$. By the definition of $\lambda$, we have $\lambda \cdot (|(P_1 \cap P_2|) \leqslant d(P_1 \cap P_2)$ (even if $P_1 \cap P_2 = \emptyset$). Combining this assertion with $\lambda \cdot (|P_1| + |P_2|) = d(P_1) + d(P_2)$ and $d(P_1 \cup P_2) + d(P_1 \cap P_2)) \leqslant d(P_1) + d(P_2)$, we have $\lambda \cdot |(P_1 \cup P_2)| \geq d(P_1 \cup P_2)$. This implies that $\lambda \cdot |(P_1 \cup P_2)| = d(P_1 \cup P_2)$. Hence $P_1 \cup P_2$ is in $\vartheta_1$.

**Subcase $\mathbf{B_2}$.** $d(P_1 \cup P_2) = n + 1$.

Then the equalities hold in (4.1) and (4.2). So we have $P_1 \cap P_2 = \emptyset$. We now show that $d(P_1) = d(P_2)$.

Indeed, suppose on contrary. Without loss of generality, we may assume that $d(P_1) > d(P_2)$. Then $d(P_1) > \frac{n+1}{2}$. Choose a set which consists of all the vectors in $P_1$ and $d(P_2) - 1$ vectors in $P_2$. Then the $\mathbb{C}$-vector space spanned by this set has the dimension at most $d(P_1) + d(P_2) - 1 \leqslant n$. The cardinality of this set is

$$\begin{aligned}|P_1| + d(P_2) - 1 &= \frac{1}{\lambda} \cdot d(P_1) + (n + 1 - d(P_1) - 1)\\ &= \left(\frac{1}{\lambda} - 1\right) \cdot d(P_1) + n\\ &> \left(\frac{2N - 2n}{n + 1}\right) \cdot \left(\frac{n + 1}{2}\right) + n = N.\end{aligned}$$

This contradicts the $N$-subgenerality. So we have $d(P_1) = d(P_2) = \frac{n+1}{2}$, and $n$ must be odd. Since $V(P_1) + V(P_2) = V(P_1 \cup P_2)$, we get

$$d(V(P_1) \cap V(P_2)) = d(V(P_1)) + d(V(P_2)) - d(V(P_1) + V(P_2)) = 0.$$

Hence Claim 1 is proved.

**Claim 2.** We have $\bigcup\{P : \ P \in \vartheta_1\} = Q$.

Indeed, put $\bigcup_{P \in \vartheta_1} P = Q_1$. If $Q_1 \neq Q$, then there exists $\lambda_1 = \min_{\{P \in \vartheta,\ P - Q_1 \neq \emptyset\}} \dfrac{d(P)}{|P|}$. Moreover, $\lambda_1 > \lambda$. Put

$$w(j) = \begin{cases} \lambda & \text{if } j \in Q_1, \\ \lambda_1 & \text{if } j \notin Q_1. \end{cases}$$

It is easy to see that the above constructed weight function satisfies the conditions (i) and (iv) in Lemma 1. According to Lemma 4, we have

$$\sum_{j=1}^{q} w(j) \cdot \delta\left(a_j, f\right) \leqslant n + 1 = \lambda \cdot \sum_{j=1}^{q} \delta\left(a_j, f\right).$$

Hence

$$0 < (\lambda_1 - \lambda) \cdot \sum_{j \in\ Q - Q_1} \delta\left(a_j, f\right) = \delta\left(a_j, f\right) \leqslant 0.$$

This is a contradiction. So we must have $Q = Q_1$ and Claim 2 is proved.

To complete the proof of Main Theorem it suffices to consider the following two cases.

**Case 1.** $n$ is even.

Due to Claim 1, the union of two elements in $\vartheta_1$ is in $\vartheta_1$, so the union of all elements of $\vartheta_1$ is also in $\vartheta_1$. By Claim 2, it implies that $Q \in \vartheta_1$. Hence $q \leqslant N+1$. This is a contradiction. Hence **Case B** cannot happen if $n$ is even. Thus, the assertion (I) of Main Theorem always holds if $n$ is even.

**Case 2.** $n$ is odd.

Let $n = 2m-1$. We must prove that the given system has a Borel distribution. To prove this it suffices to show that, for each $P \in \vartheta_1$, $|P| = N - m + 1$ and $d(P) = m(**)$.

Indeed, assume that there is $P_0 \in \vartheta_1$ such that $P_0$ does not satisfy the above assertion.

If the possibility ii) in Claim 1 never happen, then the union of two elements in $\vartheta_1$ is in $\vartheta_1$. By using the same arguments as in Case 1, we derive absurdity.

Assume that the possibility ii) in Claim 1 holds for some elements $A, B \in \vartheta_1$. Then it is easy to see that $A \cup P_0 \in \vartheta_1$ and $B \cup P_0 \in \vartheta_1$. Since $(A \cup P_0) \cap (B \cup P_0) \neq \emptyset$, we have $(A \cup P_0) \cup (B \cup P_0) \in \vartheta_1$. Hence $N + 1 \geq |(A \cup P_0) \cup (B \cup P_0)| \geq 2N - 2m + 2 > N + 1$. This is a contradiction. Thus, the assertion (**) is proved.


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