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HERMITE-HADAMARD TYPE INEQUALITIES FOR HIGHER ORDER CONVEX FUNCTIONS AND VARIOUS QUADRATURE RULES

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Abstract. Hermite-Hadamard type inequalities are given for the higher order convex functions and the general three, four and five point quadrature rules, as well as for their corrected counterparts. Specifically, Hermite-Hadamard type estimates are obtained for various classical quadrature rules, such as the Simpson, dual Simpson, Simpson 3/8, Maclaurin, the Gauss 2 and 3-point, the Lobatto 4 and 5-point rule, and their corrected variants.

1. Introduction

The well-known Hermite-Hadamard type inequality states that if $f : [0, 1] \to \mathbb{R}$ is a convex function, then the following inequalities hold

$$f\left(\frac{1}{2}\right) \leq \int_0^1 f(t)dt \leq \frac{f(0) + f(1)}{2}. \tag{1.1}$$

If f is concave, inequalities are reversed (cf. [9]).

This pair of inequalities, comparing the trapezoid and the midpoint formula, has been improved and extended in a number of ways. The aim of this paper is to give related results for the higher order convex functions and the general three, four and five point quadrature rules, as well as for their corrected counterparts. Beside values of the function in the chosen nodes, "corrected" quadrature formulas, or "quadratures with end corrections", include values of the first derivative at the end points of the interval. They have higher accuracy than the adjoint classical quadrature formulas.

Related results for the classical Gauss and Lobatto quadrature rules were considered in [1] and [2], using a different, more general approach. Some of them will be recaptured in this paper. Similar results were also given in [10], [11] and [12].

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2. Preliminaries

The first step was obtaining the general quadrature formulas. The family of general three point quadrature formulas was obtained in [4] and of general corrected three point quadrature formulas in [5]; general closed four point quadrature formulas, both classical and corrected, were considered in [6] and finally, general closed five point quadrature formulas, both classical and corrected, were derived in [7]. Namely, for $f : [0,1] \to \mathbb{R}$ such that $f^{(n-1)}$ is continuous and of bounded variation on $[0,1]$ for some $n \geq 1$, we have

$$\int_0^1 f(t)dt - Q_\alpha(x) + T_{n-1}^\alpha(x) = \frac{1}{n!} \int_0^1 F_n^\alpha(x,t) df^{(n-1)}(t), \tag{2.1}$$

for $\alpha = q3,\ cq3$ and $x \in [0,1/2)$, for $\alpha = q4,\ cq4$ and $x \in (0,1/2]$, and for $\alpha = q5,\ cq5$ and $x \in (0,1/2)$, where

$$T_{n-1}^\alpha(x) = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \frac{1}{(2k)!} G_{2k}^\alpha(x,0) \left[f^{(2k-1)}(1) - f^{(2k-1)}(0) \right] \tag{2.2}$$

$$F_n^\alpha(x,t) = G_n^\alpha(x,t) - G_n^\alpha(x,0), \tag{2.3}$$

and

$$Q_{q3}(x) = \frac{f(x) + 24B_2(x) f(1/2) + f(1-x)}{6(1-2x)^2}, \tag{2.4}$$

$$Q_{cq3}(x) = \frac{7f(x) - 480B_4(x) f(1/2) + 7f(1-x)}{30(1-2x)^2(1+4x-4x^2)}, \tag{2.5}$$

$$Q_{q4}(x) = \frac{-6B_2(x)f(0) + f(x) + f(1-x) - 6B_2(x)f(1)}{12x(1-x)}, \tag{2.6}$$

$$Q_{cq4}(x) = \frac{30B_4(x)f(0) + f(x) + f(1-x) + 30B_4(x)f(1)}{60x^2(1-x)^2}, \tag{2.7}$$

$$\begin{aligned} Q_{q5}(x) = \frac{1}{60x(1-x)(1-2x)^2} \big[& f(x) + f(1-x) \\ & - (10x^2 - 10x + 1)(1-2x)^2 (f(0) + f(1)) \\ & + 32x(1-x)(5x^2 - 5x + 1) f(1/2) \big], \end{aligned} \tag{2.8}$$

$$\begin{aligned} Q_{cq5}(x) = \frac{1}{420x^2(1-x)^2(1-2x)^2} \big[& f(x) + f(1-x) \\ & + (98x^4 - 196x^3 + 102x^2 - 4x - 1)(1-2x)^2 (f(0) + f(1)) \\ & + 64x^2(1-x)^2(14x^2 - 14x + 3) f(1/2) \big], \end{aligned} \tag{2.9}$$

and finally,

$$G_n^{q3}(x,t)=\frac{B_n^*(x-t)+24B_2(x)\cdot B_n^*(1/2-t)+B_n^*(1-x-t)}{6(1-2x)^2}, \tag{2.10}$$

$$G_n^{cq3}(x,t)=\frac{7B_n^*(x-t)-480B_4(x)\cdot B_n^*(1/2-t)+7B_n^*(1-x-t)}{30(1-2x)^2(1+4x-4x^2)}, \tag{2.11}$$

$$G_n^{q4}(x,t)=\frac{B_n^*(x-t)-12B_2(x)\cdot B_n^*(1-t)+B_n^*(1-x-t)}{12x(1-x)}, \tag{2.12}$$

$$G_n^{cq4}(x,t)=\frac{60B_4(x)\cdot B_n^*(1-t)+B_n^*(x-t)+B_n^*(1-x-t)}{60x^2(1-x)^2}, \tag{2.13}$$

$$\begin{aligned} G_n^{q5}(x,t)=&\frac{10x^2-10x+1}{30x(x-1)}\,B_n^*(1-t)+\frac{B_n^*(x-t)+B_n^*(1-x-t)}{60x(1-x)(1-2x)^2}\\ &+\frac{8(5x^2-5x+1)}{15(1-2x)^2}\,B_n^*(1/2-t), \end{aligned} \tag{2.14}$$

$$\begin{aligned} G_n^{cq5}(x,t)=&\frac{98x^4-196x^3+102x^2-4x-1}{210x^2(1-x)^2}\,B_n^*(1-t)\\ &+\frac{B_n^*(x-t)+B_n^*(1-x-t)}{420x^2(1-x)^2(1-2x)^2}+\frac{16(14x^2-14x+3)}{105(1-2x)^2}\,B_n^*(1/2-t). \end{aligned} \tag{2.15}$$

Bernoulli polynomials play an important role here, so let us recall some of their basic properties. Bernoulli polynomials $B_n(t)$ are uniquely determined by

$$B_n'(x)=nB_{n-1}(x),\quad B_n(t+1)-B_n(t)=nt^{n-1},\ n\ge 0,\quad B_0(t)=1.$$

For the nth Bernoulli polynomial we have $B_n(1-x)=(-1)^nB_n(x),\ x\in\mathbb{R},\ n\ge 1$.

The nth Bernoulli number B_n is defined by $B_n=B_n(0)$. For $n\ge 2$, we have $B_n(1)=B_n(0)=B_n$. Note that $B_{2n-1}=0,\ n\ge 2$ and $B_1(1)=-B_1(0)=1/2$.

$B_n^*(x)$ are periodic functions of period 1 defined by $B_n^*(x+1)=B_n^*(x),\ x\in\mathbb{R}$, and related to Bernoulli polynomials as $B_n^*(x)=B_n(x),\ 0\le x<1$. $B_0^*(x)$ is a constant equal to 1, while $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer. For $n\ge 2$, $B_n^*(t)$ is a continuous function. For further details on Bernoulli polynomials see [8].

Applying properties of Bernoulli polynomials, it easily follows that functions $G_n^\alpha,\ \ \alpha=q3,cq3,q4,cq4,q5,cq5,\ \ n\ge 1$, have the following properties:

$$G_n^\alpha(x,1-t)=(-1)^nG_n^\alpha(x,t),\quad t\in[0,1], \tag{2.16}$$

$$\frac{\partial^j G_n^\alpha(x,t)}{\partial t^j}=(-1)^j\frac{n!}{(n-j)!}G_{n-j}^\alpha(x,t),\ \ j=1,2,\dots,n \tag{2.17}$$

and also, that $G_{2n-1}^\alpha(x,0)=0$ for $n\ge 1$, and so $F_{2n-1}^\alpha(x,t)=G_{2n-1}^\alpha(x,t)$.

The following important lemma combines the lemmas which were the key results in [4], [5], [6] and [7].

Lemma 2.1. *Functions* $G_{2n-1}^\alpha(x,t)$*, defined in* (2.10) – (2.15)*, have no zeros in the variable* t *on* $(0,\ 1/2)$ *for*

(A1) $n\ge 2$ *and*

(1) $\alpha=q3$ *and* $x\in\{0\}\cup[1/6,\ 1/2)$

(2) $\alpha = q4$ *and* $x \in (0,\ 1/2 - \sqrt{3}/6] \cup [1/3,\ 1/2]$

(A2) $n \geq 3$ *and*

(1) $\alpha = cq3$ *and* $x \in [0,\ 1/2 - \sqrt{15}/10] \cup [1/6,\ 1/2)$

(2) $\alpha = cq4$ *and* $x \in (0,\ 1/2 - \sqrt{5}/10] \cup [1/3,\ 1/2]$

(3) $\alpha = q5$ *and* $x \in (0,\ 1/2 - \sqrt{15}/10] \cup [1/5,\ 1/2)$

(A3) $n \geq 4$ *and*

(1) $\alpha = cq5$ *and* $x \in (0,\ 1/2 - \sqrt{21}/14] \cup [3/7 - \sqrt{2}/7,\ 1/2)$.

The signs of the functions are determined by

$$(-1)^n G^{\alpha}_{2n-1}(x,t) > 0$$

for

(A1+) $n \geq 2$ *and*

(1) $\alpha = q3$ *and* $x = 0$

(2) $\alpha = q4$ *and* $x \in [1/3,\ 1/2]$

(A2+) $n \geq 3$ *and*

(1) $\alpha = cq3$ *and* $x \in [0,\ 1/2 - \sqrt{15}/10]$

(2) $\alpha = cq4$ *and* $x \in [1/3,\ 1/2]$

(3) $\alpha = q5$ *and* $x \in (0,\ 1/2 - \sqrt{15}/10]$

(A3+) $n \geq 4$ *and*

(1) $\alpha = cq5$ *and* $x \in (0,\ 1/2 - \sqrt{21}/14]$

and by

$$(-1)^n G^{\alpha}_{2n-1}(x,t) < 0$$

for

(A1–) $n \geq 2$ *and*

(1) $\alpha = q3$ *and* $x \in [1/6,\ 1/2)$

(2) $\alpha = q4$ *and* $x \in (0,\ 1/2 - \sqrt{3}/6]$

(A2–) $n \geq 3$ *and*

(1) $\alpha = cq3$ *and* $x \in [1/6,\ 1/2)$

(2) $\alpha = cq4$ *and* $x \in (0,\ 1/2 - \sqrt{5}/10]$

(3) $\alpha = q5$ *and* $x \in [1/5,\ 1/2)$

(A3–) $n \geq 4$ *and*

(1) $\alpha = cq5$ *and* $x \in [3/7 - \sqrt{2}/7,\ 1/2)$.

Remark 2.2. An important consequence of Lemma 2.1, (2.16) and (2.17) is that functions F^{α}_{2n}, defined by (2.3), are monotonous in t on $(0,1/2)$ and $(1/2,1)$, and since $F^{\alpha}_{2n}(x,0) = F^{\alpha}_{2n}(x,1) = 0$, they have constant sign on $(0,1)$. Namely, just note that

$$\frac{\partial F^{\alpha}_{2n}(x,t)}{\partial t} = \frac{\partial G^{\alpha}_{2n}(x,t)}{\partial t} = -2n\ G^{\alpha}_{2n-1}(x,t).$$

Finally, we recall the definition of the nth order divided difference of a function and an n-convex function, which are closely connected.

Definition 2.3. Let $f : [a,b] \to \mathbb{R}$. The nth order divided difference of f at distinct points $x_0, \ldots, x_n \in [a,b]$ is defined recursively by

$$[x_i]f = f(x_i), \quad i = 0, \ldots, n$$

and

$$[x_0, \ldots, x_n]f = \frac{[x_1, \ldots, x_n]f - [x_0, \ldots, x_{n-1}]f}{x_n - x_0}.$$

Definition 2.4. A function $f : [a, b] \to \mathbb{R}$ is said to be n-convex on $[a, b]$ for some $n \geq 0$ if for any choice of $n+1$ distinct points $x_0, \ldots, x_n$ from $[a, b]$, we have $[x_0, \ldots, x_n]f \geq 0$. If the inequality is reversed, then f is said to be n-concave on $[a, b]$.

If f is n-convex, then $f^{(n-2)}$ exists and is convex in the standard sense. If $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geq 0$. For more details see for example [9].

An important fact is that an n-convex function on $[a, b]$ is continuous on (a, b) and bounded on $[a, b]$, and each continuous and bounded function can be represented as a uniform limit of a sequence of the corresponding Bernstein polynomials (see for example [9]). The Bernstein polynomials of an n-convex function are also n-convex, so when stating our results for an n-convex function f, without any loss of generality, we may assume that $f^{(n)}$ exists and is continuous.

Now that the necessary preliminaries have been presented, we can proceed to the main result.

3. Main result

Theorem 3.1. *Let $f : [0, 1] \to \mathbb{R}$ be $(2n)$-convex on $[0, 1]$. Then for*

$$\begin{array}{llll} & \alpha = q3, & n \geq 2, & x \in [1/6,\ 1/2), \\ \textit{or} & \alpha = q4, & n \geq 2, & x \in (0,\ 1/2 - \sqrt{3}/6], \\ \textit{or} & \alpha = cq3, & n \geq 3, & x \in [1/6,\ 1/2), \\ \textit{or} & \alpha = cq4, & n \geq 3, & x \in (0,\ 1/2 - \sqrt{5}/10], \\ \textit{or} & \alpha = q5, & n \geq 3, & x \in [1/5,\ 1/2), \\ \textit{or} & \alpha = cq5, & n \geq 4, & x \in [3/7 - \sqrt{2}/7,\ 1/2) \end{array}$$

and

$$\begin{array}{llll} & \beta = q3, & n \geq 2, & y = 0, \\ \textit{or} & \beta = q4, & n \geq 2, & y \in [1/3,\ 1/2], \\ \textit{or} & \beta = cq3, & n \geq 3, & y \in [0,\ 1/2 - \sqrt{15}/10], \\ \textit{or} & \beta = cq4, & n \geq 3, & y \in [1/3,\ 1/2], \\ \textit{or} & \beta = q5, & n \geq 3, & y \in (0,\ 1/2 - \sqrt{15}/10], \\ \textit{or} & \beta = cq5, & n \geq 4, & y \in (0,\ 1/2 - \sqrt{21}/14] \end{array}$$

we have

(3.1)
$$(-1)^n \left(Q_\alpha(x) - T_{2n-1}^\alpha(x)\right) \leq (-1)^n \int_0^1 f(t)dt \leq (-1)^n \left(Q_\beta(y) - T_{2n-1}^\beta(y)\right),$$

where $Q_\alpha(x)$ are as in (2.4)–(2.9) and $T_{2n-1}^\alpha(x)$ as in (2.2). If f is $(2n)$-concave, the inequalities are reverse.

Proof. It suffices to give the proof for the case $\alpha = \beta$. First, recall that a bounded monotonic function is a function of bounded variation. Since f is $(2n)$-convex on $[0,1]$, this implies $f^{(2n-1)}$ is monotonic and bounded on $[0,1]$, and thus of bounded variation. Now, observe identity (2.1); more precisely, its right-hand side. By Remark 2.2, functions F_{2n}^{α} have constant sign on $[0,1]$. Let $(A1+)$, $(A2+)$ and $(A3+)$ from Lemma 2.1 hold. Under these conditions we have $(-1)^n F_{2n}^{\alpha}(x,t) < 0$. If f is $(2n)$-convex on $[0,1]$, this implies $f^{(2n)} \geq 0$ on $[0,1]$. Therefore, multiplying (2.1) by $(-1)^n$ gives

$$(-1)^n \int_0^1 f(t)dt \;\leq\; (-1)^n \left(Q_\alpha(x) - T_{2n-1}^{\alpha}(x)\right).$$

On the other hand, when $(A1-)$, $(A2-)$ and $(A3-)$ hold, we have the reversed situation, that is, we have $(-1)^n F_{2n}^{\alpha}(x,t) > 0$, which implies

$$(-1)^n \int_0^1 f(t)dt \;\geq\; (-1)^n \left(Q_\alpha(x) - T_{2n-1}^{\alpha}(x)\right).$$

If the function f is $(2n)$-concave, then we have $f^{(2n)} \leq 0$, so the inequalities are reverse. The proof is thus complete. □

Theorem 3.1 procures various interesting special cases, which we study in more detail in the following subsections.

3.1. **CASE:** $n = 2$ **and** $\alpha = q3, q4$ **and** $\beta = q3, q4$.

Corollary 3.2. *Let $f : [0,1] \to \mathbb{R}$ be 4-convex on $[0,1]$. Then*

- *for $x \in [1/6,\ 1/2)$, we have*

$$Q_{q3}(x) \;\leq\; \int_0^1 f(t)dt \;\leq\; \frac{1}{6}f(0) + \frac{2}{3}f\left(\frac{1}{2}\right) + \frac{1}{6}f(1), \tag{3.2}$$

- *for $x \in [1/6,\ 1/2)$ and $y \in [1/3,\ 1/2]$, we have*

$$Q_{q3}(x) \;\leq\; \int_0^1 f(t)dt \;\leq\; Q_{q4}(y), \tag{3.3}$$

- *for $x \in \left(0,\ 1/2 - \sqrt{3}/6\right]$, we have*

$$Q_{q4}(x) \;\leq\; \int_0^1 f(t)dt \;\leq\; \frac{1}{6}f(0) + \frac{2}{3}f\left(\frac{1}{2}\right) + \frac{1}{6}f(1), \tag{3.4}$$

- *for $x \in \left(0,\ 1/2 - \sqrt{3}/6\right]$ and $y \in [1/3,\ 1/2]$, we have*

$$Q_{q4}(x) \;\leq\; \int_0^1 f(t)dt \;\leq\; Q_{q4}(y), \tag{3.5}$$

where $Q_{q3}(x)$ and $Q_{q4}(x)$ are defined in (2.4) and (2.6), respectively. If f is 4-concave, the inequalities are reverse.

Let us now consider special choices of nodes x and y to see which well-known quadrature rules can be obtained as the lower bounds of an integral of a 4-convex function, and which as the upper.

For $x = 1/4$, the left-hand sides of (3.2) and (3.3) recapture the dual Simpson rule:

$$\text{Dual Simpson} = \frac{2}{3}f\left(\frac{1}{4}\right) - \frac{1}{3}f\left(\frac{1}{2}\right) + \frac{2}{3}f\left(\frac{3}{4}\right),$$

for $x = 1/6$ they recapture the Maclaurin rule:

$$\text{Maclaurin} = \frac{3}{8}f\left(\frac{1}{6}\right) + \frac{1}{4}f\left(\frac{1}{2}\right) + \frac{3}{8}f\left(\frac{5}{6}\right),$$

and finally, for $x = 1/2 - \sqrt{3}/6$, which is equivalent to $B_2(x) = 0$, the Gauss 2-point rule:

$$\text{Gauss 2-point} = \frac{1}{2}f\left(\frac{3-\sqrt{3}}{6}\right) + \frac{1}{2}f\left(\frac{3+\sqrt{3}}{6}\right).$$

The Gauss 2-point rule can also be produced, for the same choice of the node, from the left-hand sides of (3.4) and (3.5).

As for the upper bounds, first note that the right-hand sides of (3.2) and (3.4) are in fact the classical Simpson rule. Furthermore, for $y = 1/3$ the right-hand sides of (3.3) and (3.5) recapture the classical Simpson 3/8 rule:

$$\text{Simpson } 3/8 = \frac{1}{8}f(0) + \frac{3}{8}f\left(\frac{1}{3}\right) + \frac{3}{8}f\left(\frac{2}{3}\right) + \frac{1}{8}f(1).$$

Finally, note that inequality (3.2) for $x = 1/4$ recaptures two out of three inequalities from the Bullen-Simpson inequality

$$\begin{aligned} 0 &\leq \int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] \\ &\leq \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \int_0^1 f(t)dt \end{aligned}$$

and inequality (3.3) for $x = 1/6$ and $y = 1/3$ two out of three inequalities from the Bullen-Simpson 3/8 inequality

$$\begin{aligned} 0 &\leq \int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] \\ &\leq \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] - \int_0^1 f(t)dt. \end{aligned}$$

Both of these inequalities were derived in [3].

3.2. **CASE:** $n = 3$ **and** $\alpha = cq3, cq4, q5$ **and** $\beta = cq3, cq4, q5$**.**

Corollary 3.3. *Let* $f : [0,1] \to \mathbb{R}$ *be* 6*-convex on* $[0,1]$*. Then*

- *for* $x \in [0,\ 1/2 - \sqrt{15}/10]$ *and* $y \in [1/6,\ 1/2)$*, we have*

$$Q_{cq3}(x) - \frac{10x^2 - 10x + 1}{60(-4x^2 + 4x + 1)}[f'(1) - f'(0)] \leq \int_0^1 f(t)dt \leq Q_{cq3}(y) - \frac{10y^2 - 10y + 1}{60(-4y^2 + 4y + 1)}[f'(1) - f'(0)], \tag{3.6}$$

- *for* $x \in [0,\ 1/2 - \sqrt{15}/10]$ *and* $y \in \big(0,\ 1/2 - \sqrt{5}/10\big]$*, we have*

$$Q_{cq3}(x) - \frac{10x^2 - 10x + 1}{60(-4x^2 + 4x + 1)}[f'(1) - f'(0)] \leq \int_0^1 f(t)dt \leq Q_{cq4}(y) - \frac{5y^2 - 5y + 1}{60y(y-1)}[f'(1) - f'(0)], \tag{3.7}$$

- *for* $x \in [0,\ 1/2 - \sqrt{15}/10]$ *and* $y \in [1/5,\ 1/2)$*, we have*

$$Q_{cq3}(x) - \frac{10x^2 - 10x + 1}{60(-4x^2 + 4x + 1)}[f'(1) - f'(0)] \leq \int_0^1 f(t)dt \leq Q_{q5}(y), \tag{3.8}$$

- *for* $x \in [1/3,\ 1/2]$ *and* $y \in [1/6,\ 1/2)$*, we have*

$$Q_{cq4}(x) - \frac{5x^2 - 5x + 1}{60x(x-1)}[f'(1) - f'(0)] \leq \int_0^1 f(t)dt \leq Q_{cq3}(y) - \frac{10y^2 - 10y + 1}{60(-4y^2 + 4y + 1)}[f'(1) - f'(0)], \tag{3.9}$$

- *for* $x \in [1/3,\ 1/2]$ *and* $y \in \big(0,\ 1/2 - \sqrt{5}/10\big]$*, we have*

$$Q_{cq4}(x) - \frac{5x^2 - 5x + 1}{60x(x-1)}[f'(1) - f'(0)] \leq \int_0^1 f(t)dt \leq Q_{cq4}(y) - \frac{5y^2 - 5y + 1}{60y(y-1)}[f'(1) - f'(0)], \tag{3.10}$$

- *for* $x \in [1/3,\ 1/2]$ *and* $y \in [1/5,\ 1/2)$*, we have*

$$Q_{cq4}(x) - \frac{5x^2 - 5x + 1}{60x(x-1)}[f'(1) - f'(0)] \leq \int_0^1 f(t)dt \leq Q_{q5}(y), \tag{3.11}$$

- *for* $x \in (0,\ 1/2 - \sqrt{15}/10]$ *and* $y \in [1/6,\ 1/2)$*, we have*

$$Q_{q5}(x) \leq \int_0^1 f(t)dt \leq Q_{cq3}(y) - \frac{10y^2 - 10y + 1}{60(-4y^2 + 4y + 1)}[f'(1) - f'(0)], \tag{3.12}$$

- *for* $x \in (0,\ 1/2 - \sqrt{15}/10]$ *and* $y \in \big(0,\ 1/2 - \sqrt{5}/10\big]$*, we have*

$$Q_{q5}(x) \leq \int_0^1 f(t)dt \leq Q_{cq4}(y) - \frac{5y^2 - 5y + 1}{60y(y-1)}[f'(1) - f'(0)], \tag{3.13}$$

- *for* $x \in (0,\ 1/2 - \sqrt{15}/10]$ *and* $y \in [1/5,\ 1/2)$*, we have*

$$Q_{q5}(x) \ \le\ \int_0^1 f(t)dt \ \le\ Q_{q5}(y), \tag{3.14}$$

where $Q_{cq3}(x), Q_{cq4}(x)$ *and* $Q_{q5}(x)$ *are defined as in* (2.5), (2.7) *and* (2.8)*, respectively. If* f *is* 6*-concave, the inequalities are reverse.*

Now, similarly as before, we turn our attention to the special choices of nodes x and y. First, we take a look at the lower bounds of an integral of a 6-convex function.

For $x = 1/2 - \sqrt{15}/10$, which is equivalent to $10x^2 - 10x + 1 = 0$, the left-hand sides of (3.6), (3.7) and (3.8) become the Gauss 3-point rule:

$$\text{Gauss 3-point} \ =\ \frac{5}{18} f\left(\frac{5-\sqrt{15}}{10}\right) + \frac{4}{9} f\left(\frac{1}{2}\right) + \frac{5}{18} f\left(\frac{5+\sqrt{15}}{10}\right),$$

while for $x = 0$ they become the corrected Simpson rule:

$$\text{Corrected Simpson} \ =\ \frac{7}{30} f(0) + \frac{8}{15} f\left(\frac{1}{2}\right) + \frac{7}{30} f(1) - \frac{1}{60}[f'(1) - f'(0)].$$

The Gauss 3-point rule can also be obtained from the left-hand sides of (3.12), (3.13) and (3.14), while the corrected Simpson rule can also be obtained from the left-hand sides of (3.9), (3.10) and (3.11) for $x = 1/2$.

For $x = 1/3$, the left-hand sides of (3.9), (3.10) and (3.11) recapture the corrected Simpson 3/8 rule:

$$\begin{aligned}\text{Corrected Simpson 3/8} \ &=\ \frac{13}{80} f(0) + \frac{27}{80} f\left(\frac{1}{3}\right) + \frac{27}{80} f\left(\frac{2}{3}\right) + \frac{13}{80} f(1) \\ &\quad - \frac{1}{120}[f'(1) - f'(0)].\end{aligned}$$

Next, we consider the upper bounds. For $y = 1/4$, the right-hand sides of (3.6), (3.9) and (3.12) become the corrected dual Simpson rule:

$$\text{Corrected dual Simpson} = \frac{8}{15} f\left(\frac{1}{4}\right) - \frac{1}{15} f\left(\frac{1}{2}\right) + \frac{8}{15} f\left(\frac{3}{4}\right) + \frac{1}{120}[f'(1) - f'(0)],$$

while for $y = 1/6$, the corrected Maclaurin rule:

$$\text{Corrected Maclaurin} \ =\ \frac{27}{80} f\left(\frac{1}{6}\right) + \frac{13}{40} f\left(\frac{1}{2}\right) + \frac{27}{80} f\left(\frac{5}{6}\right) + \frac{1}{240}[f'(1) - f'(0)].$$

For $y = 1/2 - \sqrt{225 - 30\sqrt{30}}\big/30$, which is equivalent to $B_4(y) = 0$, the right-hand sides of (3.6), (3.7), (3.9), (3.10), (3.12) and (3.13) become the corrected

Gauss 2-point rule:

$$\text{Corrected Gauss 2-point} = \frac{1}{2}f\left(\frac{15-\sqrt{225-30\sqrt{30}}}{30}\right) + \frac{1}{2}f\left(\frac{15+\sqrt{225-30\sqrt{30}}}{30}\right) + \frac{7\sqrt{30}-5}{420}[f'(1)-f'(0)].$$

For $y = 1/2 - \sqrt{5}/10$, which is equivalent to $5y^2 - 5y + 1 = 0$, the right-hand sides of (3.7), (3.8), (3.10), (3.11), (3.13) and (3.14) give the Lobatto 4-point rule:

$$\text{Lobatto 4-point} = \frac{1}{12}f(0) + \frac{5}{12}f\left(\frac{5-\sqrt{5}}{10}\right) + \frac{5}{12}f\left(\frac{5+\sqrt{5}}{10}\right) + \frac{1}{12}f(1).$$

Finally, for $y = 1/4$, the right-hand sides of (3.8), (3.11) and (3.14) give the classical Boole rule:

$$\text{Boole} = \frac{7}{90}f(0) + \frac{16}{45}f\left(\frac{1}{4}\right) + \frac{2}{15}f\left(\frac{1}{2}\right) + \frac{16}{45}f\left(\frac{3}{4}\right) + \frac{7}{90}f(1).$$

3.3. **CASE:** $n = 4$ **and** $\alpha = \beta = cq5$.

Corollary 3.4. *Let $f : [0,1] \to \mathbb{R}$ be 8-convex on $[0,1]$. Then, for $x \in \left[3/7 - \sqrt{2}/7,\ 1/2\right)$ and $y \in (0,\ 1/2 - \sqrt{21}/14]$, we have*

$$(3.15)\qquad Q_{cq5}(x) - \frac{7x^2-7x+1}{420x(x-1)}[f'(1)-f'(0)] \le \int_0^1 f(t)dt \le Q_{cq5}(y) - \frac{7y^2-7y+1}{420y(y-1)}[f'(1)-f'(0)],$$

where $Q_{cq5}(x)$ is defined in (2.9). If f is 8-concave, the inequalities are reverse.

For $x = 1/2 - \sqrt{7}/14$, which is equivalent to $14x^2 - 14x + 3 = 0$, from (3.15) we get the corrected Lobatto 4-point rule as a lower bound of an integral of an 8-convex function:

Corrected Lobatto 4-point

$$= \frac{37}{270}f(0) + \frac{49}{135}f\left(\frac{7-\sqrt{7}}{14}\right) + \frac{49}{135}f\left(\frac{7+\sqrt{7}}{14}\right) + \frac{37}{270}f(1) - \frac{1}{180}[f'(1)-f'(0)],$$

and for $x = 1/4$ we get another lower bound in a form of the corrected Boole rule:

$$\text{Corrected Boole} = \frac{31}{270}f(0) + \frac{256}{945}f\left(\frac{1}{4}\right) + \frac{8}{35}f\left(\frac{1}{2}\right) + \frac{256}{945}f\left(\frac{3}{4}\right) + \frac{31}{270}f(1) - \frac{1}{252}[f'(1)-f'(0)].$$

As for the upper bounds, for $y = 1/2 - \sqrt{21}/14$, which is equivalent to $7y^2 - 7y + 1 = 0$, we get the classical Lobatto 5-point rule:

$$\text{Lobatto 5-point} = \frac{1}{20}f(0) + \frac{49}{180}f\left(\frac{7-\sqrt{21}}{14}\right) + \frac{16}{45}f\left(\frac{1}{2}\right) + \frac{49}{180}f\left(\frac{7+\sqrt{21}}{14}\right) + \frac{1}{20}f(1),$$

and finally, for $y = 1/2 - \sqrt{45 - 2\sqrt{102}}\Big/14$, which is equivalent to
$98y^4 - 196y^3 + 102y^2 - 4y - 1$
$= \left[(y-1/2)^2 - (45+2\sqrt{102})/196\right]\left[(y-1/2)^2 - (45-2\sqrt{102})/196\right] = 0,$
the corrected Gauss 3-point rule:

$$\begin{aligned}\text{Corrected Gauss 3-point} &= \frac{1977+16\sqrt{102}}{6930}\left(f\left(\frac{7-\sqrt{45-2\sqrt{102}}}{14}\right) + f\left(\frac{7+\sqrt{45-2\sqrt{102}}}{14}\right)\right)\\ &\quad + \frac{2976-32\sqrt{102}}{6930}f\left(\frac{1}{2}\right) - \frac{9-\sqrt{102}}{420}[f'(1)-f'(0)].\end{aligned}$$

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