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A NEW EXTRAGRADIENT ITERATION ALGORITHM FOR BILEVEL VARIATIONAL INEQUALITIES

PHAM NGOC ANH

Abstract. In this paper, we introduce an approximation extragradient iteration method for solving bilevel variational inequalities involving two variational inequalities and we show that these problems can be solved by projection sequences and fixed point techniques. We obtain a strong convergence of three iteration sequences generated by this method in a real Hilbert space.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, and let C be a nonempty closed convex subset of $\mathcal{H}$. We consider the *bilevel variational inequalities* (shortly BVI):

$$\text{Find } x^* \in Sol(G, C) \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0 \ \ \forall x \in Sol(G, C),$$

where $G : \mathcal{H} \to \mathcal{H}$, $Sol(G, C)$ denotes the set of solutions of the following variational inequalities:

$$\text{Find } y^* \in C \text{ such that } \langle G(y^*), y - y^* \rangle \geq 0 \ \ \forall y \in C,$$

and $F : C \to \mathcal{H}$. We denote by $Sol(BVI)$ the set of solutions of (BVI).

The problems (BVI) are also called to be *quasivariational inequalities* (see [8, 9, 10]). There problems are very interesting because they cover a class of mathematical programs with equilibrum constraints (see [12]), bilevel minimization problems (see [16]), variational inequalities and complementarity problems (see [1, 2, 5, 7, 13]).

If $F \equiv 0$, then the bilevel variational inequalities (BVI) become the following variational inequalities $\big($shortly $VI(G, C)\big)$:

$$\text{Find } x^* \in C \text{ such that } \langle G(x^*), x - x^* \rangle \geq 0 \ \ \forall x \in C.$$

Suppose that $f : \mathcal{H} \to \mathbb{R}$. It is well-known in convex programming that if f is convex and differentiable on $Sol(G, C)$ then x^* is a solution to

$$\min\{f(x) \mid x \in Sol(G, C)\}$$

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if and only if x^* is the solution to the variational inequalities $VI\big(\nabla f, Sol(G,C)\big)$, where ∇f is the differentiation of f. Then the bilevel variational inequalities (BVI) are written by a form of mathematical programs with equilibrum constraints:

$$\begin{cases} \min f(x) \\ x \in \{y^* \mid \langle G(y^*), z - y^* \rangle \geq 0 \ \ \forall z \in C\}. \end{cases}$$

If f, g are two convex and differentiable functions, then the problems (BVI) (where $F := \nabla f$ and $G := \nabla g$) become the following bilevel minimization problem (see [16]):

$$\begin{cases} \min f(x) \\ x \in \operatorname{argmin}\{g(x) \mid x \in C\}. \end{cases}$$

In recent years, variational inequalities become an attractive field for many researchers and have many important applications in electricity markets, transportations, economics, nonlinear analysis (see [6, 9, 19]). Methods for solving variational inequalities have been studied extensively. The extragradient algorithm for solving the variational inequalities $VI(G,C)$ was introduced by Korpelevich in [11], where the iteration sequence $\{x^k\}$ is defined by

$$\begin{cases} x^0 \in C, \\ y^k = Pr_C\big(x^k - c_k G(x^k)\big), \\ x^{k+1} = Pr_C\big(x^k - c_k G(y^k)\big), \end{cases}$$

and extended by many other authors (see [5, 9, 14, 18]). One of the main conditions ensures the convergence result of this method is that the cost mapping enjoys the Lipschitzian continuity property. However, such a condition is rather restrictive. In order to avoid it, the following Armijo-backtracking linesearch has been used to construct a hyperplane separating x^k from the solution set. Then the new iterate x^{k+1} is the projection of x^k onto this hyperplane. Recently, Anh and Kuno in [4] extended these results to generalized monotone nonlipschitzian multivalued variational inequalities. Precisely, the authors first used the interior proximal function to develop a convergent algorithm for the multivalued variational inequalities $VI(F,C)$, where F is a generalized monotone multifunction. Next the authors constructed an appropriate hyperplane which separates the current iterative point from the solution set. Then the next iterate is the projection of the current iterate onto the intersection of the feasible set with the halfspace containing the solution set.

Note that since the constraint set $\mathrm{Sol}(G,C)$ being the solution set of the problem $\mathrm{VI}(G,C)$ is not explicitly given, the existing algorithms for variational inequalities can not be directly applied because the subproblems can not be implemented by the available algorithms of convex programming. In this paper we extend results in [3] to the bilevel variational inequalities (BVI), but in a real Hilbert space. We are interested in finding a solution to bilevel variational inequalities (BVI) where the functions F and G satisfy the following usual conditions:

(A_1) G is monotone on C and F is β-strongly monotone on C,
(A_2) F is L_1-Lipschitz continuous on C,
(A_3) G is L_2-Lipschitz continuous on C,
(A_4) The solution set of (BVI) denoted by Sol(BVI) is nonempty.

In the next section, we give a new approximation extragradient algorithm for solving problems (BVI).

2. Preliminaries

We list some known definitions and properties of the projection under the Euclidean norm which will be required in our following analysis.

Definition 2.1. Let C be a nonempty closed convex subset in a real Hilbert space $\mathcal{H}$. We denote the projection on C by $Pr_C(\cdot)$ with images

$$Pr_C(x) = \{y \in C \mid \|y - x\| = \min_{v \in C} \|v - x\|\} \quad \forall x \in \mathcal{H}.$$

The function $\varphi : C \to \mathcal{H}$ is said to be

(i) γ-strongly monotone on C if for any $x, y \in C$, we have

$$\langle \varphi(x) - \varphi(y), x - y\rangle \geq \gamma \|x - y\|^2,$$

(ii) monotone on C if for any $x, y \in C$, we have

$$\langle \varphi(x) - \varphi(y), x - y\rangle \geq 0,$$

(iii) Lipschitz on C with constant $L > 0$ (shortly L-Lipschitz) if for any $x, y \in C$, we have

$$\|\varphi(x) - \varphi(y)\| \leq L\|x - y\|.$$

If $\varphi : C \to C$ and $L = 1$ then φ is called nonexpansive on C.

The projection $Pr_C(\cdot)$ has the following basic properties:

$(Proj_1)$ $\|Pr_C(x) - Pr_C(y)\| \leq \|x - y\| \quad \forall x, y \in \mathcal{H}$.
$(Proj_2)$ $\|Pr_C(x) - Pr_C(y)\|^2 \leq \langle Pr_C(x) - Pr_C(y), x - y\rangle \quad \forall x, y \in \mathcal{H}$.
$(Proj_3)$ $\langle x - Pr_C(x), y - Pr_C(y)\rangle \leq 0 \quad \forall y \in C, x \in \mathcal{H}$.
$(Proj_4)$ $\|Pr_C(x) - y\|^2 \leq \|x - y\|^2 - \|Pr_C(x) - x\|^2 \quad \forall y \in C, x \in \mathcal{H}$.
$(Proj_5)$ $\|Pr_C(x) - Pr_C(y)\|^2 \leq \|x - y\|^2 - \|Pr_C(x) - x + y - Pr_C(y)\|^2 \quad \forall x, y \in \mathcal{H}$.

Now we are in a position to propose a new extragradient-type algorithm for (BVI).

Algorithm 2.2. *Initialization.* Choose $k = 0, x^0 \in \mathcal{H}, 0 < \lambda \leq \frac{2\beta}{L_1^2}$, positive sequences $\{\epsilon_k\}, \{\beta_k\}, \{\gamma_k\}$, $\{\delta_k\}$, $\{\lambda_k\}$, $\{\alpha_k\}$ and $\{\bar{\epsilon}_k\}$ such that

$$\begin{cases} \{\alpha_k\} \subset [m, n] \text{ for some } m, n \in (0,1), \lambda_k \leq \frac{1}{L_2} \quad \forall k \geq 0, \\ \lim\limits_{k\to\infty} \delta_k = 0, \sum\limits_{k=0}^{\infty} \bar{\epsilon}_k < \infty, 0 < \liminf\limits_{k\to\infty} \beta_k < \limsup\limits_{k\to\infty} \beta_k < 1, \\ \epsilon_k + \beta_k + \gamma_k = 1 \quad \forall k \geq 0, \lim\limits_{k\to\infty} \epsilon_k = 0, \sum\limits_{k=0}^{\infty} \epsilon_k = \infty. \end{cases}$$

Step 1. If $x^k \in Sol(BVI)$, then stop. Otherwise compute $y^k = Pr_C\big(x^k - \lambda_k G(x^k)\big)$ and $z^k = Pr_C\big(x^k - \lambda_k G(y^k)\big)$.
Step 2. Inner iterations $j = 0, 1, \cdots$. Compute

$$\begin{cases} x^{k,0} = z^k - \lambda F(z^k), \\ y^{k,j} = Pr_C\big(x^{k,j} - \delta_j G(x^{k,j})\big), \\ x^{k,j+1} = \epsilon_j x^{k,0} + \beta_j x^{k,j} + \gamma_j Pr_C\big(x^{k,j} - \delta_j G(y^{k,j})\big). \end{cases}$$

Find h^k such that $\|h^k - \lim_{j\to\infty} x^{k,j}\| \le \bar{\epsilon_k}$ and set $x^{k+1} = \alpha_k x^k + (1-\alpha_k)h^k$.
Step 3. Increase k by 1 and go to Step 1.

Remark 2.3. If $x^{k+1} = \alpha_k x^k + (1-\alpha_k)h^k$ is substituted for $x^{k+1} = \bar{\alpha}_k u + \bar{\beta}_k x^k + \bar{\gamma}_k h^k$, where $\bar{\alpha}_k, \bar{\beta}_k, \bar{\gamma}_k \in [0,1]$ for all $k \ge 0, u \in \mathcal{R}^n$ and $\bar{\alpha}_k + \bar{\beta}_k + \bar{\gamma}_k = 1$, then Algorithm 2.2 becomes Algorithm 2.1 in $\mathcal{R}^n$ proposed by Anh et al. in [3]. Using this fixed point technique allows us to extend the result from a finite-dimensional space $\mathcal{R}^n$ to a real Hilbert space $\mathcal{H}$.

Remark 2.4. Suppose that $\alpha_k = \delta_k = \lambda = 0$. Then we can choose $h^k = z^k$ and it is easy to see that the sequence $\{x^k\}$ in Algorithm 2.2 is the well-known extragradient iteration sequence which was first introduced by Korpelevich in [11].

3. Convergence results

Let C be a nonempty closed convex subset of $\mathcal{H}$, $G : \mathcal{H} \to \mathcal{H}$ be monotone and L_2-Lipschitz on C, and $S : C \to C$ be a nonexpansive mapping such that $Sol(G, C) \cap Fix(S) \neq \emptyset$, where $Fix(S) := \{x \in C \mid S(x) = x\}$ is the set of fixed points of S. Let the sequences $\{x^k\}$ and $\{y^k\}$ be generated by

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^k = Pr_C\big(x^k - \delta_k G(x^k)\big), \\ x^{k+1} = \epsilon_k x^0 + \beta_k x^k + \gamma_k S Pr_C\big(x^k - \delta_k G(y^k)\big) \quad \forall k \ge 0, \end{cases}$$

where $\{\epsilon_k\}, \{\beta_k\}, \{\gamma_k\}$ and $\{\delta_k\}$ satisfy the following conditions:

$$\begin{cases} \delta_k > 0 \ \forall k \ge 0, \lim_{k\to\infty} \delta_k = 0, \\ \epsilon_k + \beta_k + \gamma_k = 1 \ \forall k \ge 0, \\ \sum_{k=1}^{\infty} \epsilon_k = \infty, \lim_{k\to\infty} \epsilon_k = 0, \\ 0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1. \end{cases}$$

Under these conditions, Yao et al. showed that the sequences $\{x^k\}$ and $\{y^k\}$ converge strongly to the same point $Pr_{Sol(G,C)\cap Fix(S)}(x^0)$ in [18].

Apply these iteration sequences with S being the identity mapping, we have the following lemma.

Lemma 3.1. *Suppose that the assumptions* $(\mathrm{A}_1) - (\mathrm{A}_4)$ *hold. Then the sequence* $\{x^{k,j}\}$ *generated by Algorithm 2.2 converges strongly to the point* $Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big)$ *as* $j \to \infty$. *Consequently, we have*

$$\|h^k - Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big)\| \leq \bar{\epsilon}_k \quad \forall k \geq 0.$$

Lemma 3.2. *Let sequences* $\{x^k\}$ *and* $\{z^k\}$ *be generated by Algorithm 2.2,* G *be* L_2*-Lipschitz and monotone on* C*, and* $x^* \in Sol(G,C)$*. Then, we have*

$$\|z^k - x^*\|^2 \leq \|x^k - x^*\|^2 - (1 - \lambda_k L_2)\|x^k - y^k\|^2 - (1 - \lambda_k L_2)\|y^k - z^k\|^2. \tag{3.1}$$

Proof. Let x^* be a solution to probems $VI(G,C)$, $x^* \in C$ and

$$\langle G(x^*), x - x^*\rangle \geq 0 \quad \forall x \in C.$$

Then, for each $\lambda_k > 0$, x^* is a fixed point of mapping $T(x) = Pr_C\big(x - \lambda_k G(x)\big)$ on C (see [9]), i.e.,

$$x^* = Pr_C\big(x^* - \lambda_k G(x^*)\big).$$

Substituting x by $x^k - \lambda_k G(y^k)$ and y by x^* into $(Proj_4)$, we get

$$\begin{aligned}
\|z^k - x^*\|^2 \leq & \|x^k - \lambda_k G(y^k) - x^*\|^2 - \|x^k - \lambda_k G(y^k) - z^k\|^2 \\
= & \|x^k - x^*\|^2 - 2\lambda_k\langle G(y^k), x^k - x^*\rangle + \lambda_k^2\|G(y^k)\|^2 - \|x^k - z^k\|^2 \\
& - \lambda_k^2\|G(y^k\|^2 + 2\lambda_k\langle G(y^k), x^k - z^k\rangle \\
= & \|x^k - x^*\|^2 - \|x^k - z^k\|^2 + 2\lambda_k\langle G(y^k), x^* - z^k\rangle \\
= & \|x^k - x^*\|^2 - \|x^k - z^k\|^2 + 2\lambda_k\langle G(y^k) - G(x^*), x^* - y^k\rangle \\
& + 2\lambda_k\langle G(x^*), x^* - y^k\rangle + 2\lambda_k\langle G(y^k), y^k - z^k\rangle \\
\leq & \|x^k - x^*\|^2 - \|x^k - z^k\|^2 + 2\lambda_k\langle G(y^k), y^k - z^k\rangle. \qquad (3.2)
\end{aligned}$$

The last inequality holds because $y^k \in C$, $x^* \in Sol(G,C)$ and G is monotone on C.
Substituting x by $x^k - \lambda_k G(x^k)$ and y by z^k into $(Proj_3)$, we have

$$\langle x^k - \lambda_k G(x^k) - y^k, z^k - y^k\rangle \leq 0.$$

Combining this with (3.2) and the Lipchitzian continuity of G on C with constant L_2, we obtain

$$\begin{aligned}\|z^k - x^*\|^2 \leq &\|x^k - x^*\|^2 - \|(x^k - y^k) + (y^k - z^k)\|^2 + 2\lambda_k\langle G(y^k), y^k - z^k\rangle\\ = &\|x^k - x^*\|^2 - \|x^k - y^k\|^2 - \|y^k - z^k\|^2 - 2\langle x^k - y^k, y^k - z^k\rangle\\ &+ 2\lambda_k\langle G(y^k), y^k - z^k\rangle\\ = &\|x^k - x^*\|^2 - \|x^k - y^k\|^2 - \|y^k - z^k\|^2 - 2\langle x^k - \lambda_k G(y^k) - y^k, y^k - z^k\rangle\\ = &\|x^k - x^*\|^2 - \|x^k - y^k\|^2 - \|y^k - z^k\|^2 - 2\langle x^k - \lambda_k G(x^k) - y^k, y^k - z^k\rangle\\ &+ 2\lambda_k\langle G(x^k) - G(y^k), z^k - y^k\rangle\\ \leq &\|x^k - x^*\|^2 - \|x^k - y^k\|^2 - \|y^k - z^k\|^2 + 2\lambda_k\langle G(x^k) - G(y^k), z^k - y^k\rangle\\ \leq &\|x^k - x^*\|^2 - \|x^k - y^k\|^2 - \|y^k - z^k\|^2 + 2\lambda_k\|G(x^k) - G(y^k)\|\|z^k - y^k\|\\ \leq &\|x^k - x^*\|^2 - \|x^k - y^k\|^2 - \|y^k - z^k\|^2 + 2\lambda_k L_2\|x^k - y^k\|\|z^k - y^k\|\\ \leq &\|x^k - x^*\|^2 - \|x^k - y^k\|^2 - \|y^k - z^k\|^2 + \lambda_k L_2\big(\|x^k - y^k\|^2 + \|z^k - y^k\|^2\big)\\ \leq &\|x^k - x^*\|^2 - (1 - \lambda_k L_2)\|x^k - y^k\|^2 - (1 - \lambda_k L_2)\|y^k - z^k\|^2.\end{aligned}$$

This implies (3.1). □

Lemma 3.3. *Suppose that Assumptions* $(\mathrm{A}_1) - (\mathrm{A}_4)$ *hold. Then, the sequence* $\{x^k\}$ *generated by Algorithm 2.2 is bounded.*

Proof. Suppose that x^* is a solution to problems (BVI),

$$\langle F(x^*), x - x^*\rangle \geq 0 \quad \forall x \in Sol(G, C),$$

we have

$$x^* = Pr_{Sol(G,C)}\big(x^* - \lambda F(x^*)\big).$$

Then, it follows from $(Proj_1)$, β-strongly monotonicity and L_1-Lipschitz continuity of F, and $0 < \lambda \leq \frac{2\beta}{L_1^2}$ that

$$\begin{aligned}&\|Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big) - x^*\|^2\\ &= \|Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big) - Pr_{Sol(G,C)}\big(x^* - \lambda F(x^*)\big)\|^2\\ &\leq \|z^k - \lambda F(z^k) - x^* + \lambda F(x^*)\|^2\\ &\leq \|z^k - x^*\|^2 - 2\lambda\langle F(z^k) - F(x^*), z^k - x^*\rangle + \lambda^2\|F(z^k) - F(x^*)\|^2\\ &\leq (1 - 2\beta\lambda + \lambda^2 L_1^2)\|z^k - x^*\|^2\\ &\leq \|z^k - x^*\|^2. \qquad (3.3)\end{aligned}$$

It follows from $\lambda_k \leq \frac{1}{L_2}$ and (3.1) that $\|z^k - x^*\| \leq \|x^k - x^*\|$. Combining this with (3.3) and Assumptions $0 < \lambda \leq \frac{2\beta}{L_1}$, $\sum\limits_{k=0}^{\infty} \bar{\epsilon_k} < +\infty$, we have

$$\begin{aligned}
\|x^{k+1} - x^*\| = &\|\alpha_k x^k + (1-\alpha_k)h^k - x^*\| \\
\leq &\alpha_k \|x^k - x^*\| + (1-\alpha_k)\|h^k - x^*\| \\
\leq &\alpha_k \|x^k - x^*\| + (1-\alpha_k)\|h^k - Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big)\| \\
&+ (1-\alpha_k)\|Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big) - x^*\| \\
\leq &\alpha_k \|x^k - x^*\| + (1-\alpha_k)\bar{\epsilon}_k + (1-\alpha_k)\|z^k - x^*\| \\
\leq &\alpha_k \|x^k - x^*\| + (1-\alpha_k)\bar{\epsilon}_k + (1-\alpha_k)\|x^k - x^*\| \\
= &\|x^k - x^*\| + (1-\alpha_k)\bar{\epsilon}_k \qquad (3.4)\\
< &\|x^k - x^*\| + \bar{\epsilon}_k \\
\leq &\|x^0 - x^*\| + \sum_{k=0}^{+\infty} \bar{\epsilon_k} \\
< &+\infty.
\end{aligned}$$

Therefore, the sequence $\{x^k\}$ is bounded. □

Lemma 3.4. (see [9]) *Let $\{a_k\}$ and $\{b_k\}$ be two positive real sequences such that*

$$a_{k+1} \leq a_k + b_k \quad \forall k \geq 0 \text{ and } \sum_{k=0}^{\infty} b_k < +\infty.$$

Then there exists $\lim\limits_{k\to\infty} a_k = c$.

Lemma 3.5. *Suppose that Assumptions $(\mathrm{A}_1) - (\mathrm{A}_4)$ hold and the sequences $\{x^k\}$ and $\{z^k\}$ are generated by Algorithm 2.2. Then, we have*

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 \leq &\|x^k - x^*\|^2 + 2(1-\alpha_k)\bar{\epsilon}_k\|z^k - x^*\| + (1-\alpha_k)\bar{\epsilon}_k^2 \\
&- (1-\alpha_k)(1-\lambda_k L_2)\|x^k - y^k\|^2 - (1-\alpha_k)(1-\lambda_k L_2)\|y^k - z^k\|^2. \qquad (3.5)
\end{aligned}$$

Consequently, we have

$$\lim_{k\to\infty} \|x^k - y^k\| = \lim_{k\to\infty} \|y^k - z^k\| = \lim_{k\to\infty} \|x^k - z^k\| = 0.$$

Proof. For each $k \geq 0$, Lemma 3.1 shows that there exists

$$\lim_{j\to\infty} x^{k,j} = Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big).$$

Combining this with $0 < \lambda \le \frac{2\beta}{L_1^2}$, (3.1), Lemma 3.1 and $x^* \in Sol(BVI)$, for $k \ge 0$ we have

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 =& \|\alpha_k x^k + (1-\alpha_k)h^k - x^*\|^2 \\
\le& \alpha_k \|x^k - x^*\|^2 + (1-\alpha_k)\|h^k - x^*\|^2 \\
\le& \alpha_k \|x^k - x^*\|^2 + (1-\alpha_k)\Big(\|Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big) - x^*\| + \bar{\epsilon}_k\Big)^2 \\
=& \alpha_k \|x^k - x^*\|^2 + (1-\alpha_k) \\
& \times \{\|Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big) - Pr_{Sol(G,C)}\big(x^* - \lambda F(x^*)\big)\| + \bar{\epsilon}_k\}^2 \\
\le& \alpha_k \|x^k - x^*\|^2 + (1-\alpha_k)\Big(\sqrt{1 - 2\eta\lambda + \lambda^2 L_1^2}\|z^k - x^*\| + \bar{\epsilon}_k\Big)^2 \\
\le& \alpha_k \|x^k - x^*\|^2 + (1-\alpha_k)\Big(\|z^k - x^*\| + \bar{\epsilon}_k\Big)^2 \\
=& \alpha_k \|x^k - x^*\|^2 + (1-\alpha_k)\|z^k - x^*\|^2 \\
& + 2(1-\alpha_k)\bar{\epsilon}_k\|z^k - x^*\| + (1-\alpha_k)\bar{\epsilon}_k^2 \\
\le& \alpha_k \|x^k - x^*\|^2 + 2(1-\alpha_k)\bar{\epsilon}_k\|z^k - x^*\| + (1-\alpha_k)\bar{\epsilon}_k^2 + (1-\alpha_k) \\
& \times \Big(\|x^k - x^*\|^2 - (1-\lambda_k L_2)\|x^k - y^k\|^2 - (1-\lambda_k L_2)\|y^k - z^k\|^2\Big) \\
=& \|x^k - x^*\|^2 + 2(1-\alpha_k)\bar{\epsilon}_k\|z^k - x^*\| + (1-\alpha_k)\bar{\epsilon}_k^2 \\
& - (1-\alpha_k)(1-\lambda_k L_2)\|x^k - y^k\|^2 - (1-\alpha_k)(1-\lambda_k L_2)\|y^k - z^k\|^2.
\end{aligned}$$

This implies (3.5). It follows from (3.4) that

$$\|x^{k+1} - x^*\| \le \|x^k - x^*\| + \bar{\epsilon}_k.$$

Combining this, $\sum_{k=0}^{\infty} \bar{\epsilon}_k < +\infty$ and Lemma 3.4, there exists

$$\lim_{k\to\infty} \|x^k - x^*\| = c. \tag{3.6}$$

Hence by (3.5), we have $\|x^k - y^k\| \to 0$ as $k \to \infty$. Since $\lambda_k \le \frac{1}{L_2}$, (3.5), (3.6) and $\{\alpha_k\} \subset [m, n]$ for some $m, n \in (0, 1)$, we obtain

$$\begin{aligned}
(1-\alpha_k)(1-\lambda_k L_2)\|z^k - y^k\|^2 \le& \|x^k - x^*\|^2 + 2(1-\alpha_k)\bar{\epsilon}_k\|z^k - x^*\| + (1-\alpha_k)\bar{\epsilon}_k^2 \\
& - \|x^{k+1} - x^*\|^2,
\end{aligned}$$

and hence $\|z^k - y^k\| \to 0$ as $k \to \infty$. Consequently,

$$\|x^k - z^k\| \le \|x^k - y^k\| + \|y^k - z^k\| \Rightarrow \lim_{k\to\infty} \|x^k - z^k\| = 0.$$

□

Lemma 3.6. (see [15]) *Let $\mathcal{H}$ be a real Hilbert space, $\{\alpha_k\}$ be a sequence of real numbers such that $0 < a \le \alpha_k \le b < 1$ for all $k \ge 0$, and two sequences*

$\{x^k\}, \{y^k\}$ in $\mathcal{H}$ such that

$$\begin{cases} \limsup_{k\to\infty} \|x^k\| \le c, \\ \limsup_{k\to\infty} \|y^k\| \le c, \\ \lim_{k\to\infty} \|\alpha_k x^k + (1-\alpha_k) y^k\| = c. \end{cases}$$

Then, $\lim_{k\to\infty} \|x^k - y^k\| = 0$.

Lemma 3.7. (see [17]) *Let $\mathcal{H}$ be a real Hilbert space and C be a nonempty closed convex subset of $\mathcal{H}$. Let $\{x^k\}$ be a sequence in $\mathcal{H}$. Suppose that, for all $x^* \in C$,*

$$\|x^{k+1} - x^*\| \le \|x^k - x^*\| \quad \forall k \ge 0.$$

Then, the sequence $\{Pr_C(x^k)\}$ converges strongly to some $\bar{x} \in C$.

Theorem 3.8. *Suppose that Assumptions $(A_1)-(A_4)$ hold. Then three sequences $\{x^k\}, \{y^k\}$ and $\{z^k\}$ generated by Algorithm 2.2 converge strongly to a solution x^* of problems (BVI). Moreover, we have*

$$x^* = \lim_{k\to\infty} Pr_{Sol(G,C)}(x^k).$$

Proof. It follows from (3.1), (3.3) and (3.6) that

$$\begin{aligned} \limsup_{k\to\infty} \|h^k - x^*\| &\le \limsup_{k\to\infty} \{\|h^k - Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big)\| \\ &\quad + \|Pr_{Sol(G,C)}\big(z^k - \lambda F(z^k)\big) - x^*\|\} \\ &\le \limsup_{k\to\infty} \{\bar{\epsilon}_k + \|z^k - x^*\|\} \\ &\le \limsup_{k\to\infty} \{\bar{\epsilon}_k + \|x^k - x^*\|\} \\ &= c. \end{aligned} \tag{3.7}$$

Using $x^{k+1} = \alpha_k x^k + (1-\alpha_k)h^k$ and $\{\alpha_k\} \subset [m,n] \subset (0,1)$, we have

$$\lim_{k\to\infty} \|\alpha_k(x^k - x^*) + (1-\alpha_k)(h^k - x^*)\| = \lim_{k\to\infty} \|x^{k+1} - x^*\| = c. \tag{3.8}$$

Combining Lemma 3.6, (3.7) and (3.8), we have

$$\lim_{k\to\infty} \|h^k - x^k\| = 0.$$

Consequently, we get

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = \lim_{k\to\infty} (1-\alpha_k)\|h^k - x^k\| = 0. \tag{3.9}$$

From $(Proj_1)$, it follows that

$$\begin{aligned}\|Pr_{Sol(G,C)}\big(y^k-\lambda F(y^k)\big)-x^{k+1}\|\\ &\le \|Pr_{Sol(G,C)}\big(y^k-\lambda F(y^k)\big)-Pr_{Sol(G,C)}\big(z^k-\lambda F(z^k)\big)\|\\ &\quad+\|Pr_{Sol(G,C)}\big(z^k-\lambda F(z^k)\big)-h^k\|+\|h^k-x^{k+1}\|\\ &\le(1+\lambda L_1)\|y^k-z^k\|+\bar{\epsilon}_k+\|h^k-x^{k+1}\|\\ &=(1+\lambda L_1)\|y^k-z^k\|+\bar{\epsilon}_k+\frac{\alpha_k}{1-\alpha_k}\|x^k-x^{k+1}\|.\end{aligned}$$

Then, we have

$$\begin{aligned}&\|Pr_{Sol(G,C)}\big(x^k-\lambda F(x^k)\big)-x^k\|\\ &\le\|Pr_{Sol(G,C)}\big(x^k-\lambda F(x^k)\big)-Pr_{Sol(G,C)}\big(z^k-\lambda F(z^k)\big)\|+\|x^{k+1}-x^k\|\\ &\quad+\|Pr_{Sol(G,C)}\big(y^k-\lambda F(y^k)\big)-x^{k+1}\|\\ &\quad+\|Pr_{Sol(G,C)}\big(y^k-\lambda F(y^k)\big)-Pr_{Sol(G,C)}\big(z^k-\lambda F(z^k)\big)\|\\ &\le(1+\lambda L_1)\|x^k-z^k\|+(1+\lambda L_1)\|y^k-z^k\|+\|x^{k+1}-x^k\|\\ &\quad+\|Pr_{Sol(G,C)}\big(y^k-\lambda F(y^k)\big)-x^{k+1}\|\\ &\le(1+\lambda L_1)\|x^k-z^k\|+(1+\lambda L_1)\|y^k-z^k\|+\|x^{k+1}-x^k\|\\ &\quad(1+\lambda L_1)\|y^k-z^k\|+\bar{\epsilon}_k+\frac{\alpha_k}{1-\alpha_k}\|x^k-x^{k+1}\|\\ &\le(1+\lambda L_1)\|x^k-z^k\|+2(1+\lambda L_1)\|y^k-z^k\|+\bar{\epsilon}_k+\frac{1}{1-\alpha_k}\|x^k-x^{k+1}\|.\end{aligned}\tag{3.10}$$

It follows from (3.9), (3.10) and Lemma 3.5 that

$$\lim_{k\to\infty}\|Pr_{Sol(G,C)}\big(x^k-\lambda F(x^k)\big)-x^k\|=0.\tag{3.11}$$

Lemma 3.3 shows that the sequence $\{x^k\}$ is bounded. Then, there exists $M>0$ such that

$$\|Pr_{Sol(G,C)}(x^k-\lambda F(x^k))-x^*\|\le M\quad\forall k\ge 0.\tag{3.12}$$

Since $(Proj_1)$, F is β-strongly monotone and L_1-Lipschitz continuous, we have

$$\begin{aligned}&\|Pr_{Sol(G,C)}(x^k-\lambda F(x^k))-x^*\|\\ &\quad=\|Pr_{Sol(G,C)}(x^k-\lambda F(x^k))-Pr_{Sol(G,C)}(x^*-\lambda F(x^*))\|^2\\ &\quad\le\|x^k-\lambda F(x^k)-(x^*-\lambda F(x^*))\|^2\\ &\quad=\|x^k-x^*\|^2-2\lambda\langle F(x^k)-F(x^*),x^k-x^*\rangle\\ &\qquad+\lambda^2\|F(x^k)-F(x^*)\|^2\\ &\quad\le\|x^k-x^*\|^2-2\lambda\beta\|x^k-x^*\|^2+\lambda^2L_1^2\|x^k-x^*\|^2.\end{aligned}$$

Combining this and (3.12), we have

$$\begin{aligned}\|x^k - x^*\|^2 =&\|x^k - Pr_{Sol(G,C)}(x^k - \lambda F(x^k))\|^2 + \|x^* - Pr_{Sol(G,C)}(x^k - \lambda F(x^k))\|^2\\ &+ 2\langle x^k - Pr_{Sol(G,C)}(x^k - \lambda F(x^k)), Pr_{Sol(G,C)}(x^k - \lambda F(x^k)) - x^*\rangle\\ \le&\|x^k - Pr_{Sol(G,C)}(x^k - \lambda F(x^k))\|^2\\ &+ 2M\|x^k - Pr_{Sol(G,C)}(x^k - \lambda F(x^k))\|\\ &+ \|x^k - x^*\|^2 - 2\lambda\beta\|x^k - x^*\|^2 + \lambda^2 L_1^2\|x^k - x^*\|^2.\end{aligned} \tag{3.13}$$

Using this, (3.11) and $\lambda < \frac{2\beta}{L_1^2}$, we get

$$\begin{aligned}\lambda(2\beta - \lambda L_1^2)\|x^k - x^*\|^2 \le&\|x^k - Pr_{Sol(G,C)}(x^k - \lambda F(x^k))\|^2\\ &+ 2M\|x^k - Pr_{Sol(G,C)}(x^k - \lambda F(x^k))\|\\ \to&0 \quad \text{as } k \to \infty.\end{aligned}$$

Thus, the sequence $\{x^k\}$ converges strongly to $x^* \in Sol(BVI)$. Then Lemma 3.5 implies that the sequences $\{x^k\}, \{y^k\}$ and $\{z^k\}$ must converge strongly to the unique solution x^* of problems (BVI).

Now, we set

$$t^k = Pr_{Sol(G,C)}(x^k) \text{ and } x^k \to x^* \text{ as } k \to \infty.$$

Then, it follows from $(Proj_3)$ and $x^* \in C$ that

$$\langle x^* - t^k, t^k - x^k\rangle \ge 0.$$

By Lemma 3.7 and (3.1), $\{t^n\}$ converges strongly to some $\bar{x} \in Sol(G, C)$. Therefore, we have

$$\lim_{k\to\infty} \langle \bar{x}^* - t^k, t^k - x^k\rangle \ge 0 \Rightarrow \langle x^* - \bar{x}, \bar{x} - x^*\rangle \ge 0,$$

and $x^* \equiv \bar{x}$. Thus the sequences $\{x^k\}, \{y^k\}$ and $\{z^k\}$ converge strongly to x^*, where

$$x^* = \lim_{k\to\infty} Pr_{Sol(G,C)}(x^k).$$

□

As a direct consequence of Theorem 3.8, we obtain the following corollary.

Corollary 3.9. *Let C be a nonempty closed convex subset of $\mathcal{H}$, $G : \mathcal{H} \to \mathcal{H}$ be monotone and L-Lipschitz continuous. Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by*

$$\begin{cases} x^0 \in \mathcal{H},\\ y^k = Pr_C\big(x^k - \lambda_k G(x^k)\big),\\ x^{k+1} = \alpha_k x^k + (1 - \alpha_k) SPr_C\big(x^k - \lambda_k G(y^k)\big) \quad \forall k \ge 0,\end{cases}$$

where $\{\alpha_k\}$ *and* $\{\delta_k\}$ *satisfy the following conditions:*

$$\begin{cases} 0 < \lambda_k \leq \frac{1}{L} \ \forall k \geq 0, \\ \sum\limits_{k=1}^{\infty} \alpha_k = \infty, \lim\limits_{k\to\infty} \alpha_k = 0. \end{cases}$$

Then $\{x^k\}$ *and* $\{y^k\}$ *converge strongly to the same* $\bar{x} \in Sol(G, C)$.

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Department of Scientific Fundamentals
Posts and Telecommunications Institute of Technology, Hanoi, Vietnam.
E-mail address: `anhpn@ptit.edu.vn`