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THE RATE OF INCREASE OF MEAN VALUES OF FUNCTIONS IN WEIGHTED HARDY SPACES

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Abstract. Let $0 < p < \infty$ and $0 \leq q < \infty$. For each f in the weighted Hardy space $H_{p,q}$, we show that $d\|f_r\|_{p,q}^p/dr$ grows at most like $o(1/1-r)$ as $r \to 1$.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane and $0 < p < \infty$. The Hardy space $H^p(\mathbb{D})$ is the family of all analytic functions f in $\mathbb{D}$ satisfying

$$\|f\|_p = \lim_{r\to 1} \|f_r\|_p < \infty,$$

where

$$\|f_r\|_p = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p}.$$

If $0 \leq q < \infty$, define

$$\|f_r\|_{p,q} = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p (1-r^2)^q d\theta\Big)^{1/p}.$$

The weighted Hardy space $H_{p,q}(\mathbb{D})$ is the family of all analytic functions f in $\mathbb{D}$ satisfying

$$\|f\|_{p,q} = \sup_{0<r<1} \|f_r\|_{p,q} < \infty.$$

It is trivial that $H_{p,0}$ is just the classical Hardy space H^p and $H_{p,q}$ is a Banach space for $p \geq 1$. There are many books about H^p and $H_{p,q}$ such as Duren's book [3] and Zhu's book [7]. It is well known that $\|f_r\|_p$ is a nondecreasing function of r. Furthermore, Hardy [4] showed that $\log \|f_r\|_p$ is a convex function of $\log r$. Other properties of mean values of analytic functions can be found in [1, 2, 6].

Recently, Mashreghi [5] showed that $d\|f_r\|_p/dr$ grows at most like $o(1/1-r)$ as $r \to 1$. In this paper, we generalized this result to weighted Hardy spaces.

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2. Lemmas

Suppose that $f \not\equiv 0$ is analytic in $\mathbb{D}$. Let $W(z) = |f(z)|^p(1-|z|^2)^q$ for $0 < p < \infty$, $0 \leq q < \infty$, and ∇ be the gradient operator.

Lemma 2.1. *Let $0 < r \leq 1$ and $D_r = \{z : |z| < r\}$. For $z_0 \in D_r$ and small $\varepsilon > 0$, let $D(z_0, \varepsilon) = \{z : |z - z_0| \leq \varepsilon\} \subset D_r$ and*

$$I_\varepsilon = \int_{\partial D(z_0,\varepsilon)} \Big(\log(r/|z|)\frac{\partial W}{\partial n} - W\frac{\partial \log(r/|z|)}{\partial n}\Big) d\ell.$$

If $z_0 \neq 0$ is a zero of f, then $\lim_{\varepsilon\to 0} I_\varepsilon = 0$. If $z_0 = 0$ then $\lim_{\varepsilon\to 0} I_\varepsilon = 2\pi|f(0)|^p$.

Proof. Since $W(z) = |f(z)|^p(1-|z|^2)^q$, direct computation gives

$$\nabla W(z) = (1-|z|^2)^q \nabla|f(z)|^p + |f(z)|^p \nabla(1-|z|^2)^q$$

and

$$|\nabla|f(z)|^p| \leq p|f(z)|^{p-1}|f'(z)|, \quad |\nabla(1-|z|^2)^q| = 2q|z|(1-|z|^2)^{q-1}.$$

So

$$\begin{aligned}\Big|\frac{\partial W(z)}{\partial n}\Big| &\leq \big|\nabla W(z)\big| \leq (1-|z|^2)^q|\nabla|f(z)|^p| + |f(z)|^p|\nabla(1-|z|^2)^q| \\ &\leq p|f(z)|^{p-1}|f'(z)|(1-|z|^2)^q + 2q|z||f(z)|^p(1-|z|^2)^{q-1}\end{aligned}$$

and

$$\Big|\frac{\partial \log(r/|z|)}{\partial n}\Big| \leq |\nabla \log(r/|z|)| = \frac{1}{|z|}.$$

Write

$$\begin{aligned} I_\varepsilon &= \int_{\partial D(z_0,\varepsilon)} \Big(\log(r/|z|)\frac{\partial W}{\partial n} - W\frac{\partial \log(r/|z|)}{\partial n}\Big) d\ell \\ &= \int_0^{2\pi} \log(r/|z_0+\varepsilon e^{i\theta}|)\frac{\partial W(z_0+\varepsilon e^{i\theta})}{\partial n}\varepsilon d\theta - \int_{\partial D(z_0,\varepsilon)} W(z)\frac{\partial \log(r/|z|)}{\partial n} d\ell \\ &= I_1 - I_2. \end{aligned}$$

For convenience, let $C > 0$ be a constant independent of ε, whose value may change from line to line.

If $z_0 \neq 0$ is a zero of order $k \geq 1$, then $|I_1| \leq C\varepsilon^{kp}$ and $|I_2| \leq C\varepsilon^{k+1}$. Thus $\lim_{\varepsilon\to 0} I_\varepsilon(z_0) = 0$.

If $z_0 = 0$ and $f(z_0) \neq 0$, then $|I_1| \leq \varepsilon \log(r/\varepsilon)$ and

$$-I_2 = \int_0^{2\pi} W(\varepsilon e^{i\theta}) d\theta = (1-\varepsilon^2)^q \int_0^{2\pi} |f(\varepsilon e^{i\theta})|^p d\theta.$$

Hence $\lim_{\varepsilon\to 0} I_\varepsilon = 2\pi|f(0)|^p$.

At last, if $z_0 = 0$ is a zero of f of order $k \geq 1$, then $|I_1| \leq C\varepsilon^{kp}\log(r/\varepsilon)$ and $|I_2| \leq C\varepsilon^{kp}$. We have $\lim_{\varepsilon\to 0} I_\varepsilon = 2\pi|f(0)|^p$ also. □

Lemma 2.2. *Let $f \in H_{p,\ q}(\mathbb{D})$, $0 < p < \infty$, $0 \le q < \infty$, and $f \not\equiv 0$. Then*

$$\lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|^p(1-r^2)^q d\theta - 2\pi|f(0)|^p = \int_{|z|<1} \log(1/|z|)G(z)dxdy\ ,$$

where

$$G(z) = (1-|z|^2)^q\nabla^2|f|^p + 2\nabla|f|^p\cdot\nabla(1-|z|^2)^q + |f|^p\nabla^2(1-|z|^2)^q.$$

Proof. Since $f \not\equiv 0$ is analytic in $\mathbb{D}$, for any $0 < R < 1$, we can choose r, $R < r < 1$ such that f has finitely many zeros in $D_r = \{|z| < r\}$ and no zeros on the circle ∂D_r. Let $\{z_1, z_2, \cdots, z_n\}$ be the set consisting of 0 and all zeros of f in D_r. Take $\varepsilon > 0$ so small that $\{z : |z - z_k| \le \varepsilon\} \subset D_r$ for $1 \le k \le n$, and all these disks are disjoint. Denote

$$\Omega = D_r \setminus \bigcup_{k=1}^{n}\{z : |z - z_k| \le \varepsilon\}.$$

The function $W(z) = |f(z)|^p(1 - |z|^2)^q$ is infinitely differentiable in a neighborhood of $\overline{\Omega}$. Then by Green's theorem, we have

$$\int_\Omega \log(r/|z|)\nabla^2W(z)dxdy = \int_{\partial\Omega}\Big(\log(r/|z|)\frac{\partial W}{\partial n} - W\frac{\partial\log(r/|z|)}{\partial n}\Big)d\ell. \tag{2.1}$$

Direct computation gives

$$\nabla^2W(z) = (1-|z|^2)^q\nabla^2|f|^p + 2\nabla|f|^p\cdot\nabla(1-|z|^2)^q + |f|^p\nabla^2(1-|z|^2)^q = G(z).$$

The direction of every small circle in $\partial\Omega$ is clockwise. For $\varepsilon \to 0$, by Lemma 2.1, formula (2.1) becomes

$$\int_{\partial D_r}\Big(\log(r/|z|)\frac{\partial W}{\partial n} - W\frac{\partial\log(r/|z|)}{\partial n}\Big)d\ell - 2\pi|f(0)|^p = \int_{D_r}\log(r/|z|)G(z)dxdy.$$

The left-side of the above formula is equal to

$$\int_0^{2\pi} W(re^{i\theta})d\theta - 2\pi|f(0)|^p.$$

Now we obtain the lemma as $r \to 1$. □

3. The rate of increase of $\|f_r\|_{p,q}^p$

Theorem 3.1. *Let $f \in H_{p,\ q}(\mathbb{D})$ and $f \not\equiv 0$, then*

$$\frac{d\|f_r\|_{p,q}^p}{dr} = o(1/1-r)\ \text{as}\ r \to 1.$$

Proof. As in the proof of Lemma 2.2, we choose suitable r and ε. Replacing $\log r/|z|$ by $\log 1/|z|$, by Green's theorem, we have

$$\int_\Omega \log(1/|z|)\nabla^2W(z)dxdy = \int_{\partial\Omega}\Big(\log(1/|z|)\frac{\partial W}{\partial n} - W\frac{\partial\log(1/|z|)}{\partial n}\Big)d\ell. \tag{3.1}$$

For $\varepsilon \to 0$, using Lemma 2.1 for $r = 1$, formula (3.1) turns into

$$\int_{D_r}\log(1/|z|)G(z)dxdy = \int_{\partial D_r}\Big(\log(1/|z|)\frac{\partial W}{\partial n} - W\frac{\partial\log(1/|z|)}{\partial n}\Big)d\ell - 2\pi|f(0)|^p$$

$$= \int_0^{2\pi} \Big(\log \frac{1}{r} \frac{\partial W}{\partial r}(re^{i\theta}) + \frac{W(re^{i\theta})}{r} \Big) r d\theta - 2\pi |f(0)|^p$$

$$= \int_0^{2\pi} W(re^{i\theta}) d\theta - r \log r \int_0^{2\pi} \frac{\partial W}{\partial r}(re^{i\theta}) d\theta - 2\pi |f(0)|^p$$

$$= \int_0^{2\pi} |f(re^{i\theta})|^p (1-r^2)^q d\theta - 2\pi r \log r \frac{d\|f_r\|_{p,q}^p}{dr} - 2\pi |f(0)|^p.$$

Now let $r \to 1$, by Lemma 2.2, we have

$$\lim_{r\to 1} \log r \frac{d\|f_r\|_{p,q}^p}{dr} = 0.$$

Hence

$$\lim_{r\to 1} \frac{d\|f_r\|_{p,q}^p}{dr} = o(1/\log r) = o(1/1-r).$$

□

By Theorem 3.1, if $\lim\limits_{r\to 1} \|f_r\|_{p,q} \neq 0$, or for $q = 0$, $\|f_r\|_{p,q}$ is increasing with r, then we can get $\lim\limits_{r\to 1} d\|f_r\|_{p,q}/dr = o(1/1-r)$.

In the proof of Theorem 3.1, using 1 instead of $\log 1/|z|$, similar arguments can deduce

$$2\pi r \frac{d\|f_r\|_{p,q}^p}{dr} = \int_{|z|<r} G(z) dxdy.$$

Corollary 1. *If $f \in H_{p,q}$ with $0 < p < \infty,\ 0 \le q < \infty$, then we have*

$$2 \lim_{r\to 1} \int_0^{2\pi} |f(re^{i\theta})|^p (1-r^2)^q d\theta = \int_{\mathbb{D}} (1-|z|^2) \nabla^2 \Big(|f(z)|^p (1-|z|^2)^q \Big) dxdy$$

$$+\ 4 \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^q dxdy. \tag{3.2}$$

Proof. Denote

$$J_\varepsilon = \int_{\partial D(z_0,\varepsilon)} \Big((1-|z|^2) \frac{\partial W}{\partial n} - W \frac{\partial (1-|z|^2)}{\partial n} \Big) d\ell,$$

where z_0 is a zero of f. Similar arguments as in the proof of Lemma 2.1, we have $\lim\limits_{\varepsilon\to 0} J_\varepsilon = 0$. Choosing suitable r and ε as in the proof of Theorem 3.1, by Green's theorem, we have

$$\int_\Omega (1-|z|^2) \nabla^2 W(z) - \nabla^2 (1-|z|^2) \cdot W(z) dxdy$$

$$= \int_{\partial\Omega} \Big((1-|z|^2) \frac{\partial W}{\partial n} - W \frac{\partial (1-|z|^2)}{\partial n} \Big) d\ell$$

As $\varepsilon \to 0$, the above formula turns into

$$\int_{D_r} (1-|z|^2) \nabla^2 W(z) + 4W(z) dxdy = \int_{\partial D_r} \Big((1-|z|^2) \frac{\partial W}{\partial n} - W \frac{\partial (1-|z|^2)}{\partial n} \Big) d\ell$$

$$= \int_0^{2\pi} \Big((1-r^2) \frac{\partial W}{\partial r}(re^{i\theta}) + 2rW(re^{i\theta}) \Big) r d\theta. \tag{3.3}$$

By Theorem 3.1, we have

$$\lim_{r\to 1} \int_0^{2\pi} r(1-r^2)\frac{\partial W}{\partial r}(re^{i\theta}) d\theta = 2 \lim_{r\to 1} \left[(1-r)\frac{\partial}{\partial r} \int_0^{2\pi} W(re^{i\theta}) d\theta \right] = 0 \ .$$

The proof is completed by letting $r \to 1$ in formula (3.3). □

Note that $\nabla^2 |f|^p = p^2 |f|^{p-2} |f'|^2$. Taking $q = 0$ in formula (3.2), we obtain

$$\int_0^{2\pi} |f(e^{i\theta})|^p d\theta = \frac{p^2}{2} \int_{\mathbb{D}} (1-|z|^2)|f(z)|^{p-2}|f'(z)|^2 dxdy + 2\int_{\mathbb{D}} |f(z)|^p dxdy,$$

which is reminiscent of Hardy-Stein identity

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta = |f(0)|^p + \frac{p^2}{2\pi} \int_{\mathbb{D}} \log \frac{1}{|z|} |f(z)|^{p-2} |f'(z)|^2 dxdy.$$

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