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NEW RESULTS FOR BOUNDARY VALUE PROBLEMS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH NON-SEPARATED BOUNDARY CONDITIONS

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Abstract. In this paper, a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with non-separated boundary conditions is studied. Some new existence and uniqueness results are obtained by using Leray-Schauder degree theory and fixed point theorems. Some interesting observations are presented.

1. Introduction

In this paper, we discuss the existence and uniqueness of the solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with non-separated boundary conditions. Our results are based on Leray-Schauder degree theory, the contraction mapping principle and Krasnoselskii's fixed point theorem. More precisely, we consider the problem

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)),\ t \in [0, T],\ T > 0,\ 1 < q \le 2, \\ x(0) = \lambda_1 x(T) + \mu_1, \quad x'(0) = \lambda_2 x'(T) + \mu_2,\ \lambda_1 \ne 1,\ \lambda_2 \ne 1, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, and $f : [0, T]\times\mathbb{R} \to \mathbb{R}$ and $\lambda_1,\ \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

The theory of boundary value problems for nonlinear fractional differential equations is still in its initial stages and many aspects of this theory need to be explored. For some recent results on fractional boundary value problems, see [1-9, 13, 15] and the references therein.

2. Preliminaries

Let us recall some basic definitions [10, 12, 14].

Definition 2.1. For a function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds,\ n-1 < q < n, n = [q] + 1,$$

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where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \ q > 0,$$

provided the integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a function $g(t)$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} (\frac{d}{dt})^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \ n = [q]+1,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Lemma 2.4 ([12])**.** *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, \ i = 0, 1, 2, ..., n-1$ ($n = [q]+1$).

In view of Lemma 2.4 it follows that

$$I^q \ {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1} \tag{2.1}$$

for some $c_i \in \mathbb{R}, \ i = 0, 1, 2, ..., n-1$ $(n = [q]+1)$.

Lemma 2.5. *For $h \in C[0, T]$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = h(t), \ 0 < t < T, \ 1 < q \leq 2, \\ x(0) = \lambda_1 x(T) + \mu_1, \qquad x'(0) = \lambda_2 x'(T) + \mu_2, \end{cases} \tag{2.2}$$

is given by

$$x(t) = \int_0^T G(t,s) h(s) ds + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} - \frac{\mu_1}{(\lambda_1 - 1)},$$

where $G(t, s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{\lambda_1 (T-s)^{q-1}}{(\lambda_1 - 1)\Gamma(q)} + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t](T-s)^{q-2}}{(\lambda_2 - 1)(\lambda_1 - 1)\Gamma(q-1)}, \ 0 \leq s \leq t \leq T, \\ \\ -\frac{\lambda_1 (T-s)^{q-1}}{(\lambda_1 - 1)\Gamma(q)} + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t](T-s)^{q-2}}{(\lambda_2 - 1)(\lambda_1 - 1)\Gamma(q-1)}, \ 0 \leq t \leq s \leq T. \end{cases} \tag{2.3}$$

Proof. Using (2.1), for some constants $c_0, c_1 \in \mathbb{R}$ we have

$$x(t) = I^q h(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s) ds - c_0 - c_1 t. \tag{2.4}$$

In view of the relations ${}^cD^q\ I^q x(t) = x(t)$ and $I^q\ I^p x(t) = I^{q+p}x(t)$ for $q, p > 0, x \in L(0,T)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} h(s)ds - c_1.$$

Applying the boundary conditions for (2.2) we find that

$$\begin{aligned} c_0 &= \frac{\lambda_1}{(\lambda_1-1)}\Big[\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} h(s)ds \\ &\qquad - \frac{T\lambda_2}{(\lambda_2-1)}\Big(\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} h(s)ds + \frac{\mu_2}{\lambda_2}\Big) + \frac{\mu_1}{\lambda_1}\Big], \\ c_1 &= \frac{\lambda_2}{(\lambda_2-1)}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} h(s)ds + \frac{\mu_2}{(\lambda_2-1)}. \end{aligned}$$

Substituting the values of c_0 and c_1 in (2.4) we obtain the unique solution of (2.2) given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds - \frac{\lambda_1}{(\lambda_1-1)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} h(s)ds \\ &\quad + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} h(s)ds \\ &\quad + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)} \\ &= \int_0^T G(t,s)h(s)ds + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}, \end{aligned}$$

where $G(t,s)$ is given by (2.3). This completes the proof. □

By Lemma 2.5, problem (1.1) is reduced to the fixed point problem

$$x = \mathcal{F}(x), \tag{2.5}$$

where $\mathcal{F} : C[0,T] \to C[0,T]$ is given by

$$\begin{aligned} (\mathcal{F}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &\quad + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds \\ &\quad + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] - \mu_1\xi_1,\ \ t \in [0,T], \\ &\qquad \xi_1 = \frac{1}{(\lambda_1-1)},\ \ \xi_2 = \frac{1}{(\lambda_2-1)(\lambda_1-1)}. \end{aligned}$$

Formula (2.5) will be used to prove the existence of solutions of (1.1) in Section 3. In order to establish existence and uniqueness results for (1.1) in a Banach space (Section 4), let $(X, \|.\|)$ be a Banach space and let $\mathcal{C} = \mathrm{C}([0,T], X)$ denote the Banach space of continuous functions $[0,T] \to X$ endowed with the topology of uniform convergence with the sup-norm denoted by $\|.\|$. In this case, the operator $\mathcal{F}$ in (2.5) will be defined as $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ with $f:[0,T]\times X\to X$ and $\mu_1,\mu_2\in X$. To prove Theorem 4.3, we need the operators $\mathcal{F}_1,\mathcal{F}_2:\mathcal{C}\to\mathcal{C}$ which will be defined later.

3. An existence result via Leray-Schauder criterion

Theorem 3.1. *Assume that there exist constants* $0\le\kappa<\frac{\Gamma(q+1)}{(1+|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)}$ *and* $M>0$ *such that* $|f(t,x)|\le\frac{\kappa}{T^q}|x|+M$ *for all* $t\in[0,T], x\in C[0,T]$. *Then, the boundary value problem (1.1) has at least one solution.*

Proof. In view of the fixed point problem (2.5) we just need to prove the existence of at least one solution $x\in C[0,T]$ satisfying (2.5). Define a suitable ball $B_R\subset C[0,T]$ with radius $R>0$ as

$$B_R=\{x\in C[0,T]:\max_{t\in[0,T]}|x(t)|<R\},$$

where R will be fixed later. Then it is sufficient to show that $\mathcal{F}:\overline{B}_R\to C[0,T]$ satisfies

$$x\neq\lambda\mathcal{F}x,\ \forall\ x\in\partial B_R \text{ and } \forall\ \lambda\in[0,1]. \tag{3.1}$$

Let us set

$$H(\lambda,x)=\lambda\mathcal{F}x,\ x\in C(\mathbb{R})\ \lambda\in[0,1].$$

Then, by the Arzela-Ascoli theorem, $h_\lambda(x)=x-H(\lambda,x)=x-\lambda\mathcal{F}x$ is completely continuous. If (3.1) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\lambda,B_R,0)&=\deg(I-\lambda\mathcal{F},B_R,0)=\deg(h_1,B_R,0)\\&=\deg(h_0,B_R,0)=\deg(I,B_R,0)=1\neq0,\ 0\in B_r,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have $h_1(t)=x-\lambda\mathcal{F}x=0$ for at least one $x\in B_R$. In order to prove (3.1) we assume that $x=\lambda\mathcal{F}x$ for some $\lambda\in[0,1]$ and for all $t\in[0,T]$ so that

$$\begin{aligned}|x(t)| &= |\lambda\mathcal{F}x(t)|\\
&\le \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|\xi_1\lambda_1|\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\\
&\quad+|\xi_2\lambda_2||\lambda_1T+(1-\lambda_1)t|\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds\\
&\quad+|\xi_2\mu_2||\lambda_1T+(1-\lambda_1)t|+|\mu_1\xi_1|\\
&\le \Big(\frac{\kappa}{T^q}|x|+M\Big)\Big[\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}ds+|\xi_1\lambda_1|\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}ds\\
&\quad+|\xi_2\lambda_2||\lambda_1T+(1-\lambda_1)t|\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}ds\Big]\\
&\quad+|\xi_2\mu_2||\lambda_1T+(1-\lambda_1)t|+|\mu_1\xi_1|\\
&\le \Big(\frac{\kappa}{T^q}|x|+M\Big)\frac{T^q(1+|\xi_1\lambda_1|+|\xi_2\mu_2(1+\lambda_1)|q)}{\Gamma(q+1)}+|\xi_2\mu_2(1+\lambda_1)|T+|\mu_1\xi_1|,\end{aligned}$$

which, by taking norm ($\sup_{t\in[0,T]}|x(t)| = \|x\|$) and solving for $\|x\|$, yields

$$\|x\| \leq \frac{MT^q(1+|\xi_1\lambda_1|+|\xi_2\mu_2(1+\lambda_1)|q)+(|\xi_2\mu_2(1+\lambda_1)|T+|\mu_1\xi_1|)\Gamma(q+1)}{\Gamma(q+1)-\kappa(1+|\xi_1\lambda_1|+|\xi_2\mu_2(1+\lambda_1)|q)}.$$

Setting $R = \frac{MT^q(1+|\xi_1\lambda_1|+|\xi_2\mu_2(1+\lambda_1)|q)+(|\xi_2\mu_2(1+\lambda_1)|T+|\mu_1\xi_1|)\Gamma(q+1)}{\Gamma(q+1)-\kappa(1+|\xi_1\lambda_1|+|\xi_2\mu_2(1+\lambda_1)|q)}+1$ it follows that (3.1) holds. This completes the proof. □

Example 3.2. Consider the boundary value problem

$$\begin{cases} {}^cD^q x(t) = \frac{1}{(4\pi)}\sin(\frac{2\pi x}{T^q}) + \frac{|x|}{1+|x|}, \ t\in[0,T], \ 1<q\leq 2, \\ x(0) = -\frac{1}{2}u(T)+\mu_1, \qquad x'(0) = -\frac{4}{5}u'(T)+\mu_2. \end{cases} \tag{3.2}$$

Clearly,

$$\Big|f(t,x)\Big| = \Big|\frac{1}{(4\pi)}\sin(\frac{2\pi x}{T^q}) + \frac{|x|}{1+|x|}\Big| \leq \frac{1}{2T^q}\|x\|+1$$

with $\kappa = \frac{1}{2} < \frac{\Gamma(q+1)}{(1+|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)} = \frac{27\Gamma(q+1)}{4(q+9)}$ for $1<q\leq 2$ and $M=1$. Thus, the conclusion of Theorem 3.1 applies to (3.2).

Remark 3.3. For a positive constant N_1, we can modify the assumption on the nonlinear function $f(t,x)$ in Theorem 3.1 so that

$$|f(t,x)| \leq \frac{\Gamma(q+1)N_1}{T^q(1+|\xi_1\lambda_1|+|\xi_2\mu_2(1+\lambda_1)|q)} \ \forall t\in[0,T], \ x\in[-N_1,N_1].$$

4. Existence and uniqueness results in a Banach space

Throughout this section, let $(X,\|.\|)$ be a Banach space and let $f:[0,T]\times X\to X$. Furthermore, we assume that

(A_1) $|f(t,x)-f(t,y)| \leq L|x-y|, \forall t\in[0,T], \ x,y\in X$;
(A_2) $|f(t,x)| \leq \mu(t), \ \forall (t,x)\in[0,T]\times X$, and $\mu\in C([0,T],R^+)$;
(A_3) $\gamma_1 = \frac{LT^q(1+|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)} < 1$.

Theorem 4.1. *Assume that $f:[0,T]\times X\to X$ is a jointly continuous function, and that the assumptions (A_1) and (A_3) are satisfied. Then, the boundary value problem (1.1) has a unique solution.*

Proof. Let us set $\sup_{t\in[0,T]}|f(t,0)| = M$ and choose $r \geq \frac{\gamma_2}{(1-\gamma_1)}$, where

$$\gamma_2 = \frac{MT^q(1+|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)} + |\xi_2\mu_2(1+\lambda_1)|T+|\mu_1\xi_1|,$$

and γ_1 is given by the assumption (A_3). Now we show that $\mathcal{F} B_r \subset B_r$, where $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is defined by (2.5) and $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For $x \in B_r$ we have

$$
\begin{aligned}
\|(\mathcal{F}x)(t)\| \leq\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s,x(s))\| ds + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \|f(s,x(s))\| ds \\
& + |\xi_2\lambda_2| \sup_{t\in[0,T]} |\lambda_1 T + (1-\lambda_1)t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \|f(s,x(s))\| ds \\
& + |\xi_2\mu_2| \sup_{t\in[0,T]} |\lambda_1 T + (1-\lambda_1)t| + |\mu_1\xi_1| \\
\leq\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|) ds \\
& + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} (\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|) ds \\
& + |\xi_2\lambda_2| \sup_{t\in[0,T]} |\lambda_1 T + (1-\lambda_1)t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} (\|f(s,x(s)) - f(s,0)\| \\
& + \|f(s,0)\|) ds + |\xi_2\mu_2| \sup_{t\in[0,T]} |\lambda_1 T + (1-\lambda_1)t| + |\mu_1\xi_1| \\
\leq\ & \Big(Lr + M\Big)\Big[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + |\xi_1\lambda_1| \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} ds \\
& + |\xi_2\lambda_2| \sup_{t\in[0,T]} |\lambda_1 T + (1-\lambda_1)t| \int_0^T (T-s)^{q-2} ds\Big] \\
& + |\xi_2\mu_2| \sup_{t\in[0,T]} |\lambda_1 T + (1-\lambda_1)t| + |\mu_1\xi_1| \\
\leq\ & \frac{LT^q(1 + |\xi_1\lambda_1| + |\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)} r + \frac{MT^q(1 + |\xi_1\lambda_1| + |\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)} \\
& + |\xi_2\mu_2(1+\lambda_1)|T + |\mu_1\xi_1| \\
=\ & \gamma_1 r + \gamma_2 \leq r.
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,T]$ we obtain

$$
\begin{aligned}
& \|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \\
\leq\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s,x(s)) - f(s,y(s))\| ds \\
& + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \|f(s,x(s)) - f(s,y(s))\| ds \\
& + |\xi_2\lambda_2| \sup_{t\in[0,T]} |\lambda_1 T + (1-\lambda_1)t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \|f(s,x(s)) - f(s,y(s))\| ds
\end{aligned}
$$

$$\begin{aligned}
&\leq L\|x-y\|\Big[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + |\xi_1\lambda_1|\frac{1}{2\Gamma(q)}\int_0^T (T-s)^{q-1}ds \\
&\quad +|\xi_2\lambda_2|\sup_{t\in[0,T]}|\lambda_1 T+(1-\lambda_1)t|\int_0^T (T-s)^{q-2}ds\Big] \\
&\leq \frac{LT^q(1+|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)}\|x-y\| = \gamma_1\|x-y\|.
\end{aligned}$$

Observe that γ_1 depends only on the parameters involved in the problem. As $\gamma_1 < 1$, therefore $\mathcal{F}$ is a contraction. Thus, by the contraction mapping principle (Banach fixed point theorem), it follows that the boundary value problem (1.1) has a unique solution. □

Now, we prove the existence of solutions of (1.1) by applying Krasnoselskii's fixed point theorem [11].

Theorem 4.2 (Krasnoselskii's fixed point theorem)**.** *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then, there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 4.3. *Let $f : [0,T]\times X \to X$ be a jointly continuous function mapping bounded subsets of $[0,T]\times X$ into relatively compact subsets of X, and assume that (A_1) and (A_2) hold with*

$$\frac{LT^q(|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)} < 1.$$

Then, the boundary value problem (1.1) has at least one solution on $[0,T]$.

Proof. Let us fix

$$\overline{r} \geq \frac{\|\mu\|T^q(1+|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)} + |\xi_2\mu_2(1+\lambda_1)|T + |\mu_1\xi_1|),$$

where $\|\mu\| = \sup_{t\in[0,T]}|\mu(t)|$ and consider $B_{\overline{r}} = \{x \in \mathcal{C} : \|x\| \leq \overline{r}\}$. We define the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $B_{\overline{r}}$ as

$$\begin{aligned}
(\mathcal{F}_1 x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds, \\
(\mathcal{F}_2 x)(t) &= -\xi_1\lambda_1\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds \\
&\quad +\xi_2\lambda_2[\lambda_1 T+(1-\lambda_1)t]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,u(s))ds \\
&\quad +\xi_2\mu_2[\lambda_1 T+(1-\lambda_1)t] - \mu_1\xi_1.
\end{aligned}$$

For $x, y \in B_{\overline{r}}$ we find that

$$\|\mathcal{F}_1 x+\mathcal{F}_2 y\| \leq \frac{\|\mu\|T^q(1+|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)} + |\xi_2\mu_2(1+\lambda_1)|T + |\mu_1\xi_1| \leq \overline{r}.$$

Thus, $\mathcal{F}_1x+\mathcal{F}_2y \in B_{\overline{r}}$. It follows from the assumption (A_1) that $\mathcal{F}_2$ is a contraction mapping for $\frac{LT^q(|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)} < 1$. The continuity of f implies that the operator $\mathcal{F}_1$ is continuous. Also, $\mathcal{F}_1$ is uniformly bounded on $B_{\overline{r}}$ as

$$\|\mathcal{F}_1 x\| \leq \frac{\|\mu\|T^q}{\Gamma(q+1)}.$$

Now we prove compactness of the operator $\mathcal{F}_1$. In view of (A_1), we define

$$\sup_{(t,x)\in[0,T]\times B_{\overline{r}}} |f(t,x)| = \overline{f},$$

and consequently we have

$$\begin{aligned}\|(\mathcal{F}_1x)(t_1)-(\mathcal{F}_1x)(t_2)\| &= \|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]f(s,x(s))ds \\ &\quad + \int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\| \\ &\leq \frac{\overline{f}}{\Gamma(q+1)}|2(t_2-t_1)^q+t_1^q-t_2^q|,\end{aligned}$$

which is independent of x. Thus, $\mathcal{F}_1$ is equicontinuous. Using the fact that f maps bounded subsets into relatively compact subsets, we see that $\mathcal{F}_1(\mathcal{A})(t)$ is relatively compact in X for every t, where $\mathcal{A}$ is a bounded subset of $\mathcal{C}$. So, $\mathcal{F}_1$ is relatively compact on $B_{\overline{r}}$. Hence, by the Arzela-Ascoli theorem, $\mathcal{F}_1$ is compact on $B_{\overline{r}}$. Thus, all the assumptions of Theorem 4.2 are satisfied. So, the conclusion of Theorem 4.2 implies that the boundary value problem (1.1) has at least one solution on $[0,T]$. □

Example 4.4. Consider the anti-periodic boundary value problem

$$\begin{cases} {}^cD^qx(t) = \frac{1}{(t+9)^2}\frac{|x|}{1+|x|}, \ t\in[0,2\pi], \ 1<q\leq 2, \\ x(0)=\frac{1}{4}x(2\pi)+\mu_1, \qquad x'(0)=\frac{1}{2}x'(2\pi)+\mu_2. \end{cases} \tag{4.1}$$

Here, $f(t,x)=\frac{1}{(t+9)^2}\frac{|x|}{1+|x|}$, $T=2\pi$, and μ_1,μ_2 are arbitrary elements of X. As $\|f(t,x)-f(t,y)\| \leq \frac{1}{81}\|x-y\|$, assumption (A_1) is satisfied with $L_1=\frac{1}{81}$. Further,

$$\frac{LT^q(1+|\xi_1\lambda_1|+|\xi_2\lambda_2(1+\lambda_1)|q)}{\Gamma(q+1)} = \frac{1}{243}\frac{(2\pi)^q(4+5q)}{\Gamma(q+1)} < 1.$$

Thus, by Theorem 4.1, the boundary value problem (4.1) has a unique solution on $[0,2\pi]$.

5. Discussion

In this paper, we have presented some existence and uniqueness results for nonlinear fractional differential equations of order $q\in(1,2]$ with non-separated boundary conditions. Our results give rise to various interesting situations. Some of them are listed below:

(i) The results for an anti-periodic boundary value problem of fractional differential equations of order $q \in (1, 2]$ follow as a special case by taking $\lambda_1 = -1 = \lambda_2$, $\mu_1 = 0 = \mu_2$ in (1.1). In this case, the operator $\mathcal{F} : C[0, T] \to C[0, T]$ takes the form

$$
\begin{aligned}
(\mathcal{F}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, u(s))ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, u(s))ds \\
&+ \frac{1}{4}(T-2t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s, u(s))ds, \ t \in [0, T],
\end{aligned}
$$

and Theorem 3.1 reduces to the corresponding result of [7].

(ii) For $q = 2$ we obtain new results for a second order boundary value problem with non-separated boundary conditions. In this case, the Green's function $G(t, s)$ is

$$
G(t,s) = \begin{cases} \frac{-\lambda_1(\lambda_2-1)(T-s)+\lambda_2[\lambda_1 T+(1-\lambda_1)t]}{(\lambda_1-1)(\lambda_2-1)}, & 0 \le t < s \le T, \\[2ex] \frac{(\lambda_1-1)(\lambda_2-1)(t-s)-\lambda_1(\lambda_2-1)(T-s)+\lambda_2[\lambda_1 T+(1-\lambda_1)t]}{(\lambda_1-1)(\lambda_2-1)}, & 0 \le s \le t \le T, \end{cases}
$$

which takes the following form for the second order anti-periodic boundary value problem ($\lambda_1 = -1 = \lambda_2$):

$$
G(t,s) = \begin{cases} \frac{1}{4}(-T-2t+2s), & 0 \le t < s \le T, \\[2ex] \frac{1}{4}(-T+2t-2s), & 0 \le s \le t \le T. \end{cases}
$$

(iii) The results for an initial value problem of fractional order $q \in (1, 2]$ can be obtained by taking $\lambda_1 = 0 = \lambda_2$ in the present results with the operator of the form

$$
(\mathcal{F}x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, u(s))ds + \mu_2 t + \mu_1.
$$

These results correspond to the non-impulsive part of the results of [1].

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